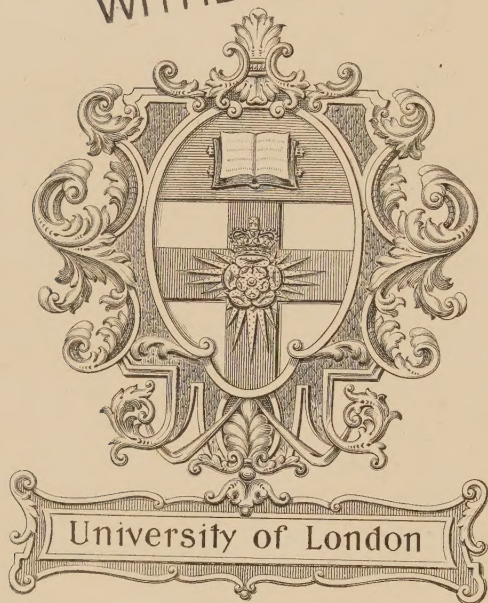


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PROCEEDINGS
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THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 46

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2.

1940



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P A P E R S

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON WIENER'S GENERAL HARMONIC ANALYSIS

By H. R. PITT.

[Received 28 July, 1938.—Read 17 November, 1938.]

1. *Introduction.*

The problem discussed in this paper is that of the harmonic representation of an arbitrary function. That is, if a function $f(x)$ is given in $(-\infty, \infty)$, we wish to determine conditions under which some meaning can be attached to the formula†

$$(1.1) \quad f(x) \sim \int e^{-iyx} ds(y).$$

The first general treatment of this problem was given by Wiener‡. Our object here is to extend his theory to a somewhat wider class of functions $f(x)$ and to investigate systematically how the formula (1.1) reduces to such classical harmonic developments as Fourier integrals and Fourier series of periodic or almost periodic functions. After that we show how certain well-known theorems on Fourier series can be extended to developments of the type (1.1).

† Integrals with unspecified limits are over $(-\infty, \infty)$.

‡ Wiener, 5, 6, and 7.

2. The definition of Wiener's integral.

We say that the function $f(x)$, defined in $(-\infty, \infty)$, belongs to the class $\mathfrak{B}$, and we write $f(x) \in \mathfrak{B}$, if $f(x)$ is integrable in any finite interval and if

$$(2.1) \quad \sum_{n=-\infty}^{\infty} F_n^p (|n|+1)^{-p} < \infty, \quad F_n = \int_n^{n+1} |f(x)| dx$$

for some p in the range $1 \leq p \leq 2$. We write $\mathfrak{B}_p$ instead of $\mathfrak{B}$ when it is necessary to indicate the particular value of p for which (2.1) holds. We write $f(x) \in B_p$ ($p \geq 1$) if

$$\frac{1}{2T} \int_{-T}^T |f(x)|^p dx$$

is bounded for all positive T . It is plain that $\mathfrak{B}_p$ includes B_p if $p > 1$ (but not if $p = 1$). Moreover, any function which is integrable and periodic, or almost periodic in the Stepanoff sense, belongs to $\mathfrak{B}$. For F_n is then bounded and the function belongs to $\mathfrak{B}_p$ for any $p > 1$.

The basis for the definition of Wiener's integral is

LEMMA 1. Suppose that $f(x) \in \mathfrak{B}_p$ for some p in the range $1 \leq p \leq 2$, and let

$$(2.2) \quad s_l(y) = \frac{1}{2\pi} \int_{-l}^l \frac{f(x)}{ix} [e^{ixy} - e^{-x^2}] dx.$$

Then, as $l \rightarrow \infty$, $s_l(y)$ converges in mean of order $\dagger p'$ over any finite interval to a function $s(y)$.

It is sufficient to show that, for any positive A ,

$$(2.3) \quad \lim_{l, l' \rightarrow \infty} \int_{-A}^A \left| \int_l^{l'} \frac{f(x) e^{ixy} dx}{x} \right|^{p'} dy = 0.$$

Let $l' > l > 2A^{-1}$. Let $p > 1$ (the case $p = 1$ requires only trivial modifications). Let

$$g(x) = f(x) \quad (l \leq x \leq l'), \quad g(x) = 0 \quad (x < l, x > l'),$$

$$G(x) = \frac{A}{2} \int_{x-A^{-1}}^{x+A^{-1}} \frac{g(u) du}{u}.$$

$\dagger$ We take

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad p' = \frac{p}{p-1}.$$

When $p = 1$, our statement is to be understood to mean that the essential upper bound of $|s(x) - s_l(x)|$ in any finite interval tends to zero.

Then an easy calculation shows that

$$\frac{A}{y} \sin \frac{y}{A} \int_l^y \frac{f(x) e^{ixy} dx}{x} = \int e^{ixy} G(x) dx.$$

Now it follows easily from (2.1) that $G(x) \in L^p$ and

$$\lim_{l \rightarrow \infty} \int |G(x)|^p dx = 0.$$

But we have, by a theorem† of Titchmarsh,

$$\int \left| \frac{A}{y} \sin \frac{y}{A} \int_l^y \frac{f(x) e^{ixy} dx}{x} \right|^{p'} dy \leq 2\pi \left[\int |G(x)|^p dx \right]^{p'-1},$$

and, since $(A/y) \sin(y/A) \geq \frac{1}{2}$ for $|y| \leq A$, the conclusion follows.

If $f(x) \in \mathfrak{B}$, we define $s(y)$ by Lemma 1 and express the relation between $f(x)$ and $s(y)$ by the formula (1.1). We call the expression on the right of (1.1) the Wiener integral of $f(x)$. In general, the integral does not exist in the ordinary Stieltjes sense, since $s(y)$ is defined only almost everywhere.

To avoid repetition, we suppose throughout this paper that $f(x) \in \mathfrak{B}_p$ and that $s(y)$ is defined by Lemma 1.

The expression

$$(2.4) \quad S_y(x) = s(y) e^{-ixy} - s(-y) e^{ixy} + ix \int_{-y}^y e^{-ixt} s(t) dt,$$

obtained formally from $\int_{-y}^y e^{-itx} ds(t)$ by partial integration, is defined for almost all positive values of y and all values of x . For any value of x , $S_y(x)$ belongs to $L^{p'}$ in any finite range of positive values of y . The Cesàro mean of $S_y(x)$ is defined for all positive values of y by

$$(2.5) \quad \sigma_y(x) = \frac{1}{y} \int_0^y S_t(x) dt = \int_{-y}^y \left[ix \left(1 - \frac{|u|}{y} \right) + \frac{|u|}{uy} \right] e^{-iux} s(u) du.$$

We also use a weaker average of $S_y(x)$, defined for any positive δ by

$$(2.6) \quad S_y^\delta(x) = \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} S_t(x) dt.$$

The relations between $S_y(x)$, $S_y^\delta(x)$, $\sigma_y(x)$ and $f(x)$ are given in the following Lemma.

† Titchmarsh, 4, 96, Theorem 74.

LEMMA 2. *We have*

$$(2.7) \quad S_y(x) \equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l f(u) \frac{\sin y(x-u)}{x-u} du,$$

the limit in mean being of order p' and being taken over any finite range of values of y . Also

$$(2.8) \quad \sigma_y(x) = \frac{1}{2\pi y} \int f(x-u) \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 du,$$

$$(2.9) \quad S_y^\delta(x) = \frac{1}{\pi\delta} \int f(x-u) \frac{\sin yu \sin \delta u}{u^2} du.$$

We have, by Lemma 1,

$$\begin{aligned} s(y)e^{-ixy} - s(-y)e^{ixy} \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{iy(u-x)} - e^{-iy(u-x)} + e^{-u^2+ixy} - e^{-u^2-ixy}] du, \\ ix \int_{-y}^y e^{-itx} s(t) dt &= ix \int_{-y}^y e^{-itx} dt \left[\text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{itu} - e^{-u^2}] du \right] \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{x}{2\pi} \int_{-l}^l \frac{f(u) du}{u} \int_{-y}^y e^{-itx} [e^{itu} - e^{-u^2}] dt. \end{aligned}$$

(Here, and frequently in the sequel, we use the result that, if $g_l(t)$ converges in mean, then its indefinite integral converges everywhere and also in mean.)

Hence

$$\begin{aligned} S_y(x) &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{u} \left[2 \sin(u-x)y + 2e^{-u^2} \sin xy \right. \\ &\quad \left. + x \int_{-y}^y e^{-itx} (e^{itu} - e^{-u^2}) dt \right] du \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l \frac{f(u)}{u} \left[1 + \frac{x}{u-x} \right] \sin(u-x)y du \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l f(u) \frac{\sin(u-x)y du}{u-x}. \end{aligned}$$

Now, using (2.7), we have

$$\begin{aligned} \int_0^y S_l(x) dt &= \int_0^y dt \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l f(u) \frac{\sin(u-x)t}{u-x} du \\ &= \frac{1}{\pi} \int \frac{f(u) du}{u-x} \int_0^y \sin(u-x)t dt = \frac{1}{2\pi} \int f(u) \left\{ \frac{\sin \frac{1}{2}y(u-x)}{\frac{1}{2}(u-x)} \right\}^2 du. \end{aligned}$$

Finally

$$\begin{aligned}
 S_y^\delta(x) &= \frac{1}{2\delta} \left\{ \int_0^{y+\delta} S_t(x) dt - \int_0^{y-\delta} S_t(x) dt \right\} \\
 &= \frac{1}{2\delta} \{ (y+\delta) \sigma_{y+\delta}(x) - (y-\delta) \sigma_{y-\delta}(x) \} \\
 &= \frac{1}{4\pi\delta} \int f(x-u) \left\{ \left(\frac{\sin \frac{1}{2}(y+\delta)u}{\frac{1}{2}u} \right)^2 - \left(\frac{\sin \frac{1}{2}(y-\delta)u}{\frac{1}{2}u} \right)^2 \right\} du \\
 &= \frac{1}{\pi\delta} \int f(x-u) \frac{\sin uy \sin u\delta}{u^2} du.
 \end{aligned}$$

3. The interpretation of classical harmonic developments as Wiener integrals.

(a) Fourier-Stieltjes transforms.

THEOREM 1. If

$$f(x) = \int e^{-iyx} ds^*(y), \quad \int |ds^*(y)| < \infty,$$

then $s(y) - s^*(y)$ is almost everywhere equal to a constant.

We have, for any real c ,

$$\begin{aligned}
 s(y) &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{iuy} - e^{-u^2}] du, \\
 s(y+c) - s(y) &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{i(y+c)u} - e^{iyu}] du.
 \end{aligned}$$

By a well-known theorem† on Fourier-Stieltjes integrals, we have

$$s^*(y+c) - s^*(y) = \lim_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{i(y+c)u} - e^{iyu}] du.$$

Hence, by a familiar theorem on mean convergence,

$$s(y+c) - s(y) = s^*(y+c) - s^*(y)$$

for almost all y and c . Interchanging y and c , and subtracting, we have

$$s(y) - s^*(y) = s(c) - s^*(c)$$

for almost all c and y , which is what we require.

† S. Bochner, 2, 66 Satz 17.

(b) *Fourier transforms*[†].

THEOREM 2. Suppose that $f(x) \in L^p(-\infty, \infty)$, $1 \leq p \leq 2$, and let $F(y)$ be its Fourier transform. Then

$$s(y) \equiv (2\pi)^{-\frac{1}{2}} \int_0^y F(u) du + C,$$

where C is constant.

We can write

$$\begin{aligned} \int_0^y F(u) du &= \int_0^y du \lim_{l \rightarrow \infty} (2\pi)^{-\frac{1}{2}} \int_{-l}^l f(t) e^{itu} dt \\ &= (2\pi)^{-\frac{1}{2}} \int f(t) dt \int_0^y e^{itu} du \\ &= (2\pi)^{-\frac{1}{2}} \int \frac{f(t)}{it} [e^{ity} - 1] dt \equiv (2\pi)^{\frac{1}{2}} [s(y) + C]. \end{aligned}$$

(c) *Almost periodic functions*[‡].

THEOREM 3. Suppose that $f(x)$ is almost periodic in the S_p sense ($p \geq 1$), and let its Fourier series be $\sum A_n e^{-i\lambda_n x}$. Suppose also that

$$(3.1) \quad \sum_{|\lambda_n| \leq \Lambda} |A_n| < \infty$$

for any finite Λ . Then

$$s(u') - s(u) \equiv \sum_{u \leq \lambda_n \leq u'} A_n.$$

In particular, if $f(x)$ is periodic and has the Fourier series $\sum A_n e^{in\pi x}$, then

$$s(u') - s(u) \equiv \sum_{u \leq n\pi \leq u'} A_n.$$

Let $\Lambda > 0$ and let

$$f_\Lambda(x) = \frac{1}{2\pi\Lambda} \int f(x-u) \left\{ \frac{\sin \frac{1}{2}\Lambda u}{\frac{1}{2}u} \right\}^2 du.$$

Then it is easy to show that $f_\Lambda(x)$ is uniformly almost periodic§ and has the Fourier series

$$\sum_{|\lambda_n| < \Lambda} A_n \left(1 - \frac{|\lambda_n|}{\Lambda} \right) e^{-i\lambda_n x}.$$

[†] Titchmarsh, 4, Ch. 4.

[‡] Besicovitch, 1, Ch. 2. For the case $p = 2$ of Theorem 3, see Wiener, 6, Theorem 20.

[§] Besicovitch, 1, Ch. 1.

Since, by hypothesis, this series is absolutely convergent, we have

$$f_{\Lambda}(x) = \sum_{|\lambda_n| < \Lambda} A_n \left(1 - \frac{|\lambda_n|}{\Lambda}\right) e^{-i\lambda_n x}.$$

By (2.5) and (2.8), we have

$$f_{\Lambda}(x) = \sigma_{\Lambda}(x) = \int_{-\Lambda}^{\Lambda} \left[ix \left(1 - \frac{|u|}{\Lambda}\right) + \frac{|u|}{\Lambda u} \right] e^{-iux} s(u) du.$$

Hence, if we define $s^*(u)$ so that

$$s^*(u') - s^*(u) = \sum_{u \leq \lambda_n < u'} A_n,$$

we have

$$\int_{-\Lambda}^{\Lambda} \left[ix \left(1 - \frac{|u|}{\Lambda}\right) + \frac{|u|}{\Lambda u} \right] e^{-iux} [s^*(u) - s(u)] du = 0,$$

and, since this holds for all Λ and x , it is plain that $s^*(u) - s(u)$ is almost everywhere equal to a constant.

We write

$$(3.2) \quad D(\lambda) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_{\lambda-\epsilon}^{\lambda+\epsilon} [s(y+\epsilon) - s(y-\epsilon)] dy$$

whenever the limit exists. Then $D(\lambda)$ is the $(C, 1)$ discontinuity of $s(y)$ at the point λ ; and, if it vanishes, we say that $s(y)$ is $(C, 1)$ continuous at y . There is a strong connection between the $(C, 1)$ discontinuities of $s(y)$ and the mean values of $f(x)e^{i\lambda x}$. Thus we have the following result.

THEOREM 4. *If $f(x)$ belongs both to $\mathfrak{B}$ and to B_1 , then*

$$(3.3) \quad D(\lambda) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(u) e^{iu\lambda} du,$$

in the sense that, if either side exists for any real λ , so does the other, and the two are equal.

We have

$$\begin{aligned} s(y) &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{2\pi} \int_{-l}^l \frac{f(u)}{iu} [e^{iyu} - e^{-u^2}] du, \\ \frac{1}{2\epsilon} \int_{\lambda-\epsilon}^{\lambda+\epsilon} [s(y+\epsilon) - s(y-\epsilon)] dy &= \frac{1}{2\pi\epsilon} \int f(u) \frac{\sin \epsilon u}{u} \int_{\lambda-\epsilon}^{\lambda+\epsilon} e^{iuy} dy \\ &= \frac{1}{\pi\epsilon} \int f(u) e^{iu\lambda} \left(\frac{\sin \epsilon u}{u} \right)^2 du. \end{aligned}$$

The conclusion now follows from the following Lemma†, which is due to Wiener.

LEMMA. If $\phi(u) \in B_1$, then

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \phi(u) du = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon} \int \phi(u) \left(\frac{\sin \epsilon u}{u} \right)^2 du$$

whenever either side exists.

As an immediate corollary of Theorem 4, we have the following result on almost periodic functions.

THEOREM 5. Suppose that $f(x)$ is B_1 almost periodic and belongs to $\mathfrak{B}$. Let its Fourier series be $\sum A_n e^{-i\lambda_n x}$. Then $s(y)$ is $(C, 1)$ continuous except at the points λ_n , where it has $(C, 1)$ discontinuities A_n .

This follows from Theorem 4 and the well-known property‡ of B almost periodic functions that

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{iu\lambda} f(u) du &= A_n \quad (\lambda = \lambda_n) \\ &= 0 \quad (\lambda \neq \lambda_n). \end{aligned}$$

4. Convergence and summability of Wiener's integral.

We now consider the convergence of $S_y^\delta(x)$ as $y \rightarrow \infty$. The following theorem gives a generalization of the well-known convergence theorems for Fourier series. In the case when $f(x)$ is periodic it follows from Theorem 3 that, when δ is sufficiently small, $S_y^\delta(x)$ takes all values of the partial sums of its Fourier series.

THEOREM 6. Suppose that $\delta > 0$, that $f(x)$ belongs to $\mathfrak{B}$ and that the Fourier series of a periodic function which coincides with $f(x)$ in an open interval containing x converges at x to the value $f(x)$. Then

$$(4.1) \quad \lim_{y \rightarrow \infty} S_y^\delta(x) = f(x).$$

† Wiener, 5, Theorem 21. Wiener actually assumes that $f(u) \geq 0$; but this assumption is used only to prove the boundedness of $\int_x^{x+1} |f(e^u)| du$, and this plainly holds when $f(x) \in B_1$.

‡ Besicovitch, 1°, 104.

Let $\eta > 0$. Then, since $f(x) \in \mathfrak{B}$, it is easy to show that $u^{-2}f(x-u)$ is integrable in $|u| \geq \eta$. It follows from the Riemann-Lebesgue theorem that

$$\lim_{y \rightarrow \infty} \frac{1}{\pi\delta} \int_{-\infty}^{-\eta} + \int_{\eta}^{\infty} [f(x-u) - f(x)] \frac{\sin uy \sin u\delta}{u^2} du = 0.$$

Therefore, if

$$\phi(u) = \phi(x, u) = f(x+u) + f(x-u) - 2f(x),$$

$$\begin{aligned} & \overline{\lim}_{y \rightarrow \infty} |S_y^\delta(x) - f(x)| \\ & \leq \overline{\lim}_{y \rightarrow \infty} \left| \frac{1}{\pi\delta} \int_{-\eta}^{\eta} [f(x-u) - f(x)] \frac{\sin uy \sin u\delta}{u^2} du \right| \\ & = \overline{\lim}_{y \rightarrow \infty} \left| \frac{1}{\pi\delta} \int_0^{\eta} \phi(u) \frac{\sin uy \sin u\delta}{u^2} du \right| \\ & \leq \overline{\lim}_{y \rightarrow \infty} \left| \int_0^{\eta} \phi(u) \frac{\sin uy}{\pi u} du \right| + \overline{\lim}_{y \rightarrow \infty} \left| \int_0^{\eta} \phi(u) \frac{\sin uy (\sin u\delta - u\delta)}{\pi\delta u^2} du \right|. \end{aligned}$$

The last term vanishes, by the Riemann-Lebesgue theorem, and the first vanishes, by hypothesis, since $f(x)$ satisfies a Fourier convergence condition.

As a corollary, we have

THEOREM 7. Suppose that $f(x)$ is S almost periodic, with the Fourier series $\sum A_n e^{-i\lambda_n x}$. Suppose that the Fourier series of a periodic function which coincides with $f(x)$ in an open interval containing x converges at x to the value $f(x)$. Suppose also that one of the following conditions is satisfied:

$$(a) \lambda_{n+1} - \lambda_n \geq \lambda > 0,$$

$$(b) P(\lambda) = \sum_{|\lambda_n| \leq \lambda} |A_n| \text{ converges for any finite } \lambda \text{ and } P(\lambda') - P(\lambda) \rightarrow 0 \\ \text{when } \lambda \rightarrow \infty, \quad 0 \leq \lambda' - \lambda \rightarrow 0.$$

Then
$$f(x) = \sum A_n e^{-i\lambda_n x} = \lim_{y \rightarrow \infty} \sum_{|\lambda_n| \leq y} A_n e^{-i\lambda_n x}.$$

By Theorem 3, we can write

$$s_y(x) = \sum_{|\lambda_n| \leq y} A_n e^{-i\lambda_n x}, \quad S_y^\delta(x) = \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} s_t(x) dt.$$

Suppose that (a) is satisfied and let $\epsilon > 0$. We can define a trigonometrical polynomial $Q(x)$ such that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x) - Q(x)| dx < \epsilon.$$

Then

$$|A_n| = \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) e^{i\lambda_n x} dx \right| \leq \epsilon + \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T Q(x) e^{i\lambda_n x} dx \right|.$$

Hence $\lim_{n \rightarrow \infty} A_n = 0$, and condition (a) implies (b).

If condition (b) is satisfied, then

$$\begin{aligned} |S_y^\delta(x) - s_y(x)| &= \left| \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} [s_t(x) - s_y(x)] dt \right| \\ &\leq P(y+\delta) - P(y-\delta). \end{aligned}$$

Hence

$$\begin{aligned} \overline{\lim}_{y \rightarrow \infty} |s_y(x) - f(x)| &\leq \overline{\lim}_{y \rightarrow \infty} |S_y^\delta(x) - f(x)| + \overline{\lim}_{y \rightarrow \infty} |P(y+\delta) - P(y-\delta)| \\ &= \overline{\lim}_{y \rightarrow \infty} |P(y+\delta) - P(y-\delta)| = \epsilon(\delta), \end{aligned}$$

say, where $\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0$. Therefore $\lim_{y \rightarrow \infty} s_y(x) = f(x)$.

The following summability theorem for Wiener's integral is an easy deduction from the Fejér-Lebesgue theorem.

THEOREM 8. *If $f(x) \in \mathfrak{B}$, then*

$$\lim_{y \rightarrow \infty} \sigma_y(x) = f(x)$$

for almost all values of x , and, in particular, for all values of x at which $f(x)$ is continuous.

We may suppose that $f(x)$ is finite. Then

$$\begin{aligned} \sigma_y(x) - f(x) &= \frac{1}{2\pi y} \int [f(x-u) - f(x)] \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 du, \\ |\sigma_y(x) - f(x)| &\leq \frac{1}{2\pi y} \int_{-1}^1 |f(x-u) - f(x)| \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 du \\ &\quad + \frac{1}{2\pi y} \int_{-1}^{-\infty} + \int_1^{\infty} |f(x-u)| \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 du + \frac{|f(x)|}{2\pi y} \int_{-\infty}^{-1} + \int_1^{\infty} \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 du. \end{aligned}$$

The first integral on the right is $o(1)$ as $y \rightarrow \infty$ for almost all x , by the Fejér-Lebesgue theorem. The remaining terms are plainly $o(1)$, since $f(x)$ is finite and $u^{-2}f(x-u)$ is integrable in $u \geq 1$.

5. *Mean convergence of Wiener's integral.*

We suppose that $p \geq 1$ and define four metrics of a function $f(x)$ as follows:

$$(5.1) \quad M_a[f(x)] = \left\{ \frac{1}{2a} \int_{-a}^a |f(x)|^p dx \right\}^{1/p},$$

$$(5.2) \quad M[f(x)] = \overline{\lim}_{a \rightarrow \infty} M_a[f(x)],$$

$$(5.3) \quad W[f(x)] = \overline{\lim}_{a \rightarrow \infty} \text{u.b.}_{-\infty < \tau < \infty} M_a[f(x+\tau)],$$

$$(5.4) \quad S_a[f(x)] = \text{u.b.}_{-\infty < \tau < \infty} M_a[f(x+\tau)].$$

We write $G[f(x)]$ when we wish to refer to one of these metrics without specifying which particular one. We say that $f(x)$ is continuous in G if $G[f(x)] < \infty$ and

$$(5.5) \quad \lim_{\epsilon \rightarrow 0} G[f(x+\epsilon) - f(x)] = 0.$$

Our next theorem is a generalization of a well known theorem* of M. Riesz on Fourier series.

THEOREM 9. *Suppose that $f(x) \in B_p$ and that*

$$\delta > 0, \quad p > 1, \quad a > 0.$$

Then

$$(5.6) \quad \lim_{y \rightarrow \infty} \int_y^{y+\delta} M_a[f(x) - S_t(x)] dt = 0.$$

If $f(x)$ is continuous in G , then

$$(5.7) \quad \lim_{y \rightarrow \infty} G[f(x) - S_y^\delta(x)] = 0.$$

We require three lemmas.

LEMMA 1. *If $f(x, t)$ is measurable and belongs to G for each t in (a, b) , then*

$$G \left[\int_a^b f(x, t) dt \right] \leq \int_a^b G[f(x, t)] dt.$$

* Zygmund, 8, 153, § 7.3.

If $f_1(x)$ and $f_2(x)$ belong to G , then

$$G[f_1(x) + f_2(x)] \leq G[f_1(x)] + G[f_2(x)].$$

This follows easily from Minkowski's inequality†.

LEMMA 2. Suppose that $\psi(x)$ has period $2a$ and belongs to L^p , $p > 1$. Then, as $l \rightarrow \infty$,

$$\frac{1}{\pi} \int_{-l}^l \psi(x-u) \frac{\sin yu}{u} du$$

converges in mean of order p' over any finite range of values of y to a function $\Psi(x, y)$. Moreover, for almost all y ,

$$M_a[\Psi(x, y)] \leq C M_a[\psi(x)],$$

where C depends only on p .

Since $\psi(x)$ belongs to B_p , the existence of $\Psi(x, y)$ follows from Lemma 2 of § 2. From the same lemma and Theorem 3, it is plain that, for almost all y , $\Psi(x, y)$ is the sum of the first $[y]$ terms of the Fourier series of $\psi(x)$ at the point x . Our conclusion is therefore equivalent to a well-known result‡ of M. Riesz.

LEMMA 3. If $f(x)$ is continuous in G , then

$$\lim_{y \rightarrow \infty} G[f(x) - \sigma_y(x)] = 0.$$

By (2.8), since

$$\frac{1}{2\pi} \int \left\{ \frac{\sin \frac{1}{2}y}{\frac{1}{2}y} \right\}^2 dy = 1,$$

we have

$$f(x) - \sigma_y(x) = \frac{1}{2\pi y} \int \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 [f(x) - f(x-u)] du,$$

and it follows from Lemma 1 that

$$\begin{aligned} (5.8) \quad G[f(x) - \sigma_y(x)] &\leq \frac{1}{2\pi y} \int \left\{ \frac{\sin \frac{1}{2}yu}{\frac{1}{2}u} \right\}^2 G[f(x) - f(x-u)] du \\ &= \frac{1}{2\pi} \int \left\{ \frac{\sin \frac{1}{2}u}{\frac{1}{2}u} \right\}^2 G \left[f(x) - f \left(x - \frac{u}{y} \right) \right] du. \end{aligned}$$

† Hardy, Littlewood, and Pólya, 3, 148, Theorem 202.

‡ Zygmund, 8, 153, § 7.3 (1).

Now, if G is M , W , or S_a , we have

$$G\left[f(x)-f\left(x-\frac{u}{y}\right)\right] \leq 2G[f(x)];$$

and, if G is M_a , it is easy to show that

$$G\left[f(x)-f\left(x-\frac{u}{y}\right)\right] \leq G[f(x)] + \left(\frac{u}{ay} + 1\right)^{1/p} \text{u.b.}_{t \geq a} M_t[f(x)].$$

Therefore the integrand on the right of (5.8) is majorised by an integrable function; and, since

$$\lim_{y \rightarrow \infty} G\left[f(x)-f\left(x-\frac{u}{y}\right)\right] = 0,$$

for each fixed u , by hypothesis, the conclusion follows.

We can now proceed to prove Theorem 9. Let

$$y' > 0, \quad \phi(x) = \phi(y', x) = f(x) - \sigma_{y'}(x).$$

Then, if $y \geq y'$,

$$\begin{aligned} S_y(x) - f(x) &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l f(x-u) \frac{\sin uy}{u} du - f(x) \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l [\phi(x-u) + \sigma_{y'}(x-u)] \frac{\sin uy}{u} du - f(x) \\ &\equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l \phi(x-u) \frac{\sin uy}{u} du - \phi(x), \end{aligned}$$

since it follows easily from (2.4), (2.5) and (2.7) that

$$\text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l \sigma_{y'}(x-u) \frac{\sin uy}{u} du \equiv \sigma_{y'}(x).$$

Now let $a' \geq a \geq 2$. Let

$$\psi(x) = \phi(x) \quad (|x| \leq 2a'), \quad \psi(x+4a') = \psi(x),$$

and define $\Psi(x, y)$ as in Lemma 2. Then, if $|x| \leq a$,

$$S_y(x) - f(x) \equiv g(x, y) - \phi(x) + \Psi(x, y),$$

where

$$g(x, y) \equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^{-a'} + \int_{a'}^l [\phi(x-u) - \psi(x-u)] \frac{\sin uy}{u} du.$$

That this limit in mean exists follows from Lemma 2 of § 2, since $\phi(x) - \psi(x)$ belongs to B_p .

Hence, for almost all y ,

$$(5.9) \quad M_a[S_y(x) - f(x)] \leq M_a[g(x, y)] + M_a[\phi(x)] + C(a' a^{-1})^{1/p} M_{a'}[\phi(x)],$$

by Lemmas 1 and 2. Here and in the sequel we use C to denote a number, not necessarily the same at each occurrence, which depends only on p and δ . By Hölder's inequality,

$$\begin{aligned} (5.10) \quad & \left\{ \frac{1}{\delta} \int_y^{y+\delta} M_a[g(x, t)] dt \right\}^{1/p} \\ & \leq \frac{1}{\delta} \int_y^{y+\delta} M_a^p[g(x, t)] dt \\ & = \frac{1}{\delta} \int_y^{y+\delta} dt \frac{1}{2a} \int_{-a}^a |g(x, t)|^p dx = \frac{1}{2\delta a} \int_{-a}^a dx \int_y^{y+\delta} |g(x, t)|^p dt. \end{aligned}$$

For any fixed x , $g(x, y)$ is the Fourier-sine transform of $G(x, u)$, where

$$\begin{aligned} G(x, u) &= [\phi(x-u) - \psi(x-u)] u^{-1} \quad (|u| \geq a'), \\ &= 0 \quad (|u| < a'). \end{aligned}$$

Therefore, if $p \geq 2$, it follows from a theorem† of Hardy and Littlewood that

$$\int |g(x, t)|^p dt \leq C \int |G(x, u)|^p |u|^{p-2} du.$$

Therefore

$$\frac{1}{\delta} \int_y^{y+\delta} |g(x, t)|^p dt \leq C \int_{-\infty}^{-a'} + \int_{a'}^{\infty} |\phi(x-u) - \psi(x-u)|^p u^{-2} du.$$

If $1 < p < 2$, we have, using another theorem of Hardy and Littlewood‡

$$\begin{aligned} \frac{1}{\delta} \int_y^{y+\delta} |g(x, t)|^p dt &= \frac{1}{\delta} \int_1^{1+\delta} |g(x, y+t-1)|^p dt \\ &\leq C \int |g(x, y+t-1)|^p |t|^{p-2} dt \leq C \int |G(x, t)|^p |t| dt \\ &= C \int_{-\infty}^{-a'} + \int_{a'}^{\infty} |\phi(x-u) - \psi(x-u)|^p |u|^{-p} du. \end{aligned}$$

† Titchmarsh, 4, 108, Theorem 79.

‡ Titchmarsh, 4, 110, Theorem 80.

Hence, for any $p > 1$, we can write

$$(5.11) \quad \frac{1}{\delta} \int_y^{y+\delta} |g(x, t)|^p dt \leq C \int_{-\infty}^{-a'} + \int_{a'}^{\infty} |\phi(x-u) - \psi(x-u)|^p u^{-1-\gamma} du,$$

where $\gamma = \min(p-1, 1)$. Since $f(x) \in B_p$, it follows that $\sigma_{y'}(x)$, $\phi(x)$, $\psi(x)$ belong to B_p , and we can define H , independent of a and a' , so that

$$(5.12) \quad \frac{1}{2x} \int_{-x}^x |\phi(x-u) - \psi(x-u)|^p du \leq H^p.$$

Then, if $|x| \leq a$, we have, by partial integration,

$$\begin{aligned} \int_{-\infty}^{-a'} + \int_{a'}^{\infty} |\phi(x-u) - \psi(x-u)|^p |u|^{-1-\gamma} du \\ \leq C \int_{-\infty}^{-a'} + \int_{a'}^{\infty} u^{-2-\gamma} du \int_{-u}^{-a'} + \int_{a'}^u |\phi(x-t) - \psi(x-t)|^p dt \\ \leq CH^p \int_{a'}^{\infty} u^{-1-\gamma} du = CH^p (a')^{-\gamma}. \end{aligned}$$

Therefore, using (5.9), (5.10) and (5.11), we have

$$(5.13) \quad \frac{1}{\delta} \int_y^{y+\delta} M_a[S_t(x) - f(x)] dt \leq CH(a')^{-\gamma/p} + M_a[\phi(x)] + C(a' a^{-1})^{1/p} M_{a'}[\phi(x)].$$

We now let $y, y' \rightarrow \infty$; it follows from Lemma 3 that, since $f(x)$ is continuous in M_a , $M_{a'}$,

$$\lim_{y \rightarrow \infty} \frac{1}{\delta} \int_y^{y+\delta} M_a[S_t(x) - f(x)] dt \leq CH(a')^{-\gamma/p},$$

and, since a' is arbitrarily large, this gives (5.6).

It follows from (5.13) and Lemma 1 that

$$(5.14) \quad M_a[S_y^\delta(x) - f(x)] \leq CH(a')^{-\gamma/p} + M_a[\phi(x)] + C(a' a^{-1})^{1/p} M_{a'}[\phi(x)].$$

If $f(x)$ is continuous in M , we put $a' = a$ and let $a \rightarrow \infty$. Then

$$M[S_y^\delta(x) - f(x)] \leq C M[\phi(x)],$$

and it follows from Lemma 3 that

$$\lim_{y \rightarrow \infty} M[S_y^\delta(x) - f(x)] = 0.$$

Now, if $f(x) \in W$, we have $S_a[f(x)] < \infty$ for some positive a , and we can carry out the preceding analysis with $f(x+\tau)$ instead of $f(x)$. The

constants C, H are independent of τ , and, instead of (5.14), we have

$$(5.15) \quad M_a[S_y^\delta(x+\tau)-f(x+\tau)] \\ \leq CH(a')^{-\gamma/p} + M_a[\phi(x+\tau)] + C(a'a^{-1})^{1/p} M a'[\phi(x+\tau)].$$

Taking the upper bound of each side for $-\infty < \tau < \infty$, we have

$$(5.16) \quad S_a[S_y^\delta(x)-f(x)] \leq CH(a')^{-\gamma/p} + S_a[\phi(x)] + C(a'a^{-1})^{1/p} S_{a'}[\phi(x)].$$

If $f(x)$ is continuous in S_a (and therefore in $S_{a'}$ also), it follows that

$$\lim_{y \rightarrow \infty} S_a[S_y^\delta(x)-f(x)] \leq CH(a')^{-\gamma/p},$$

and, if we let $a' \rightarrow \infty$, we have

$$\lim_{y \rightarrow \infty} S_a[S_y^\delta(x)-f(x)] = 0.$$

If $f(x)$ is continuous in W , we write $a' = a$ in (5.16) and let $a \rightarrow \infty$. Then

$$W[S_y^\delta(x)-f(x)] \leq CW[\phi(x)],$$

$$\lim_{y \rightarrow \infty} W[S_y^\delta(x)-f(x)] = 0.$$

6. An analogue of the Hausdorff-Young Theorem.

If $r > 1$, we write

$$V_r[s(u)] = \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{2\epsilon} \int |s(u+\epsilon) - s(u-\epsilon)|^r du \right\}^{1/r},$$

$$M_r[f(x)] = \lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-T}^T |f(x)|^r dx \right\}^{1/r}.$$

The following theorem is an analogue of the well known theorem[†] of Hausdorff for Fourier series.

THEOREM 10. *Suppose that $f(x) \in B_p$, $p > 1$, and that*

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-T}^T |f(x)|^p dx \right\}^{1/p} = M_p[f(x)]$$

[†] Zygmund, 8, 190, § 9.1.

exists. Then

$$(6.1) \quad V_{p'}[s(u)] \leq C_p M_p[f(x)] \quad (1 < p \leq 2),$$

$$(6.2) \quad C_p M_p[f(x)] \leq V_{p'}[s(u)] \quad (p \geq 2),$$

where

$$C_p = \left\{ \frac{1}{\pi} \int \left| \frac{\sin x}{x} \right|^p dx \right\}^{1/p}.$$

To prove this we require the following

LEMMA. Under the conditions of the theorem,

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi} \int \left| \frac{f(x) \sin \epsilon x}{\epsilon x} \right|^p dx = C_p^p M_p^p[f(x)].$$

This is an extension of part of the lemma of Theorem 4. The proof of the case $p = 2$ given by Bochner 2, 30, can be extended without difficulty to the general case.

To prove Theorem 10, we suppose first that $1 < p \leq 2$. We have

$$s(u + \epsilon) - s(u - \epsilon) \equiv \text{l.i.m.}_{l \rightarrow \infty} \frac{1}{\pi} \int_{-l}^l f(x) e^{iux} \frac{\sin \epsilon x}{x} dx.$$

Now $x^{-1}f(x) \sin \epsilon x$ belongs to $L^p(-\infty, \infty)$ and it follows from Titchmarsh's theorem† that its Fourier transform $s(u + \epsilon) - s(u - \epsilon)$ belongs to $L^{p'}$ and satisfies

$$\begin{aligned} \left\{ (2\pi)^{-\frac{1}{2}} \int |s(u + \epsilon) - s(u - \epsilon)|^{p'} du \right\}^{1/p'} &\leq \left\{ 2^{\frac{1}{2}(p-1)} \pi^{-\frac{1}{2}(p+1)} \int \left| \frac{f(x) \sin \epsilon x}{x} \right|^p dx \right\}^{1/p}, \\ \left\{ \frac{1}{2\epsilon} \int |s(u + \epsilon) - s(u - \epsilon)|^{p'} du \right\}^{1/p'} &\leq \left\{ \frac{\epsilon}{\pi} \int \left| \frac{f(x) \sin \epsilon x}{\epsilon x} \right|^p dx \right\}^{1/p}. \end{aligned}$$

Therefore, using the Lemma, we have

$$V_{p'}[s(u)] \leq \overline{\lim}_{\epsilon \rightarrow 0} \left\{ \frac{\epsilon}{\pi} \int \left| \frac{f(x) \sin \epsilon x}{\epsilon x} \right|^p dx \right\}^{1/p} = C_p M_p[f(x)].$$

If $p \geq 2$, a similar argument shows that

$$C_p \left\{ \frac{\epsilon}{\pi} \int \left| \frac{f(x) \sin \epsilon x}{\epsilon x} \right|^p dx \right\}^{1/p} \leq \left\{ \frac{1}{2\epsilon} \int |s(u + \epsilon) - s(u - \epsilon)|^{p'} du \right\}^{1/p'},$$

and therefore

$$C_p M_p[f(x)] \leq V_{p'}[s(u)].$$

† Titchmarsh, 4, 96, Theorem 74.

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THE FORCES ACTING ON A CIRCULAR ARC AEROFOIL IN A STREAM BOUNDED BY A PLANE WALL

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[Received 11 November, 1938.—Read 17 November, 1938.]

Introduction.

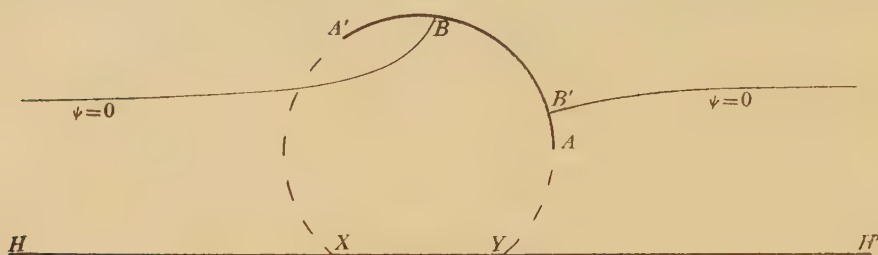
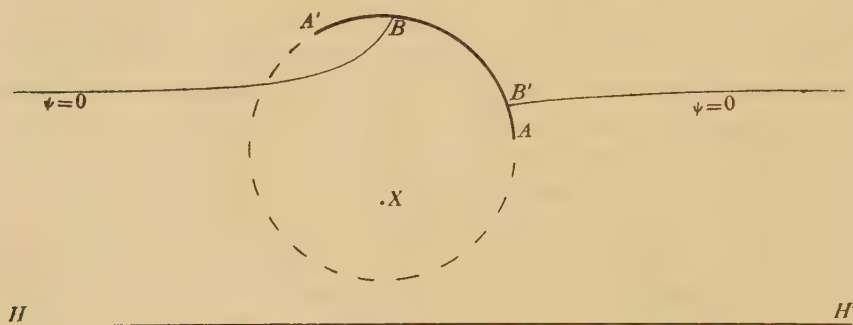
1. Tomotika and Imai (1) have recently investigated the effect of camber on the lift and moment forces acting on an aerofoil placed in a stream which is bounded by a plane wall, by evaluating the forces acting on a circular arc aerofoil which touches the ground with its trailing edge. In the present paper the forces which are acting on a circular arc aerofoil when it is in any position near a plane wall are obtained, and it is shown that the general formulae reduce to those given by Tomotika and Imai when the distance of the trailing edge from the wall tends to zero. In addition, the problem of a flat plate in the presence of a plane wall, which has been solved by Tomotika (2, 3), is shown to be a limiting case of the general problem of the circular arc aerofoil.

The general formulae are very complicated, but they assume simpler forms when the centre of the circular arc lies on the bounding wall and when the circle on which the arc lies touches the bounding wall. A few numerical results have been obtained for these cases, and, by combining them with the results obtained by Tomotika and Imai for the case when the trailing edge of the aerofoil touches the wall, I have been able to indicate the general nature of the effect of the wall on the forces acting on a circular arc aerofoil.

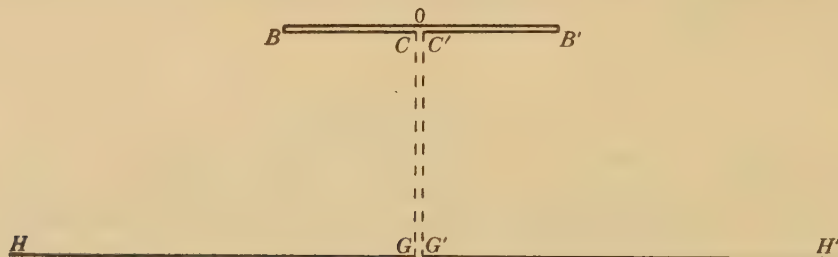
The complex velocity potential.

2. Taking the plane of the fluid motion (which is assumed to be two-dimensional) as the z -plane, we consider a steady irrotational continuous flow of an incompressible inviscid fluid past a circular arc aerofoil AA' which is placed in any position near an infinite plane wall HH' . For the present we assume that the circulation round the arc is zero. A part of

a particular stream line coincides with the surface of the aerofoil. Let this stream line be denoted by $\psi = 0$. If the fluid at infinity flows along the direction of the bounding wall with a constant velocity U , then the flow pattern is as shown in Figs. 1 or 2. The value of the stream function ψ on the wall is denoted by $-\psi_0$.

Fig. 1. z -plane.Fig. 2. z -plane.

The complex velocity potential for the flow under discussion is denoted by $w = \phi + i\psi$ and the w -plane is shown in Fig. 3. By making a cut along $CGG' C'$ as shown in the figure, we transform the w -plane on to the upper

Fig. 3. w -plane.

half of a t -plane (Fig. 4) by the method of Schwarz-Christoffel. The transformation equation is

$$\frac{dw}{dt} = M \frac{t^2 - b^2}{\sqrt{\{(t^2 - c^2)(t^2 - g^2)\}}}, \quad (1)$$

where M is a constant and where $b, -b, c, -c, g$ and $-g$ correspond to B', B, C', C, G' and G respectively.

Next, by introducing the $\wp$ function with periods $2\omega_1, 2\omega_3$, where ω_1 and ω_3/i are real and positive, we conformally transform the upper half of the t -plane into a rectangle of sides $2\omega_1, \omega_3/i$ in an s -plane (Fig. 5) by the relation

$$t^2 = \wp(s) - e_3. \quad (2)$$

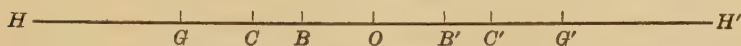


Fig. 4. t -plane.

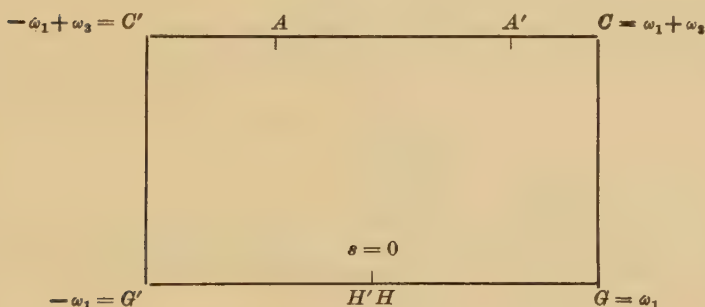


Fig. 5. s -plane.

Then the points $B, B', C, C', G, G', H, H', A, A'$ correspond to $s = \mu', -\bar{\mu}', \omega_1 + \omega_3, -\omega_1 + \omega_3, \omega_1, -\omega_1, 0, 0, s_1, s_2$ respectively.

From (1) and (2) we have

$$dw/ds = M \{\wp(s) - \wp(\mu')\}, \quad (3)$$

and, by integrating this differential equation, we have

$$w = -M \{\zeta(s) + \wp(\mu')s\}, \quad (4)$$

if we neglect an arbitrary additive constant. Since the circulation round the plate is assumed to be zero, the function w has a period $2\omega_1$. This condition gives

$$\wp(\mu') = -\eta_1/\omega_1. \quad (5)$$

Also, since the values of w at $s = \omega_1$ and at $s = \omega_1 + \omega_3$, differ by $i\psi_0$ we have

$$i\psi_0 = -M \{\eta_3 + \wp(\mu')\omega_3\}, \quad (6)$$

which, combined with (5), reduces to

$$M = 2\psi_0 \omega_1 / \pi. \quad (7)$$

3. It will be seen later that the transformation from the z -plane to the s -plane is such that dz/ds has simple zeros at the ends A and A' of the arc, and so the fluid velocity becomes infinite at these points. In order to avoid the infinity at the trailing edge A , we superpose on the above flow a rotational flow of circulation K round the plate, choosing K so that the fluid velocity at the trailing edge becomes finite. The fluid will then leave the trailing edge smoothly.

We assume that the circulation K round the plate in the z -plane takes place in the clockwise sense. This motion is then transformed into a flow with a uniform velocity $\frac{1}{2}K/\omega_1$ in the rectangle in the s -plane parallel to the negative direction of the real axis. The complex velocity potential for such a flow, which satisfies the boundary conditions, is given by

$$w' = -\frac{1}{2}Ks/\omega_1. \quad (8)$$

The total complex velocity potential Ω is then

$$\Omega = w + w' = w - \frac{1}{2}Ks/\omega_1, \quad (9)$$

and so, from (3) and (9), we have

$$d\Omega/ds = M\{\varphi(s) - \varphi(\mu')\} - \frac{1}{2}K/\omega_1. \quad (10)$$

In order that the velocity at the trailing edge may be finite, $d\Omega/ds$ must be zero when $s = s_1$. Hence

$$M\{\varphi(s_1) - \varphi(\mu')\} = \frac{1}{2}K/\omega_1, \quad (11)$$

and, by eliminating K between (10) and (11), we get

$$d\Omega/ds = M\{\varphi(s) - \varphi(s_1)\}. \quad (12)$$

The conformal transformations.

4. Transformations which are suitable for our problem have been obtained by Hodgkinson and Poole [4]. They found that the problem divides itself into three.

Problem I. The circle on which the arc aerofoil lies intersects the bounding wall at two real points X and Y (Fig. 1). Inversion with respect to Y transforms the arc into a straight line, and the bounding wall into itself. The transformation is

$$\zeta = \frac{\lambda + z}{\lambda - z}, \quad (13)$$

where 2λ is the distance XY and the mid-point of XY is the origin in the z -plane. The inverted figure in the ζ -plane is shown in Fig. 6.

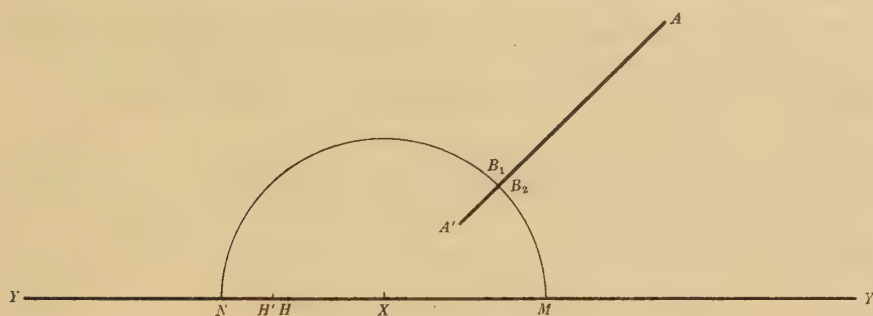


Fig. 6. ζ -plane.

Problem II. The circle on which the arc lies does not intersect the wall at real points. If X and Y are the limiting points of the system of circles coaxal with the above circle and the plane wall, and if the origin of coordinates in the z -plane (Fig. 2) is the mid-point of XY , then the inversion transformation

$$\zeta = \frac{z - \lambda' i}{z + \lambda' i}, \quad (14)$$

where $2\lambda'$ is the distance XY , transforms the arc into another circular arc and the bounding wall into a circle concentric with the arc (Fig. 7).

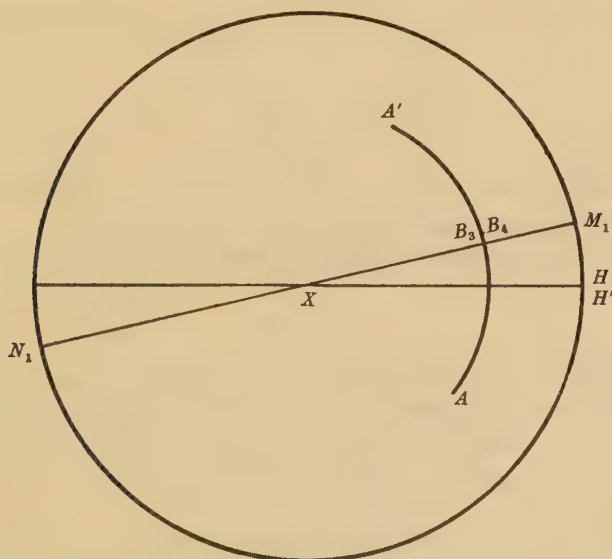


Fig. 7. ζ -plane.

Problem III. The circle on which the arc lies touches the bounding wall. This may be considered as a limiting case of either Problem I or Problem II.

We consider Problem I first. Hodgkinson and Poole (4, 398–401) showed that the ζ -plane (Fig. 6) can be transformed into the inside of a rectangle of sides $2\omega_1$, ω_3/i in a u -plane (Fig. 8) by the transformation

$$\zeta = \frac{\sigma(u_3 + u_5) \sigma(u - u_3)}{\sigma(u_3 - u_5) \sigma(u + u_3)} \exp \mu_1(u - u_5), \quad (15)$$

together with the equation

$$\mu_1 = 2\eta_1 u_3 / \omega_1 = \zeta(u_1 + u_3) - \zeta(u_1 - u_3), \quad (16)$$

where the points $u = u_1$, $2\omega_3 - u_1$, u_3 , $-u_3$, u_5 , u_5 correspond to

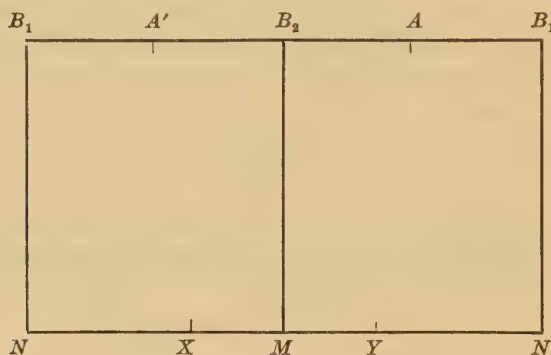


Fig. 8. u -plane.

A , A' , X , Y , H , H' respectively. The relation (16) ensures that $d\zeta/du$ has simple zeros at the points A and A' of the circular arc.

The relation between the rectangles in the u and s -planes is

$$s = u - u_5, \quad (17)$$

but for some parts of the transformation it may be necessary to adjust the value of s by adding or subtracting $2\omega_1$. If the points X and Y are denoted by $s = s_3$ and $s = s_4$ respectively in the s -plane, then, from (13) and (15), we have the following equation connecting z and s :

$$\frac{z - \lambda}{z + \lambda} = \frac{\sigma(s_3) \sigma(s - s_4)}{\sigma(s_4) \sigma(s - s_3)} \exp \left(\frac{\eta_1 \mu}{\omega_1} s \right), \quad (18)$$

where

$$\left. \begin{aligned} s_4 - s_3 &= \mu, \\ s_1 + s_2 &= s_3 + s_4 + 2\omega_3 + 2n_1\omega_1, \end{aligned} \right\} \quad (19)$$

n_1 being an integer, zero included. Also the condition (16) becomes

$$\zeta(s_1 - s_4) - \zeta(s_1 - s_3) + \eta_1 \mu / \omega_1 = 0. \quad (20)$$

It is now necessary to make a further transformation which will help to simplify the evaluation of the forces acting on the aerofoil. The s -plane is transformed conformally into an annular region in a Z -plane (Fig. 9) bounded by two concentric circles of radii 1 and q , where $q = \exp(i\pi\tau) < 1$, by the relation

$$Z = \exp\{i\pi(s + \omega_1)/\omega_1\}, \quad (21)$$

where $\tau = \omega_3/\omega_1$. Then the circular arc corresponds to the inner circle and the bounding wall to the outer circle; and the various points are transformed as follows:

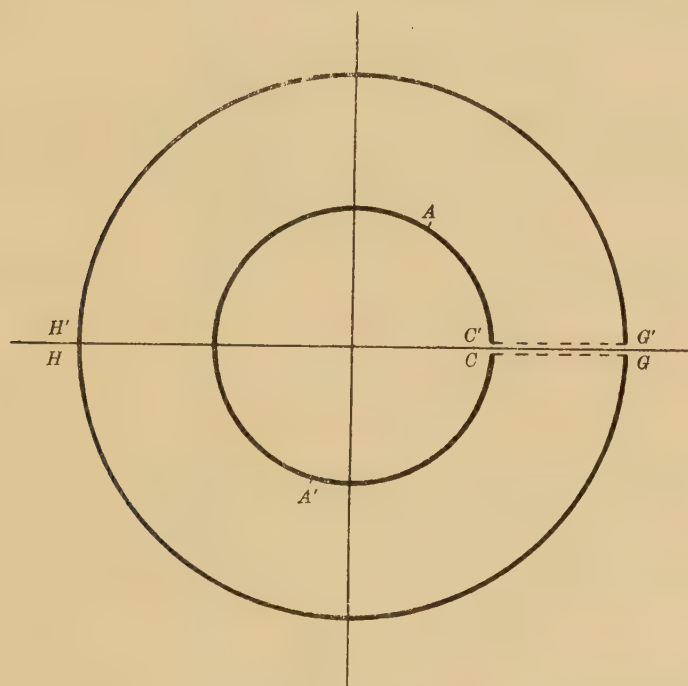


Fig. 9. Z -plane.

$$A = qe^{i\theta_1}, \quad A' = qe^{i\theta_2}, \quad H = H' = -1, \quad X = e^{i\theta_3}, \quad Y = e^{i\theta_4}.$$

We may now put

$$\left. \begin{aligned} s_1 &= -\omega_1 + \omega_3 + \omega_1 \theta_1 / \pi, \\ s_2 &= \omega_1 + \omega_3 + \omega_1 \theta_2 / \pi, \\ s_3 &= -\omega_1 + \omega_1 \theta_3 / \pi, \\ s_4 &= -\omega_1 + \omega_1 \theta_4 / \pi, \end{aligned} \right\} \quad (22)$$

and, by considering the value of (18) on the circular arc AA' , we find that $\theta_4 - \theta_3 - 2\delta'$ must be a multiple of 2π , where δ' is the angle which the chord XY subtends at any point of the arc AA' in Fig. 1. Using this result and also using (19), we may choose $\theta_1, \theta_2, \theta_3, \theta_4$ so that they satisfy the relations

$$\left. \begin{aligned} \theta_4 - \theta_3 &= 2(\delta' - \pi) = -2\delta = \pi\mu/\omega_1, \\ \theta_1 + \theta_2 &= \theta_3 + \theta_4 - 2\pi, \\ \pi &\geq \theta_1 > \theta_2 \geq -\pi, \end{aligned} \right\} \quad (23)$$

and then (20) may be put into the form

$$\frac{\partial_4' \{ \frac{1}{2}(\theta_1 - \theta_3)/\pi \}}{\partial_4 \{ \frac{1}{2}(\theta_1 - \theta_3)/\pi \}} + \frac{\partial_4' \{ \frac{1}{2}(\theta_2 - \theta_3)/\pi \}}{\partial_4 \{ \frac{1}{2}(\theta_2 - \theta_3)/\pi \}} = 0. \quad (24)$$

5. In Problem II the ζ -plane (Fig. 7) is transformed into a rectangle of sides $\omega_1', 2\omega_3'/i$ in a u' -plane (Fig. 10), where ω_1' and ω_3'/i are real and

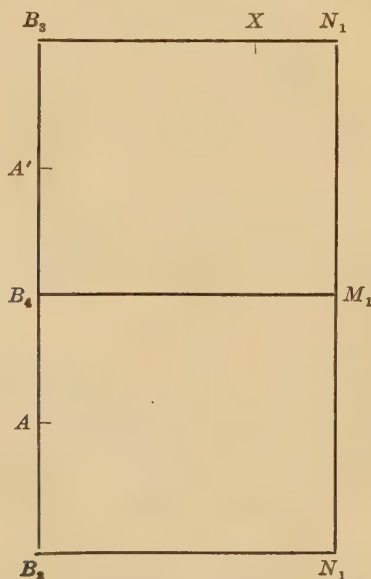


Fig. 10. u' -plane.

positive, by the equation

$$\zeta = \frac{\sigma\{(u_5' + u_3')|\omega_1', \omega_3'\} \sigma\{(u' - u_3')|\omega_1', \omega_3'\}}{\sigma\{(u_5' - u_3')|\omega_1', \omega_3'\} \sigma\{(u' + u_3')|\omega_1', \omega_3'\}} \exp \mu_2(u' - u_5'), \quad (25)$$

together with the relation

$$\mu_2 = 2\eta_3' u_3' / \omega_3' = \zeta\{(u_1' + u_3')|\omega_1', \omega_3'\} - \zeta\{(u_1' - u_3')|\omega_1', \omega_3'\}, \quad (26)$$

the sigma and zeta functions being formed from the half-periods ω_1', ω_3' as indicated. The points $u' = u_1', -2\omega_1' - u_1', u_3', u_5', u_5'$, correspond to A, A', X, H, H' respectively, where u_5' is purely imaginary and u_3' is of the form

$$u_3' = \omega_3' - \eta\omega_1' \quad (0 \leq \eta \leq 1), \quad (27)$$

η being real.

The u' -plane may be transformed into the s -plane by the equations

$$u' - u_5' = is, \quad \omega_1' = \omega_3'/i, \quad \omega_3' = i\omega_1, \quad \eta_3' = \eta_1/i, \quad (28)$$

where, as in (17), for some parts of the transformation it may be necessary to adjust the value of s by adding or subtracting $2\omega_1$. Using the results

$$\sigma(iu|\omega_1', \omega_3') = i\sigma(u|\omega_1, \omega_3), \quad \zeta(iu|\omega_1', \omega_3') = -i\zeta(u|\omega_1, \omega_3), \quad (29)$$

we find, from (14) and (15), that the relation between the z -plane (Fig. 2) and the s -plane is given by

$$\frac{z - \lambda}{z + \lambda} = \frac{\sigma(s_3)\sigma(s - s_4)}{\sigma(s_4)\sigma(s - s_3)} \exp\left(\frac{\eta_1\mu}{\omega_1}s\right), \quad (30)$$

where the point X is denoted by $s = s_3$ in the s -plane and where

$$\left. \begin{aligned} \lambda &= -i\lambda', \quad s_3 = \omega_1 + \eta\omega_3 + iu_5' + 2n_2\omega_1, \\ s_1 + s_2 &= s_3 + s_4 + 2\omega_3 + 2n_3\omega_1, \\ s_4 - s_3 &= \mu = 2iu_3', \end{aligned} \right\} \quad (31)$$

n_2 and n_3 being integers, zero included. The equation (26) becomes

$$\zeta(s_1 - s_4) - \zeta(s_1 - s_3) + \eta_1\mu/\omega_1 = 0. \quad (32)$$

The quantities s_1, s_2, s_3 , and s_4 may be expressed in the form

$$\left. \begin{aligned} s_1 &= -\omega_1 + \omega_3 + \omega_1\theta_1/\pi, \\ s_2 &= \omega_1 + \omega_3 + \omega_1\theta_2/\pi, \\ s_3 &= -\omega_1 + \omega_1\theta_3'/\pi + \eta\omega_3, \\ s_4 &= -\omega_1 + \omega_1\theta_4'/\pi - \eta\omega_3, \end{aligned} \right\} \quad (33)$$

and then $\theta_1, \theta_2, \theta_3', \theta_4'$ may be chosen so that

$$\left. \begin{aligned} \pi &\geq \theta_1 > \theta_2 \geq -\pi, \\ \theta_4' - \theta_3' &= -2\pi, \\ \theta_1 + \theta_2 &= \theta_3' + \theta_4' - 2\pi. \end{aligned} \right\} \quad (34)$$

Equation (32) now becomes

$$\frac{\partial_4' \{ \frac{1}{2}(\theta_1 - \theta_3' - \eta\tau\pi)/\pi \}}{\partial_4 \{ \frac{1}{2}(\theta_1 - \theta_3' - \eta\tau\pi)/\pi \}} + \frac{\partial_4' \{ \frac{1}{2}(\theta_2 - \theta_3' - \eta\tau\pi)/\pi \}}{\partial_4 \{ \frac{1}{2}(\theta_2 - \theta_3' - \eta\tau\pi)/\pi \}} = 0. \quad (35)$$

6. In Problem I we now put

$$\theta_1 - \theta_2 = 2\alpha, \quad \theta_3 - \delta = \theta_4 + \delta = \phi, \quad (36)$$

and in Problem II

$$\theta_1 - \theta_2 = 2\alpha, \quad \pi + \eta\tau\pi = \delta, \quad \theta_3' - \pi = \theta_4' + \pi = \phi. \quad (37)$$

Then, for both Problem I and Problem II, we have, from (23), (24), (34) and (35),

$$\left. \begin{aligned} \theta_1 &= \phi + \alpha - \pi, \quad \theta_2 = \phi - \alpha - \pi, \\ \frac{\partial_3' \{ \frac{1}{2}(\delta + \alpha)/\pi \}}{\partial_3 \{ \frac{1}{2}(\delta + \alpha)/\pi \}} &+ \frac{\partial_3' \{ \frac{1}{2}(\delta - \alpha)/\pi \}}{\partial_3 \{ \frac{1}{2}(\delta - \alpha)/\pi \}} = 0. \end{aligned} \right\} \quad (38)$$

Hence we see that the transformation (30) for Problem II is the same as the transformation (18) for Problem I provided that we generalize our definitions of λ and δ . Thus, for Problem I, λ and δ are real numbers which satisfy the conditions

$$\lambda \geq 0, \quad \pi \geq \delta \geq -\pi, \quad (39)$$

and for Problem II, $i\lambda$ and $i(\pi - \delta)$ are real numbers which satisfy the conditions

$$i\lambda \geq 0, \quad i(\pi - \delta) \geq 0. \quad (40)$$

With this extension of our definitions we may evaluate the forces acting on the aerofoil for Problem I and then use the same formulae for Problem II. The formulae (19) with $n_1 = 0$ apply for both cases.

Problem III will be considered later.

Development of the transformations, Problems I and II.

7. Let l be the length of the chord AA' , a the radius of the arc AA' , 2γ the angle which the arc AA' subtends at its centre and H the distance

of the mid-point of the chord AA' from the wall. Then, from Fig. 1, it is easily seen that

$$l/a = 2 \sin \gamma, \quad \lambda/a = \sin \delta, \quad H/a = \cos \beta \cos \gamma - \cos \delta, \quad (41)$$

and hence
$$H/l = \frac{1}{2}(\cos \beta \cos \gamma - \cos \delta)/\sin \gamma. \quad (42)$$

Let $z_A = x_A + iy_A$ and $z_{A'} = x_{A'} + iy_{A'}$ represent the complex coordinates of the ends A, A' of the arc. Then from (18) we have on reduction

$$\frac{z_A}{\lambda} = \frac{1 + bd e^{-i\delta}}{1 - bd e^{-i\delta}}, \quad \frac{z_{A'}}{\lambda} = \frac{d + b e^{-i\delta}}{d - b e^{-i\delta}}, \quad (43)$$

where

$$b = \frac{\partial_2 \left\{ \frac{1}{2}(\phi + \delta)/\pi \right\}}{\partial_2 \left\{ \frac{1}{2}(\phi - \delta)/\pi \right\}}, \quad d = \frac{\partial_3 \left\{ \frac{1}{2}(\delta + \alpha)/\pi \right\}}{\partial_3 \left\{ \frac{1}{2}(\delta - \alpha)/\pi \right\}}. \quad (44)$$

The values of H and l are given by

$$H = \frac{1}{2}(y_{A'} + y_A), \quad l e^{i(\pi - \beta)} = z_{A'} - z_A, \quad (45)$$

and so, from (43), we obtain

$$\left. \begin{aligned} \frac{H}{\lambda} &= - \frac{bd \sin \delta \{1 + b^2 + d^2 + b^2 d^2 - 4bd \cos \delta\}}{\{1 - 2bd \cos \delta + b^2 d^2\} \{b^2 - 2bd \cos \delta + d^2\}}, \\ \frac{l}{\lambda} e^{i(\pi - \beta)} &= \frac{2b(1 - d^2) \{d e^{-i\delta} + b^2 d e^{i\delta} - b(1 + d^2)\}}{\{1 - 2bd \cos \delta + b^2 d^2\} \{b^2 - 2bd \cos \delta + d^2\}}. \end{aligned} \right\} \quad (46)$$

Hence
$$\tan \beta = \frac{d(1 - b^2) \sin \delta}{d(1 + b^2) \cos \delta - b(1 + d^2)}, \quad (47)$$

and

$$\frac{H}{l} = \frac{\sin \beta}{2(1 - b^2)(1 - d^2)} \{1 + b^2 + d^2 + b^2 d^2 - 4bd \cos \delta\}. \quad (48)$$

Equations (42), (47) and (48) may be solved for d and b , giving

$$d = \left\{ \frac{\cos(\delta - \gamma) - \cos \beta}{\cos(\delta + \gamma) - \cos \beta} \right\}^{\frac{1}{2}}, \quad b = - \left\{ \frac{\cos(\delta - \beta) - \cos \gamma}{\cos(\delta + \beta) - \cos \gamma} \right\}^{\frac{1}{2}}. \quad (49)$$

If the values of β, γ and H/l are given, then equations (38), (42) and (49) are sufficient for the determination of the values of ϕ, α, δ and q .

8. It is now necessary to express ds/dz in a form suitable for use in evaluating the forces acting on the circular arc. From (18) we have

$$-2\lambda A ds/dz = F_1(s) + F_2(s) + F_3(s), \quad (50)$$

where

$$F_1(s) = 2 \frac{\sigma(s-s_3)\sigma(s-s_4)}{\sigma(s-s_1)\sigma(s-s_2)} \exp(-2\eta_3 s), \quad (51)$$

$$F_2(s) = -\frac{\sigma(s_3)\sigma^2(s-s_4)}{\sigma(s_4)\sigma(s-s_1)\sigma(s-s_2)} \exp\left(\frac{\eta_1\mu}{\omega_1} s - 2\eta_3 s\right), \quad (52)$$

$$F_3(s) = -\frac{\sigma(s_4)\sigma^2(s-s_3)}{\sigma(s_3)\sigma(s-s_1)\sigma(s-s_2)} \exp\left(-\frac{\eta_1\mu}{\omega_1} s - 2\eta_3 s\right), \quad (53)$$

and
$$A \frac{\sigma(s_1)\sigma(s_2)}{\sigma(s_3)\sigma(s_4)} = \frac{\eta_1\mu}{\omega_1} + \zeta(s_3) - \zeta(s_4). \quad (54)$$

The functions $F_1(s)$, $F_2(s)$ and $F_3(s)$ all have simple poles at the points $s = s_1, s = s_2$, but they have different periodic properties given by

$$F_1(s+2\omega_1) = F_1(s), \quad F_1(s+2\omega_3) = F_1(s), \quad (55)$$

$$F_2(s+2\omega_1) = F_2(s), \quad F_2(s+2\omega_3) = \exp(i\pi\mu/\omega_1) F_2(s), \quad (56)$$

$$F_3(s+2\omega_1) = F_3(s), \quad F_3(s+2\omega_3) = \exp(-i\pi\mu/\omega_1) F_3(s). \quad (57)$$

Hence $F_1(s)$ is an elliptic function and it can be expressed in the form

$$F_1(s) = A_0 + A_1 \zeta(s-s_1) + A_2 \zeta(s-s_2), \quad (58)$$

where A_0, A_1 and A_2 are constants. The values of A_1 and A_2 can be found immediately, since they are the residues of the function $F_1(s)$ at the poles $s = s_1, s = s_2$ respectively; and A_0 is obtained from the condition that $F_1(s)$ is zero when $s = s_3$. Thus

$$\left. \begin{aligned} A_1 = -A_2 &= \frac{2\sigma(s_1-s_3)\sigma(s_1-s_4)}{\sigma(s_1-s_2)} \exp(-2\eta_3 s_1), \\ A_0 &= A_1 \{\zeta(s_1-s_3) - \zeta(s_2-s_3)\}. \end{aligned} \right\} \quad (59)$$

By introducing the functions

$$\Phi(s) = -\frac{\sigma(s-\mu)}{\sigma(s)\sigma(\mu)} \exp\left(\frac{\eta_1\mu}{\omega_1} s\right), \quad (60)$$

$$\Psi(s) = \frac{\sigma(s+\mu)}{\sigma(s)\sigma(\mu)} \exp\left(-\frac{\eta_1\mu}{\omega_1} s\right), \quad (61)$$

which have the same periodic properties (56) and (57) as $F_2(s)$ and $F_3(s)$ respectively, and which have a simple zero at the point $s = 0$, we may

express $F_2(s)$ and $F_3(s)$ as the sum of simple elements in the form

$$F_2(s) = B_1 \Phi(s-s_1) + B_2 \Phi(s-s_2), \quad (62)$$

$$F_3(s) = C_1 \Psi(s-s_1) + C_2 \Psi(s-s_2), \quad (63)$$

where the values of the constants B_1, B_2, C_1, C_2 are given by

$$\left. \begin{aligned} B_1 &= -\frac{\sigma^2(s_1-s_4)\sigma(s_3)}{\sigma(s_1-s_2)\sigma(s_4)} \exp\left(\frac{\eta_1\mu}{\omega_1}s_1-2\eta_3s_1\right), \\ B_2 &= -\frac{\sigma^2(s_2-s_4)\sigma(s_3)}{\sigma(s_2-s_1)\sigma(s_4)} \exp\left(\frac{\eta_1\mu}{\omega_1}s_2-2\eta_3s_2\right), \\ C_1 &= -\frac{\sigma^2(s_1-s_3)\sigma(s_4)}{\sigma(s_1-s_2)\sigma(s_3)} \exp\left(-\frac{\eta_1\mu}{\omega_1}s_1-2\eta_3s_1\right), \\ C_2 &= -\frac{\sigma^2(s_2-s_3)\sigma(s_4)}{\sigma(s_2-s_1)\sigma(s_3)} \exp\left(-\frac{\eta_1\mu}{\omega_1}s_2-2\eta_3s_2\right). \end{aligned} \right\} \quad (64)$$

Since ds/dz has a double zero at the point $s=0$ we must have

$$\left. \begin{aligned} A_0 - A_1\zeta(s_1) - A_2\zeta(s_2) - B_1\Psi(s_1) - B_2\Psi(s_2) - C_1\Phi(s_1) - C_2\Phi(s_2) &= 0, \\ A_1\varphi(s_1) + A_2\varphi(s_2) - B_1\Psi'(s_1) - B_2\Psi'(s_2) - C_1\Phi'(s_1) - C_2\Phi'(s_2) &= 0, \end{aligned} \right\} \quad (65)$$

if we use the fact that $\Phi(-s) = -\Psi(s)$.

9. The fluid at infinity flows with a velocity U parallel to the wall, hence

$$(d\Omega/dz)_{s \rightarrow 0} = U. \quad (66)$$

When $s \rightarrow 0$ we have, from (18),

$$z \sim -2\lambda\{\eta_1\mu/\omega_1 + \zeta(s_3) - \zeta(s_4)\}^{-1}/s, \quad (67)$$

and hence

$$dz/ds \sim 2\lambda\{\eta_1\mu/\omega_1 + \zeta(s_3) - \zeta(s_4)\}^{-1}/s^2. \quad (68)$$

Also, for small values of s , we find from (12) that

$$d\Omega/ds \sim M/s^2. \quad (69)$$

Thus, from (66), (68) and (69), we obtain

$$M\{\eta_1\mu/\omega_1 + \zeta(s_3) - \zeta(s_4)\} = 2\lambda U. \quad (70)$$

Calculation of the lift.

10. The lift on the circular arc aerofoil can be calculated with the aid of Blasius's first formula and, as is usual for a very thin aerofoil, we do not

take into account the singularity at the leading edge. If the x - and y -components of the resultant force acting on the aerofoil are denoted by X and Y , then

$$Y+iX = -\frac{1}{2}\rho \int_C \left(\frac{d\Omega}{dz}\right)^2 dz, \quad (71)$$

where the integral is taken in the counterclockwise sense along any closed contour C surrounding the plate and ρ is the density of the fluid.

For convenience in the evaluation of the above integral we transform the integrand so that the integration takes place in the Z -plane. Thus

$$Y+iX = \frac{1}{2}\rho \frac{i\omega_1}{\pi} \int_{\Sigma} \left(\frac{d\Omega}{ds}\right)^2 \frac{ds}{dz} \frac{dZ}{Z}, \quad (72)$$

where the integration is counterclockwise along the corresponding contour Σ to C surrounding the inner circle.

11. The only contribution to the integral (72) is from the term which is independent of Z in the expression $(d\Omega/ds)^2(ds/dz)$; we now evaluate this term.

From (12) and (50) we get

$$2\lambda A (d\Omega/ds)^2 (ds/dz) = -M^2 G(s) = -M^2 \{G_1(s) + G_2(s) + G_3(s)\}, \quad (73)$$

where

$$G_1(s) = \{\wp(s) - \wp(s_1)\}^2 F_1(s), \quad (74)$$

$$G_2(s) = \{\wp(s) - \wp(s_1)\}^2 F_2(s), \quad (75)$$

$$G_3(s) = \{\wp(s) - \wp(s_1)\}^2 F_3(s). \quad (76)$$

Each of functions $G_1(s)$, $G_2(s)$ and $G_3(s)$ has a simple pole at $s = s_2$ and a pole of the fourth order at $s = 0$, and they have the same periodic properties (55), (56) and (57) as $F_1(s)$, $F_2(s)$ and $F_3(s)$ respectively. Hence we may split up each of the functions $G_1(s)$, $G_2(s)$ and $G_3(s)$ into simple elements as follows:

$$G_1(s) = M_0 + M_1 \zeta(s) + M_2 \zeta'(s) + M_3 \zeta''(s) + M_4 \zeta'''(s) + M_5 \zeta(s - s_2), \quad (77)$$

$$G_2(s) = N_1 \Phi(s) + N_2 \Phi'(s) + N_3 \Phi''(s) + N_4 \Phi'''(s) + N_5 \Phi(s - s_2), \quad (78)$$

$$G_3(s) = P_1 \Psi(s) + P_2 \Psi'(s) + P_3 \Psi''(s) + P_4 \Psi'''(s) + P_5 \Psi(s - s_2), \quad (79)$$

where all the coefficients M , N , P are constants.

We evaluate first the constants contained in the expansion of $G_1(s)$. The constant M_5 is the residue of $G_1(s)$ at the pole $s = s_2$, so that

$$M_5 = -A_1 \{\wp(s_1) - \wp(s_2)\}^2. \quad (80)$$

In order to obtain the values of M_1 , M_2 , M_3 and M_4 , we assume that s is

small and expand the various functions on both sides of (77) in ascending powers of s , and then we compare the coefficients of s^{-4} , s^{-3} , s^{-2} , s^{-1} . Thus

$$\begin{aligned} G_1(s) &= -\frac{6M_4}{s^4} + \frac{2M_3}{s^3} - \frac{M_2}{s^2} + \frac{M_1}{s} + (\text{power series in } s) \\ &= \frac{1}{s^4} [1 - 2\wp(s_1)s^2 + \dots][A_0 - A_1\zeta(s_1) - A_2\zeta(s_2) + s\{A_1\zeta'(s_1) + A_2\zeta'(s_2)\} \\ &\quad - \frac{1}{2}s^2\{A_1\zeta''(s_1) + A_2\zeta''(s_2)\} + \frac{1}{6}s^3\{A_1\zeta'''(s_1) + A_2\zeta'''(s_2)\} + \dots] \\ &= \frac{1}{s^4} \{A_0 - A_1\zeta(s_1) - A_2\zeta(s_2)\} - \frac{A_1}{s^3} \{\wp(s_1) - \wp(s_2)\} \\ &\quad + \frac{1}{s^2} \{\frac{1}{2}A_1[\wp'(s_1) - \wp'(s_2)] - 2\wp(s_1)[A_0 - A_1\zeta(s_1) - A_2\zeta(s_2)]\} \\ &\quad + \frac{A_1}{s} \{\wp(s_1) - \wp(s_2)\}^2 + (\text{power series in } s); \end{aligned}$$

this gives immediately

$$\left. \begin{aligned} M_1 &= A_1\{\wp(s_1) - \wp(s_2)\}^2, \\ M_2 &= 2\wp(s_1)\{A_0 - A_1\zeta(s_1) - A_2\zeta(s_2)\} - \frac{1}{2}A_1\{\wp'(s_1) - \wp'(s_2)\}, \\ M_3 &= -\frac{1}{2}A_1\{\wp(s_1) - \wp(s_2)\}, \\ M_4 &= -\frac{1}{6}\{A_0 - A_1\zeta(s_1) - A_2\zeta(s_2)\}. \end{aligned} \right\} \quad (81)$$

The value of M_0 may be found by observing that $G_1(s)$ has zeros at the points $s = s_1$, $s = -s_1$, so that

$$\begin{aligned} M_0 + M_1\zeta(s_1) + M_2\zeta'(s_1) + M_3\zeta''(s_1) + M_4\zeta'''(s_1) + M_5\zeta(s_1 - s_2) &= 0, \\ M_0 - M_1\zeta(s_1) + M_2\zeta'(s_1) - M_3\zeta''(s_1) + M_4\zeta'''(s_1) - M_5\zeta(s_1 + s_2) &= 0; \end{aligned}$$

by adding these equations we obtain

$$M_0 = M_2\wp(s_1) + M_4\wp''(s_1) + \frac{1}{2}M_5\{\zeta(s_1 + s_2) - \zeta(s_1 - s_2)\}. \quad (82)$$

The constants contained in the expansions of $G_2(s)$ and $G_3(s)$ may be found by a similar process. If we remember that $\Phi(-s) = -\Psi(s)$ then

$$\left. \begin{aligned} N_1 &= -2\wp(s_1)\{B_1\Psi'(s_1) + B_2\Psi'(s_2)\} + \frac{1}{6}\{B_1\Psi'''(s_1) + B_2\Psi'''(s_2)\}, \\ N_2 &= -2\wp(s_1)\{B_1\Psi(s_1) + B_2\Psi(s_2)\} + \frac{1}{2}\{B_1\Psi''(s_1) + B_2\Psi''(s_2)\}, \\ N_3 &= \frac{1}{2}\{B_1\Psi'(s_1) + B_2\Psi'(s_2)\}, \\ N_4 &= \frac{1}{6}\{B_1\Psi(s_1) + B_2\Psi(s_2)\}, \\ N_5 &= B_2\{\wp(s_1) - \wp(s_2)\}^2, \end{aligned} \right\} \quad (83)$$

and

$$\left. \begin{aligned} P_1 &= -2\wp(s_1)\{C_1\Phi'(s_1)+C_2\Phi'(s_2)\}+\frac{1}{6}\{C_1\Phi'''(s_1)+C_2\Phi'''(s_2)\}, \\ P_2 &= -2\wp(s_1)\{C_1\Phi(s_1)+C_2\Phi(s_2)\}+\frac{1}{2}\{C_1\Phi''(s_1)+C_2\Phi''(s_2)\}, \\ P_3 &= \frac{1}{2}\{C_1\Phi'(s_1)+C_2\Phi'(s_2)\}, \\ P_4 &= \frac{1}{6}\{C_1\Phi(s_1)+C_2\Phi(s_2)\}, \\ P_5 &= C_2\{\wp(s_1)-\wp(s_2)\}^2. \end{aligned} \right\} \quad (84)$$

It is now seen that the relations (65) give

$$M_3+N_3+P_3=0, \quad M_4+N_4+P_4=0;$$

this verifies the fact that $G(s)$ has only a double pole at the point $s=0$.

The function $G(s)$, which is given by the expressions (77), (78) and (79) is in a form which can be expanded in a series of positive and negative integral powers of Z , and we now proceed to obtain this expansion.

If $u=2a\omega_1+2\beta\omega_3$, where α and β are real and $|\beta|<1$, then we have (6, 101) the following expansion for the ζ function:

$$\zeta(u)=\frac{\eta_1 u}{\omega_1}+\frac{\pi}{2\omega_1}\cot\frac{\pi u}{2\omega_1}+\frac{2\pi}{\omega_1}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\frac{n\pi u}{\omega_1}. \quad (85)$$

Putting $u=s=-\omega_1+(\omega_1/i\pi)\log Z$ and then remembering that we have $|\exp(i\pi s/\omega_1)|=|Z|<1$, we get

$$\zeta(s)=\eta_1 s/\omega_1-\frac{1}{2}i\pi/\omega_1+f(Z), \quad (86)$$

where $f(Z)$ is a series of positive and negative integral powers of Z which is absolutely and uniformly convergent in the annular region

$$q+\epsilon\leq|Z|\leq 1-\epsilon \quad (\epsilon>0).$$

It can, therefore, be differentiated any number of times with respect to Z in this region. Thus, since

$$\zeta'(s)=\frac{i\pi}{\omega_1}Z\frac{d\zeta}{dZ},$$

we have, from (86),

$$\zeta'(s)=\eta_1/\omega_1+i\pi Zf'(Z)/\omega_1, \quad (87)$$

and none of the higher derivatives of $\zeta(s)$ contains a term which is independent of Z .

Next, in (85), we put $u=s-s_2=-2\omega_1-\omega_3-(\omega_1\theta_2/\pi)+(\omega_1/i\pi)\log Z$. Then, since $|\exp\{i\pi(s-s_2)/\omega_1\}|=|q^{-1}Z|>1$, we have

$$\begin{aligned} \zeta(s-s_2) &= \eta_1(s-s_2)/\omega_1+\frac{1}{2}i\pi/\omega_1 \\ &\quad +(\text{positive and negative integral powers of } Z). \end{aligned} \quad (88)$$

Hence, using the condition that $M_1 + M_5 = 0$, the expansion for $G_1(s)$ becomes

$$G_1(s) = M_0 + i\pi M_5 \omega_1 + \eta_1 (M_2 - s_2 M_5) / \omega_1 \\ + (\text{positive and negative integral powers of } Z). \quad (89)$$

The functions $G_2(s)$ and $G_3(s)$ must now be expanded in powers of Z . We have

$$\Phi(s) = \frac{1}{2\omega_1} \frac{\vartheta_1'(0) \vartheta_1(\frac{1}{2}s/\omega_1 + \delta/\pi)}{\vartheta_1(\frac{1}{2}s/\omega_1) \vartheta_1(\delta/\pi)}, \quad (90)$$

and this can be put in the following two forms, namely:

$$\Phi(s) = \frac{1}{2\omega_1} \frac{\vartheta_1'(0) \vartheta_3\{\frac{1}{2}s/\omega_1 - \frac{1}{2}(1+\tau) + \delta/\pi\}}{\vartheta_3\{\frac{1}{2}s/\omega_1 - \frac{1}{2}(1+\tau)\} \vartheta_1(\delta/\pi)} e^{-i\delta}, \quad (91)$$

$$\text{and} \quad \Phi(s) = \frac{1}{2\omega_1} \frac{\vartheta_1'(0) \vartheta_3\{\frac{1}{2}s/\omega_1 + \frac{1}{2}(1+\tau) + \delta/\pi\}}{\vartheta_3\{\frac{1}{2}s/\omega_1 + \frac{1}{2}(1+\tau)\} \vartheta_1(\delta/\pi)} e^{i\delta}. \quad (92)$$

In general, when the complex number v satisfies the relation

$$-\Re(\tau/i) < 2\Re(v/i) < \Re(\tau/i),$$

where $\Re(z)$ means "the real part of z ", we have (6, 104) the expansion formula

$$\frac{\vartheta_1'(0) \vartheta_3(v+w)}{4\pi \vartheta_3(v) \vartheta_1(w)} \\ = \frac{1}{4 \sin \pi w} + \sum_{n=1}^{\infty} (-)^n q^n \frac{\sin \pi(2nv+w) - q^{2n} \sin \pi(2nv-w)}{1 - 2q^{2n} \cos 2\pi w + q^{4n}}. \quad (93)$$

In the present problem the imaginary part of s varies from 0 to ω_3/i and so s satisfies the relation

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i} \left[\frac{s}{2\omega_1} - \frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right).$$

Thus, from (91) and (93), we get

$$\Phi(s) = \frac{\pi e^{-i\delta}}{2\omega_1 \sin \delta} + \frac{2\pi}{\omega_1} e^{-i\delta} \sum_{n=1}^{\infty} \frac{(-)^n q^n}{1 - 2q^{2n} \cos 2\delta + q^{4n}} \\ \times \left[\sin \pi \left[2n \left(\frac{s}{2\omega_1} - \frac{1+\tau}{2} \right) + \frac{\delta}{\pi} \right] - q^{2n} \sin \pi \left[2n \left(\frac{s}{2\omega_1} - \frac{1+\tau}{2} \right) - \frac{\delta}{\pi} \right] \right]; \quad (94)$$

and, when we substitute for s from (21), this becomes

$$\Phi(s) = \frac{\pi e^{-i\delta}}{2\omega_1 \sin \delta} + g(Z), \quad (95)$$

where $g(Z)$ is a series of positive and negative integral powers of Z which is absolutely and uniformly convergent in the annular region

$$q + \epsilon \leq |Z| \leq 1 - \epsilon \quad (\epsilon > 0).$$

It can therefore be differentiated any number of times with respect to Z in this region. Thus we see that, if any derivative of $\Phi(s)$ is expressed as a series in Z , it contains no term which is independent of Z .

Again, since s satisfies the relation

$$-\Re\left(\frac{\tau}{i}\right) < 2\Re\left(\frac{1}{i}\left[\frac{s-s_2}{2\omega_1} + \frac{\tau}{2}\right]\right) < \Re\left(\frac{\tau}{i}\right),$$

we have, from (92) and (93),

$$\begin{aligned} \Phi(s-s_2) &= \frac{\pi e^{i\delta}}{2\omega_1 \sin \delta} + \frac{2\pi}{\omega_1} e^{i\delta} \sum_{n=1}^{\infty} \frac{(-)^n q^n}{1 - 2q^{2n} \cos 2\delta + q^{4n}} \\ &\times \left\{ \sin \pi \left[2n \left(\frac{s-s_2}{2\omega_1} + \frac{1+\tau}{2} \right) + \frac{\delta}{\pi} \right] - q^{2n} \sin \pi \left[2n \left(\frac{s-s_2}{2\omega_1} + \frac{1+\tau}{2} \right) - \frac{\delta}{\pi} \right] \right\}, \quad (96) \end{aligned}$$

and by using (21) we obtain

$$\Phi(s-s_2) = \frac{\pi e^{i\delta}}{2\omega_1 \sin \delta} + (\text{positive and negative integral powers of } Z). \quad (97)$$

The expansion for $G_2(s)$ may now be written in the form

$$\begin{aligned} G_2(s) &= \frac{\pi N_1 e^{-i\delta}}{2\omega_1 \sin \delta} + \frac{\pi N_5 e^{i\delta}}{2\omega_1 \sin \delta} \\ &+ (\text{positive and negative integral powers of } Z). \quad (98) \end{aligned}$$

From (61) we have

$$\Psi(s) = \frac{1}{2\omega_1} \frac{\partial_1'(0) \partial_1(\frac{1}{2}s/\omega_1 - \delta/\pi)}{\partial_1(\frac{1}{2}s/\omega_1) \partial_1(-\delta/\pi)}, \quad (99)$$

and, on comparing this with (90), we see that $\Psi(s)$ may be obtained from $\Phi(s)$ by changing δ into $-\delta$. Hence, replacing δ by $-\delta$ in (98), and N_1 and N_5 by P_1 and P_5 respectively, we obtain the following expansion for $G_3(s)$:

$$\begin{aligned} G_3(s) &= -\frac{\pi P_1 e^{i\delta}}{2\omega_1 \sin \delta} - \frac{\pi P_5 e^{-i\delta}}{2\omega_1 \sin \delta} \\ &+ (\text{positive and negative integral powers of } Z). \quad (100) \end{aligned}$$

We are now able to evaluate the integral contained in (72). Using only the constant term from (89), (98) and (100) in the expansion of $G(s)$, and

substituting for M_0 from (82), we get

$$\begin{aligned} \frac{2\lambda A}{\rho\omega_1 M^2} (Y+iX) = & M_2\{\wp(s_1)+\eta_1/\omega_1\}+M_4\wp''(s_1) \\ & +\frac{1}{2}M_5\{\zeta(s_1+s_2)-\zeta(s_1-s_2)-2\eta_1s_2/\omega_1+i\pi/\omega_1\} \\ & +\frac{i\pi M_5}{2\omega_1}+\frac{\pi N_1e^{-i\delta}}{2\omega_1\sin\delta}+\frac{\pi N_5e^{i\delta}}{2\omega_1\sin\delta}-\frac{\pi P_1e^{i\delta}}{2\omega_1\sin\delta}-\frac{\pi P_5e^{-i\delta}}{2\omega_1\sin\delta}. \end{aligned} \quad (101)$$

It is now necessary to simplify the quantities contained in (101) before we can separate the real and imaginary parts. After considerable but straightforward calculations with the help of (59), (60), (61), (64), (80), (83) and (84), we find that

$$P_1+N_1=M_5, \quad (102)$$

and

$$\begin{aligned} \frac{2(P_1-N_1)}{A_1} = & \{\wp(s_1-s_3)-\wp(s_2-s_3)\}\{\wp(s_1-s_3)+\wp(s_2-s_3)+\wp(s_2)-3\wp(s_1) \\ & +\frac{2}{3}[\zeta(s_1-s_2)-\zeta(s_1-s_3)+\zeta(s_2-s_3)] \\ & \times [\zeta(s_1-s_3)-\zeta(s_2-s_3)-\zeta(s_1)+\zeta(s_2)]\}. \end{aligned} \quad (103)$$

By using known formulae (5, 6) which connect the $\wp$, ζ and σ functions with the four theta functions, and also using the relations (22) and (49), we express the whole of (101) in terms of these theta functions. The final result of this calculation is that

$$X=0, \quad YD/(l\rho U^2)=Y_1+Y_2+Y_3+Y_4+Y_5, \quad (104)$$

where

$$\begin{aligned} \frac{D \sin \delta}{2 \sin \gamma} = & \frac{\vartheta_1'(0) \vartheta_1(a/\pi) \vartheta_2\{\frac{1}{2}(\phi+\delta)/\pi\} \vartheta_2\{\frac{1}{2}(\phi-\delta)/\pi\}}{\vartheta_3(\frac{1}{2}\theta_1/\pi) \vartheta_3(\frac{1}{2}\theta_2/\pi) \vartheta_3\{\frac{1}{2}(\delta+\alpha)/\pi\} \vartheta_3\{\frac{1}{2}(\delta-\alpha)/\pi\}} \\ & \times \left\{ \frac{\vartheta_2'\{\frac{1}{2}(\phi+\delta)/\pi\}}{\vartheta_2\{\frac{1}{2}(\phi+\delta)/\pi\}} - \frac{\vartheta_2'\{\frac{1}{2}(\phi-\delta)/\pi\}}{\vartheta_2\{\frac{1}{2}(\phi-\delta)/\pi\}} \right\}^3 \\ = & -\frac{\vartheta_1'^4(0) \vartheta_1(a/\pi) \vartheta_1^3(\delta/\pi) \vartheta_3^2(\frac{1}{2}\theta_1/\pi) \vartheta_3^2(\frac{1}{2}\theta_2/\pi)}{\vartheta_2^2\{\frac{1}{2}(\phi+\delta)/\pi\} \vartheta_2^2\{\frac{1}{2}(\phi-\delta)/\pi\} \vartheta_3^4\{\frac{1}{2}(\delta+\alpha)/\pi\} \vartheta_3^4\{\frac{1}{2}(\delta-\alpha)/\pi\}}, \end{aligned} \quad (105)$$

$$\begin{aligned} Y_1 = & -\left\{ \frac{\vartheta_1'^2(0) \vartheta_4^2(\frac{1}{2}\theta_1/\pi)}{\vartheta_2^2(0) \vartheta_3^2(\frac{1}{2}\theta_1/\pi)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} \right\} \\ & \times \left\{ 4E \left[\frac{\vartheta_1'^2(0) \vartheta_4^2(\frac{1}{2}\theta_1/\pi)}{\vartheta_2^2(0) \vartheta_3^2(\frac{1}{2}\theta_1/\pi)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_1'''(0)}{3\vartheta_1'(0)} \right] \right. \\ & \left. - \vartheta_1'^3(0) \left[\frac{\vartheta_1(\theta_1/\pi)}{\vartheta_3^4(\frac{1}{2}\theta_1/\pi)} - \frac{\vartheta_2(\theta_2/\pi)}{\vartheta_3^4(\frac{1}{2}\theta_2/\pi)} \right] \right\}, \end{aligned} \quad (106)$$

$$Y_2 = -\frac{2E \vartheta_1'^3(0) \vartheta_1(\theta_1/\pi)}{3\vartheta_3^4(\frac{1}{2}\theta_1/\pi)} \left\{ \frac{\vartheta_1'(\theta_1/\pi)}{\vartheta_1(\theta_1/\pi)} - \frac{2\vartheta_3'(\frac{1}{2}\theta_1/\pi)}{\vartheta_3(\frac{1}{2}\theta_1/\pi)} \right\}, \quad (107)$$

$$Y_3 = -\frac{\vartheta_1'^4(0) \vartheta_1^2(\alpha/\pi) \vartheta_1^2\{\frac{1}{2}(\theta_1+\theta_2)/\pi\}}{\vartheta_3^4(\frac{1}{2}\theta_1/\pi) \vartheta_3^4(\frac{1}{2}\theta_2/\pi)} \left\{ \frac{\vartheta_1'\{\frac{1}{2}(\theta_1+\theta_2)/\pi\}}{\vartheta_1\{\frac{1}{2}(\theta_1+\theta_2)/\pi\}} - \frac{\vartheta_1'(\alpha/\pi)}{\vartheta_1(\alpha/\pi)} \right\}. \quad (108)$$

$$Y_4 = \frac{2\pi\vartheta_1'^4(0) \vartheta_1^2(\alpha/\pi) \vartheta_1^2\{\frac{1}{2}(\theta_1+\theta_2)/\pi\} \sin(\beta-\gamma)}{\vartheta_3^4(\frac{1}{2}\theta_1/\pi) \vartheta_3^4(\frac{1}{2}\theta_2/\pi) \{\cos\delta - \cos(\beta-\gamma)\}}, \quad (109)$$

$$Y_5 = -\frac{\pi \cot\delta \vartheta_1'^2(0) \vartheta_1(\delta/\pi) \vartheta_1(\alpha/\pi)}{\vartheta_3^2\{\frac{1}{2}(\delta+\alpha)/\pi\} \vartheta_3^2\{\frac{1}{2}(\delta-\alpha)/\pi\}} \\ \times \left\{ \frac{\vartheta_1'^2(0)}{\vartheta_2'^2(0)} \left[F + \frac{\vartheta_4^2(\frac{1}{2}\theta_2/\pi)}{\vartheta_3^2(\frac{1}{2}\theta_2/\pi)} - \frac{3\vartheta_4^2(\frac{1}{2}\theta_1/\pi)}{\vartheta_3^2(\frac{1}{2}\theta_1/\pi)} \right] \right. \\ \left. + \frac{2E}{3} \left[E - \frac{\vartheta_1'(\alpha/\pi)}{\vartheta_1(\alpha/\pi)} + \frac{\vartheta_3'(\frac{1}{2}\theta_1/\pi)}{\vartheta_3(\frac{1}{2}\theta_1/\pi)} - \frac{\vartheta_3'(\frac{1}{2}\theta_2/\pi)}{\vartheta_3(\frac{1}{2}\theta_2/\pi)} \right] \right\}, \quad (110)$$

$$E = \frac{\vartheta_3'\{\frac{1}{2}(\delta+\alpha)/\pi\}}{\vartheta_3\{\frac{1}{2}(\delta+\alpha)/\pi\}} - \frac{\vartheta_3'\{\frac{1}{2}(\delta-\alpha)/\pi\}}{\vartheta_3\{\frac{1}{2}(\delta-\alpha)/\pi\}} + \frac{\vartheta_3'(\frac{1}{2}\theta_2/\pi)}{\vartheta_3(\frac{1}{2}\theta_2/\pi)} - \frac{\vartheta_3'(\frac{1}{2}\theta_1/\pi)}{\vartheta_3(\frac{1}{2}\theta_1/\pi)} \\ = -\frac{\vartheta_1'(0) \vartheta_1(\alpha/\pi) \vartheta_2\{\frac{1}{2}(\phi+\delta)/\pi\} \vartheta_2\{\frac{1}{2}(\phi-\delta)/\pi\}}{\vartheta_3(\frac{1}{2}\theta_1/\pi) \vartheta_3(\frac{1}{2}\theta_2/\pi) \vartheta_3\{\frac{1}{2}(\delta+\alpha)/\pi\} \vartheta_3\{\frac{1}{2}(\delta-\alpha)/\pi\}}. \quad (111)$$

and

$$F = \frac{\vartheta_4^2\{\frac{1}{2}(\delta+\alpha)/\pi\}}{\vartheta_3^2\{\frac{1}{2}(\delta+\alpha)/\pi\}} + \frac{\vartheta_4^2\{\frac{1}{2}(\delta-\alpha)/\pi\}}{\vartheta_3^2\{\frac{1}{2}(\delta-\alpha)/\pi\}}. \quad (112)$$

Calculation of the couple.

12. By means of Blasius's second formula we proceed to calculate the moment of the fluid forces acting on the circular arc aerofoil. Thus, the moment Γ of the forces about the trailing edge of the plate, measured in the clockwise sense, is given by

$$\Gamma = \frac{1}{2}\rho \Re \left\{ \int \left(\frac{d\Omega}{dz} \right)^2 (z - z_A) dz \right\}, \quad (113)$$

where the integral is taken in the counterclockwise sense along a closed contour C surrounding the plate. As in the case of the lifting force we transform the integrand so that the integration takes place in the Z -plane in the counterclockwise sense along a contour Σ surrounding the inner circle. Hence

$$\Gamma = \Gamma_1 + Yx_A, \quad (114)$$

where

$$x_A/l = \frac{1}{2} \sin(\beta+\gamma)/\sin\gamma, \quad (115)$$

and

$$\Gamma_1 = -\frac{1}{2}\rho \frac{\omega_1}{\pi} \Re \left\{ i \int_{\Sigma} \left(\frac{d\Omega}{ds} \right)^2 \frac{ds}{dz} z \frac{dZ}{Z} \right\}. \quad (116)$$

Only the constant term in the expansion of the function $(d\Omega/ds)^2(ds/dz)z$ in powers of Z makes a contribution to the integral. From (18) and (50) we have

$$2A z ds/dz = F_2(s) - F_3(s), \quad (117)$$

and hence

$$2A (d\Omega/ds)^2(ds/dz)z = M^2\{G_2(s) - G_3(s)\}. \quad (118)$$

Expansions have already been obtained for $G_2(s)$ and $G_3(s)$ in (98) and (100) respectively, so that (116) and (118) give

$$\begin{aligned} \Gamma_1 &= \Re \left[\frac{\rho M^2}{2A} \left\{ \frac{\pi P_5 e^{-i\delta}}{2 \sin \delta} + \frac{\pi N_5 e^{i\delta}}{2 \sin \delta} + \frac{\pi(P_1 + N_1) \cos \delta}{2 \sin \delta} + \frac{i\pi(P_1 - N_1)}{2} \right\} \right], \\ &= \pi \rho M^2 \{P_5 e^{-i\delta} + N_5 e^{i\delta} + (P_1 + N_1) \cos \delta\} / (4A \sin \delta), \end{aligned} \quad (119)$$

that is, in terms of theta functions,

$$\begin{aligned} \frac{\Gamma_1 D}{\rho l^2 U^2} &= \frac{\pi \vartheta_1'^4(0) \vartheta_1^2(\alpha/\pi) \vartheta_1^2\{\frac{1}{2}(\theta_1 + \theta_2)/\pi\} \sin^2 \delta}{\vartheta_3^4(\frac{1}{2}\theta_1/\pi) \vartheta_3^4(\frac{1}{2}\theta_2/\pi) \sin \gamma \{\cos(\beta - \gamma) - \cos \delta\}}, \\ &= -\frac{Y_4 \sin^2 \delta}{2 \sin \gamma \sin(\beta - \gamma)}. \end{aligned} \quad (120)$$

As shown in §6, the formulae which we have obtained for the forces acting on the circular arc aerofoil apply both to Problem I and to Problem II.

Special cases of Problem I.

Centre of the circular arc on the bounding wall.

13. When the centre of the circular arc lies on the bounding wall the transformations and the formulae for the forces may be greatly simplified. In this case $\delta = \delta' = \frac{1}{2}\pi$ and the values of $\theta_1, \theta_2, \theta_3$ and θ_4 , which satisfy equations (23), may be put in the form

$$\theta_1 = \frac{1}{2}\epsilon, \quad \theta_2 = \frac{1}{2}\epsilon - \pi, \quad \theta_3 = \frac{1}{2}\epsilon + \pi, \quad \theta_4 = \frac{1}{2}\epsilon, \quad \alpha = \frac{1}{2}\pi, \quad \phi = \frac{1}{2}(\epsilon + \pi), \quad (121)$$

so that (22) gives

$$s_1 = -\omega_1 + \omega_3 + \frac{1}{2}\nu, \quad s_2 = \omega_3 + \frac{1}{2}\nu, \quad s_3 = \frac{1}{2}\nu = \frac{1}{2}\omega_1 \epsilon/\pi, \quad s_4 = -\omega_1 + \frac{1}{2}\nu. \quad (122)$$

Also, (18) becomes

$$\frac{z-a}{z+a} = \frac{\sigma(\frac{1}{2}\nu) \sigma(s - \frac{1}{2}\nu + \omega_1)}{\sigma(\frac{1}{2}\nu - \omega_1) \sigma(s - \frac{1}{2}\nu)} \exp(-\eta_1 s) = h(s), \quad (123)$$

$$i.e. \quad z/a = \{1 + h(s)\} / \{1 - h(s)\}. \quad (124)$$

Thus, z/a is an elliptic function with periods $2\omega_1, 4\omega_3$, having simple zeros at $s = \nu$ and $s = 2\omega_3$ and simple poles at $s = 0$ and $s = \nu + 2\omega_3$. It can,

therefore, be expressed in the form

$$\frac{z}{a} = A' \frac{\xi_{30}[s|\omega_1, 2\omega_3]}{\xi_{30}[(s-\nu)|\omega_1, 2\omega_3]}, \quad (125)$$

where $\xi_{30}(s) = \sigma_3(s)/\sigma(s)$. The function ξ_{30} is formed with the half-periods $\omega_1, 2\omega_3$ as indicated. By putting $s = \frac{1}{2}\nu$ in (125), the value of A' is found to be 1.

The expressions (42), (44) and (49) reduce to

$$H/l = \frac{1}{2} \cos \beta \cos \gamma / \sin \gamma, \quad (126)$$

and

$$d = \frac{\vartheta_4(0)}{\vartheta_3(0)} = \frac{(\cos \beta - \sin \gamma)^{\frac{1}{2}}}{(\cos \beta + \sin \gamma)^{\frac{1}{2}}}, \quad -b = \frac{\vartheta_1(\frac{1}{4}\epsilon/\pi)}{\vartheta_2(\frac{1}{4}\epsilon/\pi)} = \frac{(\cos \gamma - \sin \beta)^{\frac{1}{2}}}{(\cos \gamma + \sin \beta)^{\frac{1}{2}}}. \quad (127)$$

The equations (127) may be put into an alternative form. If u has the form $u = 2a\omega_1 + 2\beta\omega_3$, where a and β are real, then we have (6, 101) the following expansions:

$$\left. \begin{aligned} \log \sigma(u) &= \log \frac{2\omega_1}{\pi} + \frac{\eta_1 u^2}{2\omega_1} + \log \sin \frac{\pi u}{2\omega_1} \\ &\quad + \sum_{r=1}^{\infty} \frac{q^{2r}}{r(1-q^{2r})} \left(2 \sin \frac{r\pi u}{2\omega_1} \right)^2 \quad (|\beta| < 1), \\ \log \sigma_3(u) &= \frac{\eta_1 u^2}{2\omega_1} + \sum_{r=1}^{\infty} \frac{q^r}{r(1-q^{2r})} \left(2 \sin \frac{r\pi u}{2\omega_1} \right)^2 \quad (|\beta| < \tfrac{1}{2}). \end{aligned} \right\} \quad (128)$$

Hence, subtracting the above formulae and replacing ω_3 by $2\omega_3$ (and consequently q by q^2), we have

$$\begin{aligned} \log \xi_{30}(u|\omega_1, 2\omega_3) &= -\log \frac{2\omega_1}{\pi} - \log \sin \frac{\pi u}{2\omega_1} \\ &\quad + \sum_{r=1}^{\infty} \frac{q^{2r}}{r(1+q^{2r})} \left(2 \sin \frac{r\pi u}{2\omega_1} \right)^2 \quad (|\beta| < 1), \end{aligned} \quad (129)$$

provided that u still has the form $2a\omega_1 + 2\beta\omega_3$.

On the circular arc we have

$$s = -\omega_1 + \omega_3 + \omega_1 \theta / \pi,$$

and so, from (125) and (129), the value of the polar angle ϕ' for points on the arc AA' may, after considerable reduction, be expressed in the form

$$\phi' = \tfrac{1}{2}\epsilon + 4 \sum_{r=1}^{\infty} \frac{(-)^r q^r}{r(1+q^{2r})} \sin \tfrac{1}{2}r\epsilon \cos r(\theta - \tfrac{1}{2}\epsilon). \quad (130)$$

At A , $\theta = \frac{1}{2}\epsilon$, $\phi' = \frac{1}{2}\pi - \beta - \gamma$; and at A' , $\theta = \pi + \frac{1}{2}\epsilon$, $\phi = \frac{1}{2}\pi - \beta + \gamma$. Hence, from (130),

$$\left. \begin{aligned} \beta &= \frac{1}{2}\pi - \frac{1}{2}\epsilon - \sum_{r=1}^{\infty} \frac{2q^{2r} \sin r\epsilon}{r(1+q^{4r})}, \\ \gamma &= 4 \sum_{r=1}^{\infty} \frac{q^{2r+1} \sin(r+\frac{1}{2})\epsilon}{(2r+1)(1+q^{4r+2})}. \end{aligned} \right\} \quad (131)$$

For given values of β and γ we may find the corresponding values of ϵ and q either from equations (127) or from equations (131).

Finally, after a little reduction, the expressions (104) and (120) for the lift and the couple become

$$YD/(l\rho U^2) = Y_1 + Y_2 + Y_3 + Y_4, \quad \Gamma = \Gamma_1 + \frac{1}{2}Yl \sin(\beta + \gamma)/\sin \gamma, \quad (132)$$

where

$$D = - \frac{2\pi^4 \vartheta_2^8(0) \vartheta_3^2(\frac{1}{4}\epsilon/\pi) \vartheta_4^2(\frac{1}{4}\epsilon/\pi) \sin \gamma}{\vartheta_1^2(\frac{1}{4}\epsilon/\pi) \vartheta_2^2(\frac{1}{4}\epsilon/\pi)}, \quad (133)$$

$$E = \frac{\pi \vartheta_2^2(0) \vartheta_1(\frac{1}{4}\epsilon/\pi) \vartheta_2(\frac{1}{4}\epsilon/\pi)}{\vartheta_3(\frac{1}{4}\epsilon/\pi) \vartheta_4(\frac{1}{4}\epsilon/\pi)}, \quad (134)$$

$$\begin{aligned} Y_1 = & - \left\{ \frac{\vartheta_1'^2(0) \vartheta_4^2(\frac{1}{4}\epsilon/\pi)}{\vartheta_2^2(0) \vartheta_3^2(\frac{1}{4}\epsilon/\pi)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} \right\} \\ & \times \left\{ 4E \left[\frac{\vartheta_1'^2(0) \vartheta_4^2(\frac{1}{4}\epsilon/\pi)}{\vartheta_2^2(0) \vartheta_3^2(\frac{1}{4}\epsilon/\pi)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} - \frac{\vartheta_1'''(0)}{3\vartheta_1'(0)} \right] \right. \\ & \left. - \vartheta_1'^3(0) \vartheta_1(\frac{1}{2}\epsilon/\pi) \left[\frac{1}{\vartheta_3^4(\frac{1}{4}\epsilon/\pi)} + \frac{1}{\vartheta_4^4(\frac{1}{4}\epsilon/\pi)} \right] \right\}, \quad (135) \end{aligned}$$

$$Y_2 = - \frac{2E \vartheta_1'^3(0) \vartheta_1(\frac{1}{2}\epsilon/\pi)}{3\vartheta_3^4(\frac{1}{4}\epsilon/\pi)} \left(\frac{\vartheta_1'(\frac{1}{2}\epsilon/\pi)}{\vartheta_1(\frac{1}{2}\epsilon/\pi)} - \frac{2\vartheta_3'(\frac{1}{4}\epsilon/\pi)}{\vartheta_3(\frac{1}{4}\epsilon/\pi)} \right), \quad (136)$$

$$Y_3 = \frac{\vartheta_1'^4(0) \vartheta_2^2(0) \vartheta_2(\frac{1}{2}\epsilon/\pi) \vartheta_2'(\frac{1}{2}\epsilon/\pi)}{\vartheta_3^4(\frac{1}{4}\epsilon/\pi) \vartheta_4^4(\frac{1}{4}\epsilon/\pi)}, \quad (137)$$

$$Y_4 = - \frac{2\pi \vartheta_1'^4(0) \vartheta_2^2(0) \vartheta_2^2(\frac{1}{2}\epsilon/\pi) \sin(\beta - \gamma)}{\vartheta_3^4(\frac{1}{4}\epsilon/\pi) \vartheta_4^4(\frac{1}{4}\epsilon/\pi) \cos(\beta - \gamma)}, \quad (138)$$

$$\text{and} \quad \frac{\Gamma_1 D}{\rho l^2 U^2} = - \frac{Y_4}{2 \sin \gamma \sin(\beta - \gamma)}. \quad (139)$$

The circular arc touching the bounding wall with its trailing edge.

14. Tomotika and Imai (1) have evaluated the forces acting on a circular arc aerofoil which touches the ground with its trailing edge. This problem is a limiting case of our general problem I and it is of theoretical

interest to show that our general results reduce to those given by Tomotika and Imai as the distance of the trailing edge of the arc from the wall tends to zero. The limiting process is similar to that used by Tomotika and Imai (7) for the corresponding problem of a flat plate in the presence of a plane wall.

We now consider the limiting case of problem I in which $q \rightarrow 1$. We put

$$x = \omega_3/(i\pi\omega_1) = \tau/(i\pi), \quad (140)$$

so that x is real and positive and

$$x \rightarrow 0 \quad \text{as} \quad q \rightarrow 1.$$

Using x in place of τ , we have (5, 264) the following general formulae:

$$\left. \begin{aligned} \vartheta_1(v|x) &= \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-(v-\frac{1}{2}+n)^2/x}, \\ \vartheta_2(v|x) &= \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-(v+n)^2/x}, \\ \vartheta_3(v|x) &= \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} e^{-(v+n)^2/x}, \\ \vartheta_4(v|x) &= \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} e^{-(v-\frac{1}{2}+n)^2/x}. \end{aligned} \right\} \quad (141)$$

Tomotika and Imai have obtained the following formulae which are approximately true when $0 < v < \frac{1}{2}$ and x is very small;

$$\left. \begin{aligned} \vartheta_1(v|x) &= \vartheta_4(v|x) = \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} e^{-(\frac{1}{2}-v)^2/x} \{1 + O(e^{-2v/x})\}, \\ \vartheta_2(v|x) &= \vartheta_3(v|x) = \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} e^{-v^2/x} \{1 + O(e^{-(1-2v)/x})\}, \\ \vartheta_1'(v|x) &= \vartheta_4'(v|x) = \frac{1-2v}{\pi^{\frac{1}{2}} x^{\frac{3}{2}}} e^{-(\frac{1}{2}-v)^2/x} \{1 + O(e^{-2v/x})\}, \\ \vartheta_2'(v|x) &= \vartheta_3'(v|x) = -\frac{2v}{\pi^{\frac{1}{2}} x^{\frac{3}{2}}} e^{-v^2/x} \{1 + O(e^{-(1-2v)/x})\}, \end{aligned} \right\} \quad (142).$$

and, when $v = 0$,

$$\left. \begin{aligned} \vartheta_1(0|x) &= 0, \\ \vartheta_2(0|x) &= \vartheta_3(0|x) = \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{2}}} \{1 + O(e^{-1/x})\}, \\ \vartheta_4(0|x) &= \vartheta_1'(0|x) = \frac{2}{\pi^{\frac{1}{2}} x^{\frac{3}{2}}} e^{-1/4x} \{1 + O(e^{-2/x})\}. \end{aligned} \right\} \quad (143)$$

Further, if v is small and of the same order of magnitude as x , so that we may write approximately

$$v = cx,$$

where c is a constant, we have, from (141),

$$\left. \begin{aligned} \vartheta_1(cx|x) &= \frac{1}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-(1/4x)-c^2x} \{e^c - e^{-c} + O(e^{-2/x})\}, \\ \vartheta_2(cx|x) &= \vartheta_3(cx|x) = \frac{1}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-c^2x} \{1 + O(e^{-1/x})\}, \\ \vartheta_4(cx|x) &= \frac{1}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-(1/4x)-c^2x} \{e^c + e^{-c} + O(e^{-2/x})\}. \end{aligned} \right\} \quad (144)$$

Differentiating these with respect to c , we obtain

$$\left. \begin{aligned} \vartheta_1'(cx|x) &= \frac{1}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-(1/4x)-c^2x} \{e^c + e^{-c} - 2cx(e^c - e^{-c}) + O(e^{-2/x})\}, \\ \vartheta_2'(cx|x) &= \vartheta_3'(cx|x) = -\frac{2c}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-c^2x} \{1 + O(e^{-1/x})\}, \\ \vartheta_4'(cx|x) &= \frac{1}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}} e^{-(1/4x)-c^2x} \{e^c - e^{-c} - 2cx(e^c + e^{-c}) + O(e^{-2/x})\}. \end{aligned} \right\} \quad (145)$$

15. The quantities θ_1 , θ_2 , θ_3 and θ_4 may now be regarded as functions of x , and for small values of x we assume the following expansions:

$$\left. \begin{aligned} \frac{1}{2}\theta_1/\pi &= a_0 + 2b_0 - 2 + \delta'/\pi + (c+k)x, \\ \frac{1}{2}\theta_2/\pi &= -a_0 + (c-k)x, \\ \frac{1}{2}\theta_3/\pi &= b_0 + cx, \\ \frac{1}{2}\theta_4/\pi &= b_0 - 1 + \delta'/\pi + cx, \end{aligned} \right\} \quad (146)$$

where powers of x above the first are neglected and where a_0 , b_0 , c and k are constants independent of x . The expansions are chosen so that the first two equations in (23) are satisfied. Since δ' is the angle subtended at any point of the arc AA' by the chord XY , we have $0 < \delta'/\pi < 1$. Also, since $\pi \geq \theta_1 \geq \theta_2 \geq -\pi$, we have, in the limiting case when $x \rightarrow 0$,

$$\frac{1}{2} \geq a_0 > 2 - \delta'/\pi - a_0 - 2b_0 \geq -\frac{1}{2}. \quad (147)$$

The inequalities (147) give

$$\frac{1}{2} > a_0 + b_0 - 1 > -\frac{1}{2}$$

and

$$\frac{1}{2} > \frac{3}{2} - \delta'/\pi - a_0 - b_0 > -\frac{1}{2};$$

hence, from (142), we obtain

$$\begin{aligned} \frac{\vartheta_4' \{ \frac{1}{2}(\theta_3 - \theta_1)/\pi \}}{\vartheta_4 \{ \frac{1}{2}(\theta_3 - \theta_1)/\pi \}} &= \frac{\vartheta_3' (\frac{3}{2} - \delta'/\pi - a_0 - b_0 - kx)}{\vartheta_3 (\frac{3}{2} - \delta'/\pi - a_0 - b_0 - kx)} \\ &= -\frac{2(\frac{3}{2} - \delta'/\pi - a_0 - b_0 - kx)}{x} \\ &\quad \times \left[1 + O \left\{ \exp \left(-\frac{1 - 2(\frac{3}{2} - \delta'/\pi - a_0 - b_0)}{x} \right) \right\} \right], \quad (148) \end{aligned}$$

and

$$\begin{aligned} \frac{\vartheta_4' \{ \frac{1}{2}(\theta_3 - \theta_2)/\pi \}}{\vartheta_4 \{ \frac{1}{2}(\theta_3 - \theta_2)/\pi \}} &= \frac{\vartheta_4' \{ a_0 + b_0 - 1 + kx \}}{\vartheta_4 \{ a_0 + b_0 - 1 + kx \}} \\ &= \frac{1 - 2(a_0 + b_0 - 1 + kx)}{x} \{ 1 + O(e^{-2(a_0 + b_0 - 1)/x}) \} \quad (\frac{1}{2} > a_0 + b_0 - 1 > 0), \\ &= -\frac{1 + 2(a_0 + b_0 - 1 + kx)}{x} \{ 1 + O(e^{-2(1 - a_0 - b_0)/x}) \} \quad (0 > a_0 + b_0 - 1 > -\frac{1}{2}). \end{aligned} \quad (149)$$

Thus, substituting from (148) and (149) in equation (24), we have

$$\frac{2\delta'/\pi}{x} = 0 \quad (\frac{1}{2} > a_0 + b_0 - 1 > 0),$$

and

$$\frac{2(1 - \delta'/\pi)}{x} = 0 \quad (0 > a_0 + b_0 - 1 > -\frac{1}{2}),$$

as $x \rightarrow 0$, which is impossible. We must, therefore, have

$$a_0 + b_0 = 1, \quad (150)$$

and, using (144) and (145), we find that the condition (24) now gives

$$\tanh k = 1 - 2\delta'/\pi. \quad (151)$$

From (18) we obtain

$$\frac{z_A - \lambda}{z_A + \lambda} = \frac{\vartheta_2(\frac{1}{2}\theta_3/\pi) \vartheta_4\{\frac{1}{2}(\theta_3 - \theta_2)/\pi\}}{\vartheta_2(\frac{1}{2}\theta_4/\pi) \vartheta_4\{\frac{1}{2}(\theta_3 - \theta_1)/\pi\}} e^{-i\delta} \quad (152)$$

and

$$\frac{z_{A'} - \lambda}{z_{A'} + \lambda} = \frac{\vartheta_2(\frac{1}{2}\theta_3/\pi) \vartheta_4\{\frac{1}{2}(\theta_3 - \theta_1)/\pi\}}{\vartheta_2(\frac{1}{2}\theta_4/\pi) \vartheta_4\{\frac{1}{2}(\theta_3 - \theta_2)/\pi\}} e^{-i\delta}. \quad (153)$$

From (147) and (150) we get

$$\left. \begin{aligned} \frac{1}{2} > b_0 - 1 &\geq -\frac{1}{2}, \\ \frac{1}{2} \geq b_0 - 1 + \delta'/\pi &> -\frac{1}{2}. \end{aligned} \right\} \quad (154)$$

By considering (152) and (153) for all values of the inequality (154) it is easily found that, if the distance of the trailing edge of the aerofoil from the wall is to tend to zero while the distance of the leading edge from the wall is to remain finite, then $b_0 = \frac{1}{2}$; and with (150) this gives

$$a_0 = b_0 = \frac{1}{2}. \quad (155)$$

Hence, as $x \rightarrow 0$, $z_A \rightarrow \lambda$ (or $y_A \rightarrow 0$), and from (153) we have

$$\left| \frac{z_{A'} - \lambda}{z_{A'} + \lambda} \right| \rightarrow \frac{\sin \gamma}{\sin \beta} = \frac{e^c - e^{-c}}{e^k + e^{-k}} \exp \{ (1 - 2\delta'/\pi)(k - c) \}. \quad (156)$$

We now make a change in notation in order to reduce our results to the forms given by Tomotika and Imai. The relation (151) may be written in the form

$$e^{2k} = (1 - n)/n, \quad (157)$$

where

$$n = \delta'/\pi = (\pi - \delta)/\pi, \quad (158)$$

and we put

$$e^{2c} = (1 + m)/m, \quad (159)$$

so that the equation (156) becomes

$$\frac{\sin \gamma}{\sin \beta} = \left(\frac{n}{m} \right)^n \left(\frac{1 - n}{1 + m} \right)^{1-n}. \quad (160)$$

Except for the difference in notation the relations (158) and (160) are the same as those found by Tomotika and Imai.

16. With the help of the various formulae for the theta-functions which are given in § 14, and also using the results of § 15, we may now

obtain the limiting forms of the general formulae (104) and (114) for the forces acting on the aerofoil. After considerable calculation we find that

$$\left. \begin{aligned} 2(Y_1 + Y_3)/D &\rightarrow C', & Y_2/D &\rightarrow 0, \\ 2Y_4/D &\rightarrow A' - B', & 2Y_5/D &\rightarrow D', \end{aligned} \right\} \quad (161)$$

where

$$\left. \begin{aligned} A' &= B' \frac{\sin^2 \beta}{\sin^2 \gamma} = \frac{\pi m^{2+2n} (1+m)^{4-2n}}{(m+n)^6 \sin \beta} \left(\frac{n}{1-n} \right)^{1-2n}, \\ C' &= \frac{2 \sin \delta}{\sin \gamma} \frac{m^2 (1+m)^2}{(m+n)^2} \left\{ \frac{2mn+n-m}{2m(1+m)(m+n)^3} + \frac{1}{6m^2(1+m)^2(m+n)} \right. \\ &\quad \left. - \frac{n(1-n)}{(m+n)^4} \log \left[\frac{m(1-n)}{n(1+m)} \right] \right\}, \\ D' &= \frac{\cos \delta}{\sin \gamma} \frac{\pi n(1-n) \{6m^2 + (1+n)(n+4m)\}}{3(m+n)^4}, \end{aligned} \right\} \quad (162)$$

$$\text{and so} \quad Y/(\tfrac{1}{2}\rho l U^2) = A' - B' + C' + D'. \quad (163)$$

$$\text{Also} \quad \frac{\Gamma_1}{\tfrac{1}{2}\rho l^2 U^2} \rightarrow -\frac{A' + B'}{2} \frac{\sin \delta}{\sin \gamma} - A' \frac{\sin \delta \cos \delta}{\sin \beta}, \quad (164)$$

$$\text{whence} \quad \frac{\Gamma}{\tfrac{1}{2}\rho l^2 U^2} = -A' \frac{\sin \delta \cos \delta}{\sin \beta} - B' \frac{\sin \delta}{\sin \gamma} + \frac{C' + D'}{2} \frac{\sin \delta}{\sin \gamma}. \quad (165)$$

The formulae (163) and (165) are equivalent to those given by Tomotika and Imai.

The flat plate in the presence of a plane wall.

17. The forces acting on a flat plate in the presence of a plane wall have been evaluated by Tomotika (2 and 3), and in this section we briefly indicate how his results may be deduced from our general formulae. In this case

$$\delta = \beta, \quad \lambda \rightarrow \infty, \quad s_3 = 0, \quad s_4 = \mu = -2\omega_1 \beta / \pi, \quad s_1 + s_2 = s_4 + 2\omega_3, \quad (166)$$

$$\text{i.e.} \quad \theta_3 = \pi, \quad \theta_4 = \pi - 2\beta, \quad \theta_1 + \theta_2 = -2\beta, \quad (167)$$

and from (7) and (70) we see that s_3 must tend to zero and λ must tend to infinity in such a manner that

$$U\lambda/\zeta(s_3) \rightarrow \psi_0 \omega_1 / \pi, \quad (168)$$

$$\text{i.e.} \quad 2U\lambda \vartheta_2(\tfrac{1}{2}\theta_3/\pi)/\vartheta_1'(0) \rightarrow -\psi_0/\pi. \quad (169)$$

With the values (167) for θ_3 and θ_4 the relation (24) becomes

$$\frac{\vartheta_3'(\frac{1}{2}\theta_1/\pi)}{\vartheta_3(\frac{1}{2}\theta_1/\pi)} + \frac{\vartheta_3'(\frac{1}{2}\theta_2/\pi)}{\vartheta_3(\frac{1}{2}\theta_2/\pi)} = 0, \quad (170)$$

and (48) reduces to

$$H/l = \frac{1}{2} \sin \beta (1+d^2)/(1-d^2), \quad (171)$$

where

$$d = \vartheta_3(\frac{1}{2}\theta_2/\pi)/\vartheta_3(\frac{1}{2}\theta_1/\pi). \quad (172)$$

The only contribution to the lift (104) arises from the term Y_4 , which gives

$$Y = \frac{\psi_0 \rho U}{2 \sin \beta} \frac{\vartheta_1'(0) \vartheta_1^2(\beta/\pi) \vartheta_1\{\frac{1}{2}(\theta_1 - \theta_2)/\pi\}}{\vartheta_3(\frac{1}{2}\theta_2/\pi) \vartheta_3^3(\frac{1}{2}\theta_1/\pi)}. \quad (173)$$

Also, we find that

$$\left. \begin{aligned} \lambda x_A Y_2 / (l \Gamma_1) &\rightarrow 0, \quad \Gamma_1 / \lambda \rightarrow -Y, \\ \frac{\lambda}{l} \left(\frac{x_A Y_4}{\Gamma_1} + 1 \right) &\rightarrow \frac{1+d^2}{1-d^2} \cos \beta, \quad \frac{\lambda x_A Y_5}{l \Gamma_1} \rightarrow -\frac{\cos \beta}{1-d^2}, \\ -\frac{\lambda x_A Y_1}{l \Gamma_1} &\rightarrow -\frac{\sin \beta}{\pi(1-d^2)} \frac{\vartheta_1'(\beta/\pi) \vartheta_3^2(\frac{1}{2}\theta_1/\pi) \vartheta_3^2(\frac{1}{2}\theta_2/\pi)}{\vartheta_1'^2(0) \vartheta_1^2(\beta/\pi) \vartheta_1\{\frac{1}{2}(\theta_1 - \theta_2)/\pi\}} \\ &\quad \times \left\{ \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_1'^2(0) \vartheta_4^2(\frac{1}{2}\theta_1/\pi)}{\vartheta_2^2(0) \vartheta_3^2(\frac{1}{2}\theta_1/\pi)} \right\} = M_1, \\ -\frac{\lambda x_A Y_3}{l \Gamma_1} &\rightarrow \frac{\sin \beta}{2\pi(1-d^2)} \left\{ \frac{\vartheta_1'(\beta/\pi)}{\vartheta_1(\beta/\pi)} + \frac{\vartheta_1\{\frac{1}{2}(\theta_1 - \theta_2)/\pi\}}{\vartheta_1\{\frac{1}{2}(\theta_1 - \theta_2)/\pi\}} \right\} = M_2, \end{aligned} \right\} \quad (174)$$

where d is given by (172). Thus, using equations (114), (120) and (174), we find that the moment M of the fluid forces about the centre of the flat plate, measured in the clockwise sense, which is given by

$$M = \Gamma - \frac{1}{2} l Y \cos \beta, \quad (175)$$

becomes

$$M/(lY) = M_1 + M_2 - \frac{1}{2}(1+d^2) \cos \beta / (1-d^2). \quad (176)$$

The results (167), (170), (171), (172), (173) and (176) are the same as those given by Tomotika (2 and 3), if we allow for the differences in the notation.

The circular arc in an infinite stream.

18. The circular arc in an infinite stream is a limiting case of Problem II, and we now briefly indicate how the results for this case may be deduced

from our general formulae. From equations (37), (38), (41), (42) and (49) we find that, when $H/l \rightarrow \infty$,

$$\left. \begin{aligned} q \rightarrow 0, \quad \lambda \rightarrow -i\infty, \quad 2q \cos \delta \rightarrow -\sin \tfrac{1}{2}\gamma, \\ 2q \sin \delta \rightarrow -i \sin \tfrac{1}{2}\gamma, \quad \lambda/\sin \delta \rightarrow a, \\ \theta_1 = \tfrac{1}{2}\pi - \beta - \tfrac{1}{2}\gamma, \quad \theta_2 = -\tfrac{1}{2}\pi - \beta + \tfrac{1}{2}\gamma, \quad \theta_3' = 2\pi - \beta, \quad \theta_4' = -\beta, \\ b \rightarrow -e^{-i\theta_3'}, \quad d \rightarrow -e^{-2i(\theta_1 - \theta_3')}, \end{aligned} \right\} \quad (177)$$

and the values of the various quantities contained in equations (104)–(112) and (120) are approximately

$$\left. \begin{aligned} D &= 2^8 \pi^4 q \sin \tfrac{1}{2}\gamma, \\ E &= 4\pi \tan \tfrac{1}{2}\gamma + 8\pi q \cos \beta \cos \tfrac{1}{2}\gamma, \\ F &= 2 - 8 \tan^2 \tfrac{1}{2}\gamma, \\ Y_1 + Y_2 &= O(q^2), \quad Y_3 = O(q^2), \quad Y_4 = O(q^3), \\ Y_5 &= 2^8 \pi^5 q \sin \tfrac{1}{2}\gamma \sin (\beta + \tfrac{1}{2}\gamma) / \cos \tfrac{1}{2}\gamma, \\ \Gamma_1 / (\rho l^2 U^2) &= -\pi \sin^2 \beta \cos \tfrac{1}{2}\gamma / (4 \sin \tfrac{1}{2}\gamma). \end{aligned} \right\} \quad (178)$$

Hence we obtain the following well-known expressions for the lift and couple acting on a circular arc aerofoil in an infinite stream, namely

$$Y / (\tfrac{1}{2}\rho l U^2) = 2\pi \sin (\beta + \tfrac{1}{2}\gamma) / \cos \tfrac{1}{2}\gamma, \quad (179)$$

$$\text{and} \quad \frac{\Gamma}{\rho l^2 U^2} = \frac{Y}{4\rho l U^2} \left[\frac{2 \sin (\beta + \gamma)}{\sin \gamma} - \frac{\sin^2 \beta \cos^2 \tfrac{1}{2}\gamma}{\sin (\beta + \tfrac{1}{2}\gamma) \sin \tfrac{1}{2}\gamma} \right]. \quad (180)$$

Problem III.

19. In this case the circle on which the arc lies touches the bounding wall and a suitable transformation has been obtained by Hodgkinson and Poole (4, 402). Also, we may deduce the required transformation from that used in Problem I by a limiting process. The quantities λ and δ' tend to zero in such a manner that

$$\lambda/\delta' \rightarrow a,$$

where a is the radius of the circular arc. Hence, if we expand the left-hand side of (18) in powers of λ and the right-hand side in powers of δ' , we find

that the transformation may be put in the form

$$a\pi/z\omega_1 = \zeta(s-s_3) + \zeta(s_3) - \eta_1 s/\omega_1. \quad (181)$$

Similarly, the relations (23) and (24) become

$$\left. \begin{aligned} \theta_1 + \theta_2 &= \theta_3 + \theta_4 - 2\pi = 2\theta_3 - 4\pi, \\ \vartheta_1'^2(0) \vartheta_3'^2\{\tfrac{1}{2}(\theta_1 - \theta_3)/\pi\} + \vartheta_2(0) \vartheta_2''(0) \vartheta_4'^2\{\tfrac{1}{2}(\theta_1 - \theta_3)/\pi\} &= 0, \end{aligned} \right\} \quad (182)$$

that is

$$\left. \begin{aligned} s_1 + s_2 &= s_3 + s_4 + 2\omega_3, \\ s_4 - s_3 &= -2\omega_1, \\ \wp(s_1 - s_3) + \eta_1/\omega_1 &= 0. \end{aligned} \right\} \quad (183)$$

The limiting forms of the equations (42), (44) and (49) are

$$H/l = \tfrac{1}{2}(\cos \beta \cos \gamma + 1)/\sin \gamma, \quad (184)$$

$$\frac{\vartheta_2'(\tfrac{1}{2}\theta_3/\pi)}{\vartheta_2(\tfrac{1}{2}\theta_3/\pi)} = \frac{\pi \sin \beta}{\cos \beta + \cos \gamma}, \quad (185)$$

$$\frac{\vartheta_4'(\tfrac{1}{2}a/\pi)}{\vartheta_4(\tfrac{1}{2}a/\pi)} = \frac{\pi \sin \gamma}{\cos \beta + \cos \gamma}, \quad (186)$$

where we still have $\theta_1 - \theta_2 = 2a$. These results may also be obtained by direct use of the transformation (181).

20. By using the above limiting process we have deduced the values of the forces acting on the aerofoil from the values obtained in Problem I. The work required for this calculation is very long and complicated and will not be given here. Instead, we now evaluate the forces directly, using the transformation (181); it is sufficient to state that the results are identical with those obtained by considering Problem III as a limiting case of Problem I.

The stream function for this problem is still given by (12); and, since the stream at infinity flows with a constant velocity parallel to the wall, we find that the value of the constant becomes

$$M\omega_1\{\wp(s_3) + \eta_1/\omega_1\} = a\pi U. \quad (187)$$

Using equations (12) and (181) and Blasius's two formulae, we find that the forces acting on the aerofoil are given by

$$\left. \begin{aligned} Y+iX &= -\frac{\rho M^2 \omega_1}{2a\pi} \int_{C'C} \frac{\{\varphi(s)-\varphi(s_1)\}^2 \{F(s)\}^2}{\varphi(s-s_3)+\eta_1/\omega_1} ds, \\ \Gamma &= \Gamma_1 + x_A Y, \\ \Gamma_1 &= \frac{1}{2} \rho M^2 \Re \left\{ \int_{C'C} \frac{\{\varphi(s)-\varphi(s_1)\}^2 F(s)}{\varphi(s-s_3)+\eta_1/\omega_1} ds \right\}, \end{aligned} \right\} \quad (188)$$

where

$$F(s) = \zeta(s-s_3) + \zeta(s_3) - \eta_1 s/\omega_1, \quad (189)$$

and where the couple Γ is measured in the clockwise sense about the trailing edge A of the arc. It is found to be convenient to perform the integrations along the line $C'C$ in the s -plane and, as before, we do not take into account the pole at the point $s = s_2$. The function $F(s)$ has the following periodic properties which will be used later, namely

$$F(s-2\omega_1) = F(s), \quad F(s-2\omega_3) = F(s) + i\pi/\omega_1. \quad (190)$$

We now extend the s -plane (Fig. 5) so as to include the rectangle $C_1'A_1A_1'C_1GGG'$ which is the reflection of the rectangle $C'AA'CGG'$ in the line $G'G$ (Fig. 11). This new region in the s -plane corresponds to the

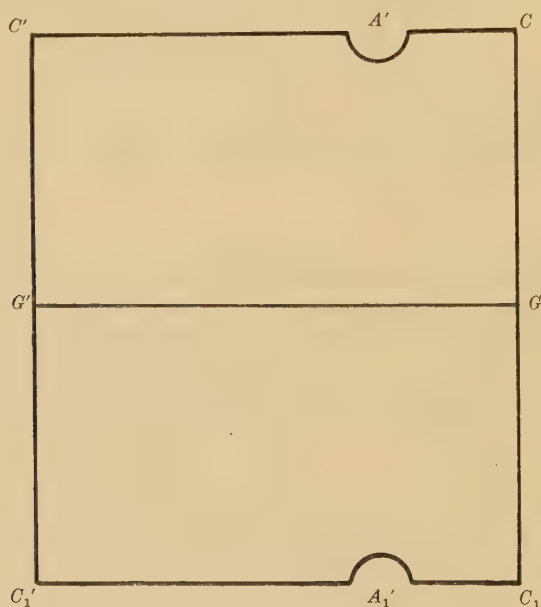


Fig. 11.

reflection of the z -plane in the bounding wall HH' . We surround the points A' and A_1' by small semi-circles r_1 and r_2 respectively in the s -plane and we denote the closed circuit consisting of the lines CGC_1 , $C_1'G'C'$, $C'C$, and C_1C_1' (excluding the small regions near the points A' and A_1'), and the semi-circles r_1 and r_2 by Σ' . By evaluating residues at points inside the contour Σ' , we obtain the following results:

$$\begin{aligned} \int_{\Sigma'} \frac{\{\varphi(s) - \varphi(s_1)\}^2 F(s)}{\varphi(s - s_3) + \eta_1/\omega_1} ds \\ = \frac{\pi i}{3\{\varphi(s_3) + \eta_1/\omega_1\}} \{\varphi''(s_1 - s_3) + 3[\varphi(s_3) + \eta_1/\omega_1][\varphi(s_2) - 3\varphi(s_1) - 2\eta_1/\omega_1]\}, \end{aligned} \quad (191)$$

$$\int_{\Sigma'} \frac{\{\varphi(s) - \varphi(s_1)\}^2 \{F(s)\}^2}{\varphi(s - s_3) + \eta_1/\omega_1} ds = 0, \quad (192)$$

$$\int_{\Sigma'} \frac{\{\varphi(s) - \varphi(s_1)\}^2 \{F(s)\}^3}{\varphi(s - s_3) + \eta_1/\omega_1} ds = -2\pi i \{\varphi(s_1) + \eta_1/\omega_1\} \{\varphi(s_1) - 2\varphi(s_3) - \eta_1/\omega_1\}. \quad (193)$$

In addition, by considering the integrals of the above functions along the semi-circles r_1 and r_2 , we have

$$\int_{r_1+r_2} \frac{\{\varphi(s) - \varphi(s_1)\}^2 F(s)}{\varphi(s - s_3) + \eta_1/\omega_1} ds = \frac{i\pi\{\varphi(s_1) - \varphi(s_2)\}^2}{\varphi'(s_2 - s_3)} \{F(s_2) + F(s_2 - 2\omega_3)\}, \quad (194)$$

$$\int_{r_1+r_2} \frac{\{\varphi(s) - \varphi(s_1)\}^2 \{F(s)\}^2}{\varphi(s - s_3) + \eta_1/\omega_1} ds = \frac{i\pi\{\varphi(s_1) - \varphi(s_2)\}^2}{\varphi'(s_2 - s_3)} \{F^2(s_2) + F^2(s_2 - 2\omega_3)\}, \quad (195)$$

$$\int_{r_1+r_2} \frac{\{\varphi(s) - \varphi(s_1)\}^2 \{F(s)\}^3}{\varphi(s - s_3) + \eta_1/\omega_1} ds = \frac{i\pi\{\varphi(s_1) - \varphi(s_2)\}^2}{\varphi'(s_2 - s_3)} \{F^3(s_2) + F^3(s_2 - 2\omega_3)\}. \quad (196)$$

The contributions to any of the integrals contained in (191), (192) or (193) from the lines CGC_1 and $C_1'G'C'$ cancel each other because of the relations (190) and the perfect periodicity of the φ function. Also, the integrals along the line C_1C_1' may be expressed in terms of integrals along the line $C'C$. Hence, using equations (188)–(196) and remembering that we ignore the pole at the point A' when evaluating the integrals in (188),

we obtain

$$\begin{aligned} & \frac{a\pi(Y+iX)}{\rho M^2 \omega_1^2} \\ &= \frac{\{\wp(s_1) - \wp(s_2)\}^2}{3\wp'(s_1 - s_3)} \left\{ [F(s_2) + \frac{1}{2}\pi i/\omega_1]^3 + \frac{\pi^2}{4\omega_1^2} [F(s_2) + \frac{1}{2}\pi i/\omega_1] \right\} \\ & \quad - \frac{1}{3}\{\wp(s_1) + \eta_1/\omega_1\} \{\wp(s_1) - 2\wp(s_3) - \eta_1/\omega_1\} \\ & \quad - \frac{\pi^2}{36\omega_1^2 \{\wp(s_3) + \eta_1/\omega_1\}} \{\wp''(s_1 - s_3) + 3[\wp(s_3) + \eta_1/\omega_1] \\ & \quad \times [\wp(s_2) - 3\wp(s_1) - 2\eta_1/\omega_1]\}, \quad (197) \end{aligned}$$

$$\text{and} \quad \Gamma_1 = -\frac{\rho M^2 \omega_1 \{\wp(s_1) - \wp(s_2)\}^2}{2\wp'(s_1 - s_3)} \left\{ F^2(s_2) + \frac{\pi i}{\omega_1} F(s_2) \right\}. \quad (198)$$

These results, in terms of theta functions, become

$$\begin{aligned} \frac{2YH_5^2 \sin \gamma}{\rho l \pi U^2} &= \frac{1}{6}H_1 H_6 (H_1^2 + \frac{1}{4}\pi^2) - \frac{1}{3}H_3 (H_3 - 2H_5) \\ & \quad - \frac{\pi^2}{36H_5} \{H_2 + 3H_5(H_4 - 3H_3)\}, \quad (199) \end{aligned}$$

$$X = 0, \quad \Gamma = \Gamma_1 + \frac{1}{2}Yl \sin(\beta + \gamma)/\sin \gamma, \quad (200)$$

$$16\Gamma_1 H_5^2 \sin^2 \gamma / (\rho l^2 U^2) = -\pi^2 H_6 (H_1^2 + \frac{1}{4}\pi^2), \quad (201)$$

where

$$2H_1 = \frac{\wp_2'(\frac{1}{2}\theta_3/\pi)}{\wp_2(\frac{1}{2}\theta_3/\pi)} - \frac{\wp_4'(\frac{1}{2}\alpha/\pi)}{\wp_4(\frac{1}{2}\alpha/\pi)}, \quad (202)$$

$$H_2 = \frac{\wp_1'^3(0) \wp_1(\alpha/\pi)}{8\wp_4^4(\frac{1}{2}\alpha/\pi)} \left\{ \frac{\wp_1'(\alpha/\pi)}{\wp_1(\alpha/\pi)} - \frac{2\wp_4'(\frac{1}{2}\alpha/\pi)}{\wp_4(\frac{1}{2}\alpha/\pi)} \right\}, \quad (203)$$

$$-4H_3 = \frac{\wp_1'^2(0) \wp_4^2(\frac{1}{2}\theta_1/\pi)}{\wp_2'^2(0) \wp_3^2(\frac{1}{2}\theta_1/\pi)} + \frac{\wp_2''(0)}{\wp_2(0)}, \quad (204)$$

$$-4H_4 = \frac{\wp_1'^2(0) \wp_4^2(\frac{1}{2}\theta_2/\pi)}{\wp_2'^2(0) \wp_3^2(\frac{1}{2}\theta_2/\pi)} + \frac{\wp_2''(0)}{\wp_2(0)}, \quad (205)$$

$$4H_5 = \frac{\wp_1'^2(0) \wp_1^2(\frac{1}{2}\theta_3/\pi)}{\wp_2'^2(0) \wp_2^2(\frac{1}{2}\theta_3/\pi)} - \frac{\wp_2''(0)}{\wp_2(0)}, \quad (206)$$

$$\text{and} \quad H_6 = \frac{\wp_1'(0) \wp_1(\alpha/\pi) \wp_1^2(\theta_3/\pi) \wp_4^4(\frac{1}{2}\alpha/\pi)}{\wp_3^4(\frac{1}{2}\theta_1/\pi) \wp_3^4(\frac{1}{2}\theta_2/\pi)}, \quad (207)$$

Numerical discussion.

21. Tomotika and Imai have carried out calculations for the case when the trailing edge of the aerofoil touches the plane wall, by taking γ to be equal to 22° ; this makes the camber of the aerofoil equal to 0.097 approximately. Their results show that, for all values of the angle of attack β less than 85° , except the limiting value $\beta \rightarrow 0$, the lift and moment coefficients are smaller when the trailing edge of the aerofoil touches the wall than when the aerofoil is in an infinite stream. The case $\beta \rightarrow 0$ when the trailing edge of the aerofoil touches the wall is unimportant, since the lift and moment coefficients are then indeterminate.

A few calculations have been made for the cases $\beta = 5^\circ$, $\beta \rightarrow 0^\circ$, by evaluating the lift and moment coefficients in the comparatively simple cases (§§ 13, 19, 20) when the centre of the circular arc lies on the wall and when the circle on which the arc lies touches the wall. The results are shown in Tables I and II.

TABLE I.
 $\beta = 5^\circ$.

l/H	$Y/(\frac{1}{2}\rho l U^2)$	Y/Y_0	$\Gamma/(\frac{1}{2}\rho l^2 U^2)$	Γ/Γ_0
0	1.7643	1	1.0077	1
0.3895	1.6831	0.9540	0.9612	0.9539
0.8111	1.6246	0.9208	0.9274	0.9203
*22.95	1.4902	0.8446	0.8282	0.8218

TABLE II.
 $\beta \rightarrow 0^\circ$.

l/H	$Y/(\frac{1}{2}\rho l U^2)$	Y/Y_0	$\Gamma/(\frac{1}{2}\rho l^2 U^2)$	Γ/Γ_0
0	1.2213	1	0.6107	1
0.3888	1.1828	0.9684	0.5914	0.9684
0.8080	1.1558	0.9464	0.5779	0.9464

The suffix 0 indicates the lift and moment coefficients for a circular arc in an unbounded stream.

From Tables I and II it is seen that, for values of β between 0° and 5° , the lift and moment coefficients are decreased by the presence of the

* This is the value of l/H when the trailing edge of the arc touches the wall.

bounding wall for all values of l/H less than about 0.8. Although detailed calculations have not been carried out for other values of β and for values of l/H which are greater than 0.8, it is probable that the lift and moment coefficients are always decreased by the presence of the wall for all values of l/H and for all values of β less than 85° , including the small values which occur in practice.

For a flat plate Tomotika found that, for small values of β (e.g. $\beta = 5^\circ, 10^\circ$), as the plate approaches the wall the lift coefficient first decreases and then increases to values which are greater than the values for a flat plate in an unbounded stream. Similar results hold for the moment coefficient. Tomotika indicated that his theoretical results were in qualitative agreement with experiment. The present work, which takes into account the camber of the aerofoil, gives results which differ from those for a flat plate and also from experiment (except when the distance of the aerofoil from the ground is large).

The calculations were made for an arc whose camber is just less than $1/10$, which approximates to a practical case, but it seems probable that, in order to get an accurate theoretical account of the forces acting on an aerofoil near a wall, we must include the thickness of the aerofoil or get a closer approximation to the outline of an aerofoil than that given by a circular arc.

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ON THE ABSOLUTE SUMMABILITY OF THE SUCCESSIVELY DERIVED SERIES OF A FOURIER SERIES AND ITS ALLIED SERIES

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[Received 8 September, 1938.—Read 17 November, 1938.]

1. Introduction.

1.1. We begin with some definitions which are fundamental so far as this paper is concerned.

Suppose that $f(x)$ is a periodic function, with period 2π , and integrable in the Lebesgue sense over $(-\pi, \pi)$, and suppose also that a polynomial

$$P(t) = \sum_{i=0}^{r-1} \frac{\theta_i}{i!} t^i$$

can be found such that, for $-\pi \leq t \leq \pi$,

$$g(t) = \frac{\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}}{2t^r},$$

$$h(t) = \frac{\{f(x+t) - P(t)\} - (-1)^r \{f(x-t) - P(-t)\}}{2t^r}$$

are integrable in $(-\pi, \pi)$. The functions $g(t)$ and $h(t)$ are defined by periodicity outside the range $(-\pi, \pi)$.

The r -th generalized derivative of $f(x)$ at x has been defined* to be the limit

$$\lim_{t \rightarrow 0} r! g(t).$$

* de la Vallée Poussin, 19,

1.2. The functions $G_a(t)$ and $g_a(t)$ are defined by the relations

$$G_a(t) = \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} g(u) du \quad (a > 0),$$

$$G_0(t) = g(t),$$

$$g_a(t) = \Gamma(a+1) t^{-a} G_a(t) \quad (a \geq 0),$$

and the functions $H_a(t)$, $h_a(t)$ similarly. The functions $G_a(t)$ and $H_a(t)$ are thus the Riemann-Liouville integrals of order a of $g(t)$ and $h(t)$ respectively.

1.3. The series Σa_n is said* to be absolutely summable (C, k) , or summable $|C, k|$, if

$$\Sigma |a_n^{(k)}| = \Sigma |c_n^{(k)} - c_{n-1}^{(k)}| < \infty,$$

where $c_n^{(k)}$ is the n -th Cesàro mean of order k for the series Σa_n .

The series Σa_n is said† to be absolutely summable (R, n, k) , or summable $|R, n, k|$, if $C_k(\omega)$ is of bounded variation in $(0, \infty)$, where $C_k(\omega)$ is the Rieszian mean of order k and type n for the series Σa_n .

It has been proved‡ that, for $k \geq 0$, these methods are equivalent in the sense that, if a series is summable by one of them, it is also summable by the other.

1.4. In this paper the following theorems will be proved.

THEOREM 1. *If $g_a(t)$ is of bounded variation in $(0, \pi)$, then the r -th derived series of the Fourier series of $f(t)$, at $t = x$, is summable $|C, \alpha + r + \delta|$, where $\alpha \geq 0$, $\delta > 0$.*

THEOREM 2. *If $H_a(+0) = 0$ and*

$$\int_0^\pi u^{-\alpha} |dH_a(u)| < \infty,$$

then the r -th derived series of the allied series of $f(t)$, at $t = x$, is summable $|C, \alpha + r + \delta|$, where $\alpha > 0$, $\delta > 0$.

* Fekete, 8; Kogbetliantz, 14.

† Obreschkoff, 15.
Hyslop, 13.

THEOREM 3. *If the r -th derived series of the Fourier series of $f(t)$, at $t = x$, is summable $|C, \alpha|$, then $g_{\alpha-r+1+\delta}(t)$ is of bounded variation in $(0, \pi)$, where $\alpha \geq r \geq 0$, $\delta > 0$.*

THEOREM 4. *If the r -th derived series of the allied series of $f(t)$, at $t = x$, is summable $|C, \alpha|$, then*

$$\int_0^\pi t^{-1} |h_{\alpha-r+\delta}(t)| dt < \infty,$$

where $\alpha \geq r \geq 0$, $\delta > 0$.

Theorems 1 and 2 will be referred to as the direct theorems and Theorems 3 and 4 as the converse theorems. It should be noted that in each case it is to be understood that $g(t)$, or $h(t)$, is integrable over $(0, \pi)$. This extra hypothesis is contained implicitly in the enunciations of the direct theorems but not in those of the converse theorems. All four theorems are extensions of known results* to which they reduce in the special case $r = 0$. Theorems 1 and 2 may also be regarded as analogues for absolute summability of the extensions which Wang† has given of the well-known theorems‡ of Hardy and Littlewood completed by Bosanquet and Paley, while Theorem 3 is the corresponding analogue of an unpublished result of Bosanquet the statement of which he communicated to me in 1936.

2. Fundamental lemmas.

2.1. It is convenient to state at the outset in a form convenient for reference certain results to which constant appeal has to be made. We have in the first place two lemmas on Riemann-Liouville integrals. It is assumed that $\phi(t)$ is integrable in $(0, \pi)$ and that $\Phi_\alpha(t)$ and $\phi_\alpha(t)$ are defined as in §1.2.

LEMMA 1. *If $\beta > \alpha \geq 0$, we have the inequalities§*

$$\int_0^\pi |d\phi_\beta(t)| \leq \int_0^\pi |d\phi_\alpha(t)|,$$

$$\int_0^\pi t^{-1} |\phi_\beta(t)| dt \leq \int_0^\pi t^{-1} |\phi_\alpha(t)| dt.$$

* Bosanquet, 5, 6; Bosanquet and Hyslop, 7.

† Wang, 20, 21. See also Zygmund, 22.

‡ Bosanquet, 1; Paley, 17; Hardy and Littlewood, 9, 10.

§ Bosanquet, 3, 6.

LEMMA 2. If* $\beta > \alpha > 0$, $\Phi_\alpha(t)$ is of bounded variation in $(0, \pi)$ and $\Phi_\alpha(+0) = 0$, then $\Phi_\beta(t)$ is an integral in $(0, \pi)$ and, for almost all values of t ,

$$\Phi_\beta'(t) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t-u)^{\beta-\alpha-1} d\Phi_\alpha(u).$$

2.2. We make extensive use of the functions $\gamma_\alpha(x)$ and $\bar{\gamma}_\alpha(x)$ where, for $\alpha > 0$,

$$\gamma_\alpha(x) + i\bar{\gamma}_\alpha(x) = \int_0^1 (1-u)^{\alpha-1} e^{ixu} du.$$

LEMMA 3. The functions $\gamma_\alpha(x)$ and $\bar{\gamma}_\alpha(x)$ possess the following properties†.

(i) If $\sigma = \min(\alpha, r+2)$ and $\tau = \min(\alpha, r+1)$, then

$$|\gamma_\alpha^{(r)}(x)| < A(1+x)^{-\sigma}, \quad |\bar{\gamma}_\alpha^{(r)}(x)| < A(1+x)^{-\tau}.$$

(ii) If $r < \alpha$,

$$x^r \{\gamma_\alpha^{(r)}(x) + i\bar{\gamma}_\alpha^{(r)}(x)\} = \sum_{\nu=0}^r (-1)^\nu \binom{r}{\nu} \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha-r+\nu)} \{\gamma_{\alpha-r+\nu}(x) + i\bar{\gamma}_{\alpha-r+\nu}(x)\}.$$

(iii) If $0 < \alpha < 1$ and

$$\gamma_{\alpha,\epsilon}(x) + i\bar{\gamma}_{\alpha,\epsilon}(x) = \int_0^{1-\epsilon} (1-u)^{\alpha-1} e^{ixu} du,$$

then, for $x \geq 0$, $\gamma_\alpha(x) - \gamma_{\alpha,\epsilon}(x)$ and $\bar{\gamma}_\alpha(x) - \bar{\gamma}_{\alpha,\epsilon}(x)$ are each $O(\epsilon^\alpha)$ uniformly in x , while, for large values of x , these differences are $O(x^{-\alpha})$ uniformly in ϵ .

2.3. Let $\kappa_\alpha(n, t)$ and $\bar{\kappa}_\alpha(n, t)$ denote the n -th Cesàro means of order α for the sequences $2\pi^{-1} \cos nt$ and $2\pi^{-1} \sin nt$ respectively. We have the following lemma.

LEMMA 4. The functions $\kappa_\alpha(n, t)$ and $\bar{\kappa}_\alpha(n, t)$ satisfy the inequality‡

$$\left| \left(\frac{d}{dt} \right)^r \{\kappa_\alpha(n, t) + i\bar{\kappa}_\alpha(n, t)\} \right| < An^r(1+nt)^{-\eta},$$

where $\eta = \min(\alpha, r+1)$.

* Bosanquet, 4.

† Hobson, 12, 565; Bosanquet, 2; Bosanquet and Hyslop, 7.

‡ Obreschkoff, 16.

2.4. Concerning summability $|C, k|$ we have the following results.

LEMMA 5. If $a_n^{(k)}$ is defined as in § 1.3, and $d_n^{(k)}$ denotes the n -th Cesàro mean of order k for the sequence $b_n = na_n$, then*, for $k \geq 0$,

$$d_n^{(k)} = na_n^{(k)}.$$

LEMMA 6. If† the series Σa_n is summable $|C, k|$, then the series $\Sigma n^{-r} a_n$ is summable $|C, k-r|$, where $0 \leq r \leq k$.

LEMMA 7. For $k \geq 0$, the series $\Sigma (-1)^n n^k$ is summable $|C, k+1+\delta|$, where $\delta > 0$.

A word of proof is perhaps necessary in this case. Let $A_n^{(k)}$ denote the n -th Cesàro sum of order k for the series $\Sigma (-1)^n n^{k+1}$ and $E_n^{(k)}$ the binomial coefficient $\binom{k+n}{n}$. By Lemma 5, it is sufficient to prove that, for $0 < \delta < 1$,

$$\sum_{n=1}^{\infty} n^{-k-\delta-2} \left| \sum_{\nu=0}^n E_{n-\nu}^{(k+\delta)} (-1)^\nu \nu^{k+1} \right| < \infty,$$

that is

$$\sum_{n=1}^{\infty} n^{-k-\delta-2} |A_n^{(k+\delta)}| < \infty.$$

Now the series $\Sigma (-1)^n n^{k+1}$ is bounded $(C, k+1)$, whence

$$A_n^{(k+\delta)} = O(n^{k+1}),$$

and the result follows.

2.5. Finally, we require two important results connected with Rieszian summability. We denote by $A_a(\omega)$ the a -th Rieszian sum of type n for the series Σa_n , and by $C_a(\omega)$ the corresponding Rieszian mean. Also $B_a(\omega)$ denotes the a -th Rieszian sum of type n for the series Σb_n , where $b_n = na_n$.

LEMMA 8. If $a > -1$, $\delta > 0$ we have‡

$$A_{a+\delta}(\omega) = \frac{\Gamma(a+\delta+1)}{\Gamma(a+1)\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} A_a(u) du.$$

* Kogbetliantz, 14.

† Kogbetliantz, 14.

‡ Hardy and Riesz, 11.

LEMMA 9. *If* $\alpha > 1$, then, for all positive values of ω ,*

$$C'_\alpha(\omega) = \alpha\omega^{-\alpha-1} B_{\alpha-1}(\omega) = \alpha\omega^{-1}\{C_{\alpha-1}(\omega) - C_\alpha(\omega)\}.$$

If $0 < \alpha \leq 1$, this result still holds when ω is not an integer.

3. The direct theorems.

3.1. In order to keep the proofs of Theorems 1 and 2 free, so far as possible, from algebraical complications, it seems advisable to perform certain of the necessary calculations separately. This we do in the lemmas below.

LEMMA 10. *If r is any positive integer or zero,*

$$0 \leq p = [\alpha] \leq \alpha < \alpha + \delta < p + 1,$$

and

$$J(n, u) = \int_u^\pi (t-u)^{p-\alpha} \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \kappa_{r+\alpha+\delta}(n, t) \right\} dt,$$

then, for $0 < u \leq \pi$,

$$|J(n, u)| < An^\alpha(1+nu)^{-\alpha-\delta}.$$

A similar result holds for $\bar{J}(n, u)$, where $\bar{J}(n, u)$ is $J(n, u)$ with $\kappa_{r+\alpha+\delta}(n, t)$ replaced by $\bar{\kappa}_{r+\alpha+\delta}(n, t)$.

Let
$$J(n, u) = \left(\int_u^{u+n^{-1}} + \int_{u+n^{-1}}^\pi \right) dt = J_1 + J_2,$$

and, for the sake of brevity, write

$$\begin{aligned} \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \kappa_{r+\alpha+\delta}(n, t) \right\} &= \sum_{\mu=0}^s e_\mu t^{r-\mu} \left(\frac{d}{dt}\right)^{r+1+p-\mu} \kappa_{r+\alpha+\delta}(n, t), \\ \left(\frac{d}{dt}\right)^{p-1} \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \kappa_{r+\alpha+\delta}(n, t) \right\} &= \sum_{\mu=0}^q f_\mu t^{r-\mu} \left(\frac{d}{dt}\right)^{r+p-\mu} \kappa_{r+\alpha+\delta}(n, t), \end{aligned}$$

where $s = \min(r, p)$ and $q = \min(r, p-1)$. Then

$$\begin{aligned} |J_1| &= \left| \sum_{\mu=0}^s e_\mu \int_u^{u+n^{-1}} (t-u)^{p-\alpha} t^{r-\mu} \left(\frac{d}{dt}\right)^{r+1+p-\mu} \kappa_{r+\alpha+\delta}(n, t) dt \right| \\ &\leq J_{1,1} + J_{1,2} + J_{1,3}, \end{aligned}$$

* See, for example, Bosanquet and Hyslop, 7.

where, by Lemma 4,

$$\begin{aligned}
 J_{1,1} &= \left| e_0 \int_u^{u+n^{-1}} (t-u)^{p-a} t^r \left(\frac{d}{dt} \right)^{r+1+p} \kappa_{r+a+\delta}(n, t) dt \right| \\
 &< A(u+n^{-1})^r n^{r+1+p} (1+nu)^{-r-a-\delta} \int_u^{u+n^{-1}} (t-u)^{p-a} dt \\
 &< An^a(1+nu)^{-a-\delta}, \\
 J_{1,2} &= \left| e_1 \int_u^{u+n^{-1}} (t-u)^{p-a} t^{r-1} \left(\frac{d}{dt} \right)^{r+p} \kappa_{a+r+\delta}(n, t) dt \right| \\
 &< A(u+n^{-1})^{r-1} n^{r+p} (1+nu)^{-r-a-\delta} \int_u^{u+n^{-1}} (t-u)^{p-a} dt \\
 &< An^a(1+nu)^{-a-\delta-1}, \\
 J_{1,3} &= \left| \sum_{\mu=2}^q e_\mu \int_u^{u+n^{-1}} (t-u)^{p-a} t^{r-\mu} \left(\frac{d}{dt} \right)^{r+1+p-\mu} \kappa_{r+a+\delta}(n, t) dt \right| \\
 &< A \sum_{\mu=2}^q (u+n^{-1})^{r-\mu} n^{r+1+p-\mu} (1+nu)^{-r-2-p+\mu} \int_u^{u+n^{-1}} (t-u)^{p-a} dt \\
 &< An^a(1+nu)^{-p-2}.
 \end{aligned}$$

It follows that

$$|J_1| < An^a(1+nu)^{-a-\delta}.$$

Applying the mean value theorem to J_2 , we obtain

$$|J_2| \leq n^{a-p} \max_{\zeta} \left| \int_{u+n^{-1}}^{\zeta} \left(\frac{d}{dt} \right)^p \left\{ t^r \left(\frac{d}{dt} \right)^{r+1} \kappa_{r+a+\delta}(n, t) \right\} dt \right|,$$

where $u+n^{-1} < \zeta < \pi$. That is,

$$|J_2| \leq n^{a-p} \max_{\zeta} \left| \left[\sum_{\mu=0}^q f_\mu t^{r-\mu} \left(\frac{d}{dt} \right)^{r+p-\mu} \kappa_{r+a+\delta}(n, t) \right]_{u+n^{-1}}^{\zeta} \right|.$$

By Lemma 4, that term in the summation for which $\mu=0$ is less in absolute value than

$$A(u+n^{-1})^r n^{r+p} (1+nu)^{-r-a-\delta} < An^p(1+nu)^{-a-\delta},$$

while those terms for which $1 \leq \mu \leq q$ are less in absolute value than

$$A(u+n^{-1})^{r-\mu} n^{r+p-\mu} (1+nu)^{-r-p-1+\mu} < An^p(1+nu)^{-p-1}.$$

It follows that

$$|J_2| < An^a(1+nu)^{-a-\delta},$$

so that the lemma is proved.

A similar proof applies in the case of the function $\bar{J}(n, u)$.

LEMMA 11. *If, with the same conditions as Lemma 10 and the additional restriction that $\alpha > 0$,*

$$\bar{K}(n, u) = \int_0^u v^{a-1} \bar{J}(n, v) dv,$$

then, for $0 < u < \pi$,

$$|\bar{K}(n, u)| < An^a u^a (1+nu)^{-a-\delta}.$$

If $0 < nu < 1$ we have, by Lemma 10,

$$|\bar{K}(n, u)| < A \int_0^u v^{a-1} n^a dv < An^a u^a.$$

We now majorize $\bar{K}(n, u)$ when $nu > 1$. We have

$$\bar{K}(n, \pi) = \int_0^\pi v^{a-1} dv \int_v^\pi (t-v)^{p-a} \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) \right\} dt,$$

and, interchanging the order of integration, which is immediately justified by Fubini's theorem, we obtain

$$\bar{K}(n, \pi) = \frac{\Gamma(\alpha) \Gamma(p+1-\alpha)}{\Gamma(p+1)} \int_0^\pi t^p \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) \right\} dt.$$

Integrating by parts, we have

$$\begin{aligned} \int_0^\pi t^p \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) \right\} dt \\ = \left[\sum_{\nu=0}^{p-1} (-1)^\nu \frac{p!}{(p-\nu)!} t^{p-\nu} \left(\frac{d}{dt}\right)^{p-\nu-1} \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) \right\} \right]_{t=0}^{t=\pi} \\ + (-1)^p p! \int_0^\pi t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) dt. \end{aligned}$$

The integrated terms vanish at $t=0$; and at $t=\pi$ they are

$$O(n^{p-a-\delta}) + O(n^{-1}),$$

by Lemma 4. Hence

$$\bar{K}(n, \pi) = O(n^{p-a-\delta}) + (-1)^p \Gamma(\alpha) \Gamma(p+1-\alpha) \int_0^\pi t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) dt.$$

Now

$$\int_0^\pi t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+a+\delta}(n, t) dt = \left[\sum_{\mu=0}^r (-1)^\mu \frac{r!}{(r-\mu)!} t^{r-\mu} \left(\frac{d}{dt}\right)^{r-\mu} \bar{\kappa}_{r+a+\delta}(n, t) \right]_0^\pi.$$

When $t = \pi$, each integrated term is certainly $O(n^{p-a-\delta})$. When $t = 0$, those terms for which $0 \leq \mu \leq r-1$ are clearly equal to zero, while the term for which $\mu = r$ is

$$(-1)^r r! \bar{\kappa}_{r+a+\delta}(n, 0) = 0.$$

It follows that

$$\bar{K}(n, \pi) = O(n^{p-a-\delta}).$$

We now have

$$\bar{K}(n, u) = \bar{K}(n, \pi) - \int_u^\pi v^{a-1} J(n, v) dv,$$

so that, if $nu > 1$,

$$\begin{aligned} |\bar{K}(n, u)| &< A \left\{ n^{p-a-\delta} + \int_u^\pi v^{a-1} n^{-\delta} v^{-a-\delta} dv \right\} \\ &< A n^{-\delta} u^{-\delta}. \end{aligned}$$

The lemma is therefore proved.

3.2. *Proof of Theorem 1.* Suppose that the Fourier series of $f(x)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx),$$

and let $a_{n,0}$ denote the n -th term of the r -th derived series.

If r is even, equal say to $2m$, we have

$$\begin{aligned} a_{n,0} &= (-1)^m n^{2m} (a_n \cos nx + \beta_n \sin nx) \\ &= (-1)^m n^{2m} 2\pi^{-1} \int_0^\pi \frac{1}{2} \{f(x+t) + f(x-t)\} \cos nt dt \\ &= (-1)^m n^{2m} 2\pi^{-1} \int_0^\pi [t^{2m} g(t) + \frac{1}{2} \{P(t) + P(-t)\}] \cos nt dt \\ &= a_n + a_{n,1}, \end{aligned}$$

where

$$\begin{aligned} a_n &= (-1)^m n^{2m} 2\pi^{-1} \int_0^\pi t^{2m} g(t) \cos nt dt, \\ a_{n,1} &= (-1)^m n^{2m} 2\pi^{-1} \sum_{\mu=1}^{m-1} \frac{\theta_{2\mu}}{(2\mu)!} \int_0^\pi t^{2\mu} \cos nt dt. \end{aligned}$$

Now

$$\int_0^\pi t^{2\mu} \cos nt \, dt = (-1)^n \sum_{\rho=1}^{\mu} (-1)^{\rho-1} \frac{(2\mu)!}{(2\mu-2\rho+1)!} \pi^{2\mu-2\rho+1} n^{-2\rho},$$

so that

$$a_{n,1} = (-1)^n 2\pi^{-1} \sum_{\rho=1}^{m-1} (-1)^{m+\rho-1} n^{2m-2\rho} \sum_{\mu=\rho}^{m-1} \frac{\theta_{2\mu}}{(2\mu-2\rho+1)!} \pi^{2\mu-2\rho+1}.$$

By Lemma 7 it is clear that $\Sigma a_{n,1}$ is summable $|C, 2m-1+\delta|$, and, *a fortiori*, summable $|C, 2m+\alpha+\delta|$. Hence it is necessary to prove only that Σa_n is summable $|C, 2m+\alpha+\delta|$. For this series, using the notation of Lemma 5, we have

$$d_n^{(2m+\alpha+\delta)} = \int_0^\pi t^{2m} g(t) \left(\frac{d}{dt} \right)^{2m+1} \bar{\kappa}_{2m+\alpha+\delta}(n, t) \, dt.$$

Suppose now that r is odd, equal say to $2m+1$. Then

$$\begin{aligned} a_{n,0} &= (-1)^m n^{2m+1} (\beta_n \cos nx - \alpha_n \sin nx) \\ &= (-1)^m n^{2m+1} 2\pi^{-1} \int_0^\pi \frac{1}{2} \{f(x+t) - f(x-t)\} \sin nt \, dt \\ &= (-1)^m n^{2m+1} 2\pi^{-1} \int_0^\pi [t^{2m+1} g(t) + \frac{1}{2} \{P(t) - P(-t)\}] \sin nt \, dt \\ &= a_n + a_{n,2}, \end{aligned}$$

where

$$\begin{aligned} a_n &= (-1)^m n^{2m+1} 2\pi^{-1} \int_0^\pi t^{2m+1} g(t) \sin nt \, dt, \\ a_{n,2} &= (-1)^m n^{2m+1} 2\pi^{-1} \sum_{\mu=1}^m \frac{\theta_{2\mu-1}}{(2\mu-1)!} \int_0^\pi t^{2\mu-1} \sin nt \, dt. \end{aligned}$$

By evaluating the integral in $a_{n,2}$ it is easy to see that $\Sigma a_{n,2}$ is summable $|C, 2m+1+\delta|$, and therefore summable $|C, 2m+\alpha+1+\delta|$. For the series Σa_n we have

$$d_n^{(2m+1+\alpha+\delta)} = - \int_0^\pi t^{2m+1} g(t) \left(\frac{d}{dt} \right)^{2m+2} \bar{\kappa}_{2m+1+\alpha+\delta}(n, t) \, dt.$$

It therefore follows, by Lemma 5, that whether r is odd or even, we have to prove that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^\pi t^r g(t) \left(\frac{d}{dt} \right)^{r+1} \bar{\kappa}_{r+\alpha+\delta}(n, t) \, dt \right| < \infty.$$

Let p and δ be defined as in Lemma 10, and let $\alpha > 0$. Then integrating by parts p times, we obtain

$$\begin{aligned} I(n) &= \int_0^\pi t^r g(t) \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+\alpha+\delta}(n, t) dt \\ &= \left[\sum_{\mu=1}^p (-1)^{\mu-1} G_\mu(t) \left(\frac{d}{dt}\right)^{\mu-1} \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+\alpha+\delta}(n, t) \right\} \right]_0^\pi \\ &\quad + (-1)^p \int_0^\pi G_p(t) \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+\alpha+\delta}(n, t) \right\} dt \\ &= I_1(n) + I_2(n). \end{aligned}$$

Clearly

$$I_1(n) = O(n^{p-\alpha-\delta}) + O(n^{-1}),$$

so that

$$\sum n^{-1} |I_1(n)| < \infty.$$

We must now show that a similar relation holds for $I_2(n)$.

Since $g_\alpha(u)$ is of bounded variation in $(0, \pi)$, the same is true of $G_\alpha(u)$ and $G_\alpha(+0) = 0$. Hence, by Lemma 2,

$$\begin{aligned} &(-1)^p \Gamma(p+1-\alpha) I_2(n) \\ &= \int_0^\pi \left(\frac{d}{dt}\right)^p \left\{ t^r \left(\frac{d}{dt}\right)^{r+1} \bar{\kappa}_{r+\alpha+\delta}(n, t) \right\} dt \int_0^t (t-u)^{p-\alpha} dG_\alpha(u) \\ &= \int_0^\pi \bar{J}(n, u) dG_\alpha(u) \\ &= \left[\bar{J}(n, u) G_\alpha(u) \right]_0^\pi - \int_0^\pi G_\alpha(u) \frac{d}{du} \bar{J}(n, u) du \\ &= O(n^{-\delta}) - \frac{1}{\Gamma(\alpha+1)} \int_0^\pi u^\alpha g_\alpha(u) \frac{d}{du} \bar{J}(n, u) du. \end{aligned}$$

Let

$$I_3(n) = \int_0^\pi u^\alpha \frac{d}{du} \bar{J}(n, u) g_\alpha(u) du.$$

Then it is sufficient to show that

$$\sum n^{-1} |I_3(n)| < \infty.$$

Now

$$\begin{aligned} \int_0^u v^\alpha \frac{d}{dv} \bar{J}(n, v) dv &= \left[v^\alpha \bar{J}(n, v) \right]_0^u - \alpha \bar{K}(n, u) \\ &= u^\alpha \bar{J}(n, u) - \alpha \bar{K}(n, u). \end{aligned}$$

Hence, on integrating by parts, we obtain

$$I_3(n) = \left[\{u^\alpha \bar{J}(n, u) - \alpha \bar{K}(n, u)\} g_\alpha(u) \right]_0^\pi - \int_0^\pi \{u^\alpha \bar{J}(n, u) - \alpha \bar{K}(n, u)\} dg_\alpha(u).$$

The integrated term is $O(n^{-\delta})$ and, by Lemmas 10 and 11,

$$\begin{aligned} \Sigma n^{-1} \left| \int_0^\pi \{u^a \bar{J}(n, u) - \alpha \bar{K}(n, u)\} dg_a(u) \right| \\ \leq \int_0^\pi |dg_a(u)| \left(\sum_{n < u^{-1}} + \sum_{n > u^{-1}} \right) |\{u^a \bar{J}(n, u) - \alpha \bar{K}(n, u)\} n^{-1}| \\ < A \int_0^\pi |dg_a(u)| \left\{ \sum_{n < u^{-1}} n^{a-1} u^a + \sum_{n > u^{-1}} n^{-\delta-1} u^{-\delta} \right\} \\ < A \int_0^\pi |dg_a(u)| < \infty. \end{aligned}$$

The theorem is therefore proved for $a > 0$ and the case $a = 0$ is at once deducible from this result.

3.3. *Proof of Theorem 2.* It is easy to see, by an argument similar to that used at the beginning of the proof of Theorem 1, that it is sufficient to prove that the series Σa_n , where

$$\begin{aligned} a_n &= (-1)^{\frac{1}{2}r} n^r 2\pi^{-1} \int_0^\pi t^r h(t) \sin nt dt \quad (r \text{ even}), \\ &= (-1)^{\frac{1}{2}(r+1)} n^r 2\pi^{-1} \int_0^\pi t^r h(t) \cos nt dt \quad (r \text{ odd}), \end{aligned}$$

is summable $|C, r + \alpha + \delta|$. For this series,

$$d_n^{(r+\alpha+\delta)} = (-1)^{r+1} \int_0^\pi t^r h(t) \left(\frac{d}{dt} \right)^{r+1} \kappa_{r+\alpha+\delta}(n, t) dt.$$

Referring to the proof of Theorem 1, we easily see that, in the last resort, the truth of the theorem depends on whether or not the series

$$\Sigma n^{-1} |E(n)|,$$

where

$$E(n) = \int_0^\pi J(n, u) dH_a(u),$$

is convergent. By Lemma 10,

$$\begin{aligned} \sum_{n=1}^\infty n^{-1} |E(n)| &\leq \int_0^\pi |dH_a(u)| \left\{ \sum_{n < u^{-1}} + \sum_{n > u^{-1}} \right\} n^{-1} |J(n, u)| \\ &< A \int_0^\pi u^{-a} |dH_a(u)| \left\{ \sum_{n < u^{-1}} n^{a-1} u^a + \sum_{n > u^{-1}} n^{-\delta-1} u^{-\delta} \right\} \\ &< \infty, \end{aligned}$$

by hypothesis. The theorem is therefore proved.

It is, perhaps, worth remarking that, when $\alpha > 1$, the theorem may be written in a slightly different form.

THEOREM 2A. *If $\beta \geq 0$ and if*

$$\int_0^\pi t^{-1} |h_\beta(t)| dt < \infty,$$

then the r -th derived series of the allied series of $f(x)$, at x , is summable

$$|C, \beta + r + 1 + \delta|.$$

This is obtained from Theorem 2 by writing $\beta + 1$ for α .

4. The converse theorems.

4.1. Following the practice adopted in §3, we give parts of the proofs of the theorems in the form of lemmas. First we have two elementary results on integrals.

LEMMA 12. *If $t^{-\rho} f(t)$ is of bounded variation in $(0, a)$, where $\rho \geq 0$, $a > 0$, so also is*

$$t^{-\rho-1} \int_0^t f(u) du.$$

LEMMA 13. *If, for $\rho \geq 0$, $a > 0$,*

$$\int_0^a t^{-\rho-1} |f(t)| dt < \infty,$$

then

$$\int_0^a t^{-\rho-2} \left| \int_0^t f(u) du \right| dt < \infty.$$

It will be observed that these results are merely slight extensions of the special case $\beta - 1 = \alpha = 0$ of Lemma 1. The proofs are quite straightforward.

4.2. We now obtain a lemma analogous to one which was proved by Takahashi* when discussing the ordinary summability of the first derived series.

* Takahashi, 18. I am indebted to a Referee for this proof which is much more elegant than my original one.

LEMMA 14. *If, for $a > 0$, $\lambda > -1$, $\sigma \geq 2$, $g(u)$ is integrable over $(0, a)$ and if*

$$t^{-\lambda-\sigma} \int_0^t ug(u) (t-u)^\lambda du$$

is of bounded variation over $(0, a)$, then so also is

$$t^{1-\lambda-\sigma} \int_0^t g(u)(t-u)^\lambda du.$$

Let

$$f(t) = t^{-\lambda-2} \int_0^t ug(u)(t-u)^\lambda du,$$

$$F(t) = \int_0^t f(v) dv.$$

Then, by hypothesis, $t^{-\sigma+2}f(t)$ is of bounded variation over $(0, a)$ and, by using the fact that $|g(u)|$ is integrable, it is not difficult to see that $F(+0) = 0$. It follows from Lemma 12 that $t^{-\sigma+1}F(t)$ is of bounded variation in $(0, a)$. But

$$\begin{aligned} t^{-\sigma+1}F(t) &= t^{-\sigma+1} \int_0^t ug(u) du \int_u^t v^{-\lambda-2}(v-u)^\lambda dv \\ &= t^{-\lambda-\sigma} \int_0^t g(u)(t-u)^{\lambda+1} du \\ &= t^{1-\lambda-\sigma} \int_0^t g(u)(t-u)^\lambda du - t^{-\lambda-\sigma} \int_0^t ug(u)(t-u)^\lambda du, \end{aligned}$$

and, since, by hypothesis, the second expression is of bounded variation in $(0, a)$, it follows that

$$t^{1-\lambda-\sigma} \int_0^t g(u)(t-u)^\lambda du$$

is also of bounded variation in $(0, a)$.

A similar result which is proved in the same way is

LEMMA 15. *If $a > 0$, $\lambda > -1$, $\sigma \geq 2$, $g(u)$ is integrable over $(0, a)$ and*

$$\int_0^a t^{-\lambda-\sigma-1} \left| \int_0^t ug(u)(t-u)^\lambda du \right| dt < \infty,$$

then

$$\int_0^a t^{-\lambda-\sigma} \left| \int_0^t g(u)(t-u)^\lambda du \right| dt < \infty.$$

4.3. We turn now to lemmas similar to those proved in § 3.1.

LEMMA 16. *If r is a positive integer,*

$$p = [\alpha] \leq \alpha < p+1, \quad 0 \leq \alpha - r < \rho < p+1-r,$$

and

$$L(v, t) + i\bar{L}(v, t) = t^p \int_v^\infty (u-v)^{p-\alpha} \{ \gamma_{\rho+r}^{(r+p+1)}(ut) + i\bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) \} du,$$

then, for $t > 0$, $v > 0$,

$$|L(v, t) + i\bar{L}(v, t)| < At^{\alpha-1}(1+vt)^{-\rho-r}.$$

It is sufficient to prove the lemma for $\bar{L}(v, t)$, since the proof in the other case is similar.

Write

$$\begin{aligned} \bar{L}(v, t) &= \left(\int_v^{v+t^{-1}} + \int_{v+t^{-1}}^\infty \right) \{ t^p (u-v)^{p-\alpha} \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) \} du \\ &= \bar{L}_1 + \bar{L}_2. \end{aligned}$$

By Lemma 3,

$$|\bar{L}_1| < At^p(1+vt)^{-\rho-r} \int_v^{v+t^{-1}} (u-v)^{p-\alpha} du < At^{\alpha-1}(1+vt)^{-\rho-r}.$$

Also, by the mean value theorem and Lemma 3,

$$\begin{aligned} |\bar{L}_2| &\leq t^\alpha \max_{\xi} \left| \int_{v+t^{-1}}^\xi \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du \right| \\ &< At^{\alpha-1}(1+vt)^{-\rho-r}. \end{aligned}$$

The result therefore follows.

Another lemma of the same type is

LEMMA 17. *If α , p , r , and ρ are defined as in Lemma 16 and*

$$Q(v, t) + i\bar{Q}(v, t) = t^{p+1} \int_v^\infty (u-v)^{p-\alpha} \{ \gamma_{\rho+r+1}^{(p+r+2)}(ut) + i\bar{\gamma}_{\rho+r+1}^{(p+r+2)}(ut) \} du,$$

then, for $v > 0$, $t > 0$,

$$|Q(v, t) + i\bar{Q}(v, t)| < At^\alpha(1+vt)^{-\rho-r-1}.$$

LEMMA 18. *If α , p , r , and ρ are defined as in Lemma 16, with the additional restriction that $\alpha > 0$, and*

$$M(v, t) + i\bar{M}(v, t) = \int_0^v x^{\alpha-1} \{ L(x, t) + i\bar{L}(x, t) \} dx,$$

then, when r is odd,

$$|M(v, t)| < At^{a-1}v^a(1+vt)^{-\rho-r},$$

and, when r is even,

$$|\bar{M}(v, t)| < At^{a-1}v^a(1+vt)^{-\rho-r}.$$

When $0 < vt < 1$ we have, from Lemma 16,

$$|M(v, t)| < A \int_0^v x^{a-1} t^{a-1} dx < At^{a-1}v^a.$$

We proceed to majorize $M(v, t)$ when $vt > 1$.

We have

$$\begin{aligned} M(\infty, t) &= t^p \int_0^\infty x^{a-1} dx \int_x^\infty (u-x)^{p-a} \gamma_{\rho+r}^{(r+p+1)}(ut) du \\ &= t^p \int_0^\infty \gamma_{\rho+r}^{(r+p+1)}(ut) du \int_0^u x^{a-1} (u-x)^{p-a} dx, \end{aligned}$$

so that, if the interchange of the order of integration can be justified*,

$$\begin{aligned} \frac{\Gamma(p+1)}{\Gamma(a)\Gamma(p+1-a)} M(\infty, t) &= t^p \int_0^\infty u^p \gamma_{\rho+r}^{(r+p+1)}(ut) du \\ &= \left[\sum_{\nu=0}^p (-1)^\nu \frac{p!}{(p-\nu)!} t^{p-\nu-1} u^{p-\nu} \gamma_{\rho+r}^{(r+p-\nu)}(ut) \right]_{u=0}^{u \rightarrow \infty}. \end{aligned}$$

As $u \rightarrow \infty$ each term tends to zero by Lemma 3. At $u = 0$, those terms for which $0 \leq \nu \leq p-1$ are obviously zero, while the term $\nu = p$ is

$$(-1)^p p! t^{-1} \gamma_{\rho+r}^{(r)}(0) = 0,$$

since r is odd. Hence $M(\infty, t) = 0$ and therefore

$$M(v, t) = - \int_v^\infty x^{a-1} L(x, t) dx.$$

It follows from Lemma 16 that, if $vt > 1$,

$$|M(v, t)| < A \int_v^\infty x^{a-1-\rho-r} t^{a-1-\rho-r} dx < At^{a-1-\rho-r} v^{a-\rho-r}.$$

The lemma is therefore proved for the case when r is odd. A similar proof holds when r is even in the case of $\bar{M}(v, t)$.

4.4. *Proof of Theorem 4.* Throughout the proofs of the converse theorems we find it more convenient to use Rieszian rather than Cesàro means.

* The justification of a similar interchange will be given in the proof of Theorem 4.

Suppose first that r is even.

The series $\sum_{n=1}^{\infty} (\beta_n \cos nx - \alpha_n \sin nx) \sin nu$

is the Fourier series of

$$\frac{1}{2} \{f(x+u) - f(x-u)\},$$

that is, of

$$u^r h(u) + \sum_{\mu=1}^{\frac{1}{2}r} \frac{\theta_{2\mu-1}}{(2\mu-1)!} u^{2\mu-1},$$

An elementary calculation shows that the Fourier series of

$$\sum_{\mu=1}^{\frac{1}{2}r} \frac{\theta_{2\mu-1}}{(2\mu-1)!} u^{2\mu-1}$$

is $\sum_{n=1}^{\infty} a_{n,0} \sin nu$, where

$$a_{n,0} = 2 \sum_{\mu=1}^{\frac{1}{2}r} \theta_{2\mu-1} \sum_{\nu=1}^{\mu} \frac{(-1)^{\nu}}{(2\mu-2\nu+1)!} \pi^{2\mu-2\nu} (-1)^n n^{1-2\nu}.$$

By Lemma 7 the series $\sum n^r a_{n,0}$ is summable $|C, \alpha|$ if $\alpha > r$. Hence the Fourier series of $u^r h(u)$ may be written in the form

$$\sum_{n=1}^{\infty} n^{-r} a_n \sin nu,$$

where $\sum a_n$ is summable $|C, \alpha|$, $\alpha > r$, and $a_n = o(n^r)$.

Suppose that p and ρ are defined as in Lemma 16. Then, if $\alpha - r \geq 1$, the function

$$\sum_{i=0}^r \binom{r}{i} (-1)^i \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} (1-u/t)^{\rho+i-1}$$

is of bounded variation for $0 \leq u \leq t$. Hence*

$$\begin{aligned} \int_0^t u^r h(u) \left\{ \sum_{i=0}^r \binom{r}{i} (-1)^i \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} (1-u/t)^{\rho+i-1} \right\} du \\ = \sum_{n=1}^{\infty} a_n n^{-r} \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} \int_0^t (1-u/t)^{\rho+i-1} \sin nu \, du \\ = t^{r+1} \sum_{n=1}^{\infty} a_n (nt)^{-r} \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} \bar{\gamma}_{\rho+i}(nt) \\ = t^{r+1} \sum_{n=1}^{\infty} a_n \bar{\gamma}_{\rho+r}^{(r)}(nt), \end{aligned}$$

by Lemma 3. If, on the other hand, $0 < \alpha - r < \rho < 1$, we have by the

* By a theorem of Hardy and W. H. Young. See Hobson, 12, 583.

above argument,

$$\begin{aligned} \int_0^t u^r h(u) \left\{ \sum_{i=1}^r (-1)^i \binom{r}{i} \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} (1-u/t)^{\rho+i-1} \right\} du \\ = t^{r+1} \sum_{n=1}^{\infty} a_n (nt)^{-r} \sum_{i=1}^r (-1)^i \binom{r}{i} \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} \bar{\gamma}_{\rho+i}(nt). \end{aligned}$$

Now the function $(1-u/t)^{\rho-1}$ is of bounded variation for $0 \leq u \leq t-t\epsilon$, so that

$$\begin{aligned} \int_0^{t-t\epsilon} u^r h(u) (1-u/t)^{\rho-1} du &= \sum_{n=1}^{\infty} a_n n^{-r} \int_0^{t-t\epsilon} (1-u/t)^{\rho-1} \sin nu \, du \\ &= t^{r+1} \sum_{n=1}^{\infty} a_n (nt)^{-r} \bar{\gamma}_{\rho, \epsilon}(nt). \end{aligned}$$

For fixed positive t , we have, by Lemma 3,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{\infty} a_n n^{-r} \{ \bar{\gamma}_{\rho, \epsilon}(nt) - \bar{\gamma}_{\rho}(nt) \} \\ = \lim_{\epsilon \rightarrow 0} \left[O \left\{ \epsilon^{\rho} \sum_{n=1}^N |a_n| n^{-r} \right\} + O \left\{ \sum_{N+1}^{\infty} |a_n| n^{-r-\rho} \right\} \right] \\ = O \left\{ \sum_{N+1}^{\infty} |a_n| n^{-r-\rho} \right\}, \end{aligned}$$

which, by Lemma 6, may be made arbitrarily small by choice of N since $\rho > \alpha - r$. It follows that

$$\int_0^t u^r h(u) (1-u/t)^{\rho-1} du = t^{r+1} \sum_{n=1}^{\infty} a_n (nt)^{-r} \bar{\gamma}_{\rho}(nt),$$

if we interpret the integral on the left in the Cauchy sense.

In both cases therefore we have

$$\sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\Gamma(\rho+r+i)}{\Gamma(\rho+i)} t^{-\rho-r-i} \int_0^t u^r h(u) (t-u)^{\rho+i-1} du = \sum_{n=1}^{\infty} a_n \bar{\gamma}_{\rho+r}^{(r)}(nt),$$

that is, for r even, $\rho > \alpha - r > 0$ and $t > 0$,

$$\left(\frac{d}{dt} \right)^r \left\{ t^{-\rho-r} \int_0^t u^r h(u) (t-u)^{\rho+r-1} du \right\} = \sum_{n=1}^{\infty} a_n \bar{\gamma}_{\rho+r}^{(r)}(nt).$$

For fixed positive t , $A_N \bar{\gamma}_{\rho+r}^{(r)}(Nt) \rightarrow 0$ as $N \rightarrow \infty$ since, when $\rho > 1$,

$$A_N \bar{\gamma}_{\rho+r}^{(r)}(Nt) = o(N^{r+1}) O(N^{-r-1}) = o(1),$$

and, when $0 < \rho < 1$,

$$A_N \bar{\gamma}_{\rho+r}^{(r)}(Nt) = o(N^{\alpha}) O(N^{-\rho-r}) = o(1).$$

Hence, by partial summation,

$$t^{-1} \sum_{n=1}^{\infty} a_n \bar{\gamma}_{\rho+r}^{(r)}(nt) = - \sum_{n=1}^{\infty} A_n \int_n^{n+1} \bar{\gamma}_{\rho+r}^{(r+1)}(ut) du = - \int_0^{\infty} A(u) \bar{\gamma}_{\rho+r}^{(r+1)}(ut) du,$$

and, on integration by parts p times, this becomes

$$\left[\sum_{\nu=1}^p \frac{(-1)^\nu}{\nu!} t^{\nu-1} A_\nu(u) \bar{\gamma}_{\rho+r}^{(r+\nu)}(ut) \right]_0^\infty + \frac{(-1)^{p+1}}{p!} \int_0^\infty A_p(u) t^p \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du.$$

The integrated terms vanish at $u=0$ and, as $u \rightarrow \infty$,

$$\begin{aligned} A_\nu(u) \bar{\gamma}_{\rho+r}^{(r+\nu)}(ut) &= o(u^{r+\nu+1}) O(u^{-r-\nu-1}) = o(1) \quad (\rho > \nu+1), \\ &= o(u^\alpha) O(u^{-\rho-r}) = o(1) \quad (\rho < \nu+1). \end{aligned}$$

Hence we have, by a slight modification of Lemma 8,

$$\begin{aligned} t^{-1} \sum_{n=1}^{\infty} a_n \bar{\gamma}_{\rho+r}^{(r)}(nt) &= \frac{(-1)^{p+1} t^p}{\Gamma(\alpha+1) \Gamma(p+1-\alpha)} \int_0^\infty \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du \int_0^u (u-v)^{p-\alpha} dA_\alpha(v) \\ &= \frac{(-1)^{p+1}}{\Gamma(\alpha+1) \Gamma(p+1-\alpha)} \int_0^\infty \bar{L}(v, t) dA_\alpha(v), \end{aligned}$$

provided that the change in the order of integration is permissible*.

The interchange will be justified if we show that, as $X \rightarrow \infty$,

$$\int_0^X dA_\alpha(v) \int_X^\infty (u-v)^{p-\alpha} \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du \rightarrow 0.$$

Write

$$\int_X^\infty (u-v)^{p-\alpha} \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du = \int_X^{X+1} + \int_{X+1}^\infty = N_1 + N_2.$$

Then, for $0 < v < X$, and fixed $t > 0$,

$$N_1 = O \left\{ \int_X^{X+1} (u-v)^{p-\alpha} u^{-\rho-r} du \right\} = O(X^{-\rho-r}),$$

and, by the second mean value theorem,

$$|N_2| \leq (X+1-v)^{p-\alpha} \max_{X' > X+1} \left| \int_{X+1}^{X'} \bar{\gamma}_{\rho+r}^{(r+p+1)}(ut) du \right| = O(X^{-\rho-r}).$$

* For the justification of a similar interchange see Bosanquet and Hyslop, 7. It is reproduced here for the sake of completeness.

Also

$$\begin{aligned} X^{-\rho-r} \int_0^X |dA_a(v)| &= X^{-\rho-r} O\left\{\int_0^X v^a |dC_a(v)|\right\} + X^{-\rho-r} O\left\{\int_0^X v^{-1} |A_a(v)| dv\right\} \\ &= O(X^{a-\rho-r}) \int_0^\infty |dC_a(v)| + O(X^{-\rho-r}) \int_0^X v^{a-1} dv \\ &= O(X^{a-\rho-r}) = o(1), \end{aligned}$$

since Σa_n is summable $|C, a|$ and $\rho > a-r$.

We now show that

$$\int_0^\pi \left| \int_0^\infty \bar{L}(v, t) dA_a(v) \right| dt < \infty.$$

On integration by parts we obtain

$$\int_0^\infty \bar{L}(v, t) dA_a(v) = \left[\bar{L}(v, t) A_a(v) \right]_0^\infty - \int_0^\infty v^a C_a(v) \frac{d}{dv} \bar{L}(v, t) dv.$$

The integrated term vanishes by Lemma 16 and

$$\int_0^v x^a \frac{d}{dx} \bar{L}(x, t) dx = v^a \bar{L}(v, t) - a \bar{M}(v, t).$$

Thus, remembering that r is even, we have, by Lemmas 16 and 18,

$$\begin{aligned} \int_0^\infty \bar{L}(v, t) dA_a(v) &= - \left[\{v^a \bar{L}(v, t) - a \bar{M}(v, t)\} C_a(v) \right]_0^\infty \\ &\quad + \int_0^\infty \{v^a \bar{L}(v, t) - a \bar{M}(v, t)\} dC_a(v) \\ &\quad - \int_0^\infty \{v^a \bar{L}(v, t) - a \bar{M}(v, t)\} dC_a(v), \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^\pi \left| \int_0^\infty \bar{L}(v, t) dA_a(v) \right| dt &\leq \int_0^\infty |dC_a(v)| \left(\int_0^{v-1} + \int_{v-1}^\pi \right) |v^a \bar{L}(v, t) - a \bar{M}(v, t)| dt \\ &< A \int_0^\infty |dC_a(v)| \left\{ \int_0^{v-1} t^{a-1} v^a dt + \int_{v-1}^\infty t^{a-1-\rho-r} v^{a-\rho-r} dt \right\} \\ &< A \int_0^\infty |dC_a(v)| < \infty. \end{aligned}$$

Collecting our results, we have proved that, when r is even and $\rho > a-r > 0$,

$$\int_0^\pi t^{-1} \left| \left(\frac{d}{dt} \right)^r \left\{ t^{-\rho-r} \int_0^t u^r h(u) (t-u)^{\rho+r-1} du \right\} \right| dt < \infty,$$

and an inspection of the argument shows that this result is also true when r is odd, the functions γ , L , and M replacing respectively $\bar{\gamma}$, $\bar{L}$, and $\bar{M}$.

The theorem is now proved for the case $a > r = 0$. We proceed to discuss the case $a > r \geq 1$.

If $\nu \geq 1$, we have

$$\begin{aligned} & \left| \left(\frac{d}{dt} \right)^{r-\nu} \left\{ t^{-\rho-r} \int_0^t u^r h(u) (t-u)^{\rho+r-1} du \right\} \right| \\ &= \left| \sum_{i=0}^{r-\nu} \binom{r-\nu}{i} \frac{\Gamma(\rho+2r-i-\nu)}{\Gamma(\rho+r-i)} t^{-\rho-2r+\nu+i} \int_0^t u^r h(u) (t-u)^{\rho+r-i-1} du \right| \\ &< A t^{-r+\nu-1} \int_0^t u^r |h(u)| du \\ &< A t^{\nu-1} \int_0^t |h(u)| du = o(1), \end{aligned}$$

as $t \rightarrow 0$ since $h(u)$ is integrable over $(0, \pi)$. It therefore follows from Lemma 13 that, for $\nu = 0, 1, \dots, r$,

$$\int_0^\pi |D_\nu(t)| dt = \int_0^\pi \left| t^{-1-\nu} \left(\frac{d}{dt} \right)^{r-\nu} \left\{ t^{-\rho-r} \int_0^t u^r h(u) (t-u)^{\rho+r-1} du \right\} \right| dt < \infty.$$

Writing for brevity

$$\Omega_\mu = \Omega_\mu(t) = t^{-\rho-r-\mu-1} \int_0^t u^r h(u) (t-u)^{\rho+\mu-1} du,$$

we have the $r+1$ equations

$$\begin{aligned} \alpha_{0,0} \Omega_0 + \alpha_{0,1} \Omega_1 + \dots + \alpha_{0,r} \Omega_r &= D_0, \\ \alpha_{1,1} \Omega_1 + \dots + \alpha_{1,r} \Omega_r &= D_1, \\ \dots & \dots \dots \dots \\ \alpha_{r,r} \Omega_r &= D_r, \end{aligned}$$

where none of the constants $\alpha_{\mu,\nu}$ is zero. It follows at once that $\Omega_0(t)$ is expressible in the form

$$e_0 D_0 + e_1 D_1 + \dots + e_r D_r,$$

from which we obtain

$$\int_0^\pi |\Omega_0(t)| dt = \int_0^\pi \left| t^{-\rho-r-1} \int_0^t u^r h(u) (t-u)^{\rho-1} du \right| dt < \infty.$$

By applying Lemma 15 we then have in turn

$$\begin{aligned} \int_0^\pi t^{-\rho-r} \left| \int_0^t u^{r-1} h(u) (t-u)^{\rho-1} du \right| dt &< \infty, \\ \dots & \dots \dots \dots \dots \dots \dots \\ \int_0^\pi t^{-\rho-1} \left| \int_0^t h(u) (t-u)^{\rho-1} du \right| dt &< \infty, \end{aligned}$$

that is,
$$\int_0^\pi t^{-1} |h_\rho(t)| dt < \infty.$$

Since $\rho > \alpha - r$, the theorem is proved when $\alpha > r \geq 0$, and the case $\alpha = r$ is at once deducible from this result.

4.5. *Proof of Theorem 3.* Following the proof of Theorem 4 it is easy to see that, when r is even and $\alpha > r$, the Fourier series of $u^r g(u)$ is of the form

$$a_0 + \sum_{n=1}^{\infty} n^{-r} a_n \cos nu,$$

where $\sum a_n$ is summable $[C, \alpha]$ and $a_n = o(n^r)$. It also follows that, for $\rho > \alpha - r$, $t > 0$,

$$\begin{aligned} t^{-r-2} \int_0^t u^r g(u) \left\{ \sum_{i=0}^{r+1} (-1)^i \binom{r+1}{i} \frac{\Gamma(\rho+r+1+i)}{\Gamma(\rho+i)} \left(1 - \frac{u}{t}\right)^{\rho+i-1} \right\} du \\ = \sum_{n=1}^{\infty} b_n \gamma_{\rho+r+1}^{(r+1)}(nt), \end{aligned}$$

where $b_n = na_n$, and therefore

$$\left(\frac{d}{dt}\right)^{r+1} \left\{ t^{-\rho-r-1} \int_0^t u^r g(u) (t-u)^{\rho+r} du \right\} = \sum_{n=0}^{\infty} b_n \gamma_{\rho+r+1}^{(r+1)}(nt).$$

Defining p and ρ as in Lemma 16 and integrating by parts as in the proof of Theorem 4, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} b_n \gamma_{\rho+r+1}^{(r+1)}(nt) &= (-1)^{p+1} \frac{t^{p+1}}{\Gamma(p+1)} \int_0^\infty B_p(u) \gamma_{\rho+r+1}^{(r+p+2)}(ut) du \\ &= \frac{(-1)^{p+1} t^{p+1}}{\Gamma(\alpha) \Gamma(p+1-\alpha)} \int_0^\infty \gamma_{\rho+r+1}^{(r+p+2)}(ut) du \int_0^u (u-v)^{p-\alpha} B_{\alpha-1}(v) dv \\ &= \frac{(-1)^{p+1}}{\Gamma(\alpha) \Gamma(p+1-\alpha)} \int_0^\infty B_{\alpha-1}(v) Q(v, t) dv. \end{aligned}$$

Hence, from Lemmas 9 and 17,

$$\begin{aligned} \int_0^\pi \left| \left(\frac{d}{dt} \right)^{r+1} \left\{ t^{\rho-r-1} \int_0^t u^r g(u) (t-u)^{\rho+r} du \right\} \right| dt \\ < A \int_0^\infty |C_a'(v)| dv \int_0^\pi v^{\alpha+1} |Q(v, t)| dt \\ < A \int_0^\infty |C_a'(v)| dv \left\{ \int_0^{v-1} v^{\alpha+1} t^\alpha dt + \int_{v-1}^\infty v^{\alpha-\rho-r} t^{\alpha-\rho-r-1} dt \right\} \\ < \infty, \end{aligned}$$

and the same result holds when r is odd.

It follows by Lemma 12 that, for $v = 0, 1, 2, \dots, r$,

$$M_v = t^{-v} \left(\frac{d}{dt} \right)^{r-v} \left\{ t^{\rho-r-1} \int_0^t u^r g(u) (t-u)^{\rho+r} du \right\}$$

is of bounded variation in $(0, \pi)$. Writing

$$\chi_\mu = t^{\rho-r-\mu-1} \int_0^t u^r g(u) (t-u)^{\rho+\mu} du,$$

we see, as in the proof of Theorem 4, that χ_0 is expressible in the form

$$c_0 M_0 + c_1 M_1 + \dots + c_r M_r,$$

so that χ_0 is of bounded variation in $(0, \pi)$. Hence, by repeated application of Lemma 14,

$$t^{\rho-1} \int_0^t g(u) (t-u)^\rho du$$

is of bounded variation in $(0, \pi)$. Since $\rho > \alpha - r$ the theorem follows.

5. Concluding remarks.

5.1. Since $h(t)$ is odd, periodic and integrable in $(0, \pi)$, the limit

$$\lim_{X \rightarrow \infty} \frac{2}{\pi} \int_t^X t^{-1} h(t) dt$$

exists and defines, for $t > 0$, a function which we shall denote by $k(t)$. It may then be proved* that, for $\alpha \geq 0$,

$$k_\alpha(t) = \frac{2}{\pi} \int_t^\infty t^{-1} h_\alpha(t) dt,$$

* For a complete account of the results assumed in this section see Bosanquet and Hyslop, 7, where further references are also given.

where $t^a k_a(t)/\Gamma(a+1)$ is the Riemann-Liouville integral of order a for the function $k(t)$. Theorem 4 may thus be written in the following form.

THEOREM 4A. *If $a \geq r \geq 0$ and the r -th derived series of the allied series of $f(t)$ is summable $|C, a|$ at $t = x$, then $k_{a-r+\delta}(t)$ is of bounded variation in $(0, \pi)$ for every $\delta > 0$.*

Moreover, it can be shown that Theorem 2 may be written in the form :

THEOREM 2B. *If $a > 0$ and if $h_a(t)$ is of bounded variation in $(0, \pi)$ and $k_\lambda(t)$ is of bounded variation in $(0, \pi)$ for some positive λ , then the r -th derived series of the allied series of $f(t)$ at $t = x$ is summable $|C, a+r+\delta|$ for every $\delta > 0$.*

That this theorem is true also when $a = 0$ follows at once from the case $a > 0$ and Lemma 1. Theorem 2 itself, however, is not true when $a = 0$.

For simplicity, suppose that $r = 1$ and let $f(x)$ be even and equal to $x(\log 2\pi/x)^{-1}$ when $0 < x \leq \pi$. Then, at $x = 0$,

$$h(t) = \left(\log \frac{2\pi}{t} \right)^{-1},$$

which is of bounded variation in $(0, \pi)$. On the other hand $t^{-1}|h_\lambda(t)|$ is not integrable in $(0, \pi)$ for any value of λ , so that, by Theorem 4A, the first derived series of the allied series of $f(x)$ at $x = 0$ cannot be summable $|C, 1+\delta|$ for any $\delta > 0$.

The above assertion is therefore proved for $a = 0$, $r = 1$. In the case of general r , we make use of the function

$$f(x) = x^r \left(\log \frac{2\pi}{x} \right)^{-1}$$

and adopt the same type of argument. In this case the algebra is more complicated owing to the presence of terms involving the constants θ_i , and we therefore omit a detailed discussion.

5.2. Theorems 1, 2B, 3, and 4 are all known* to be "best possible" for the particular case $a = r = 0$ in the sense that the δ of each conclusion cannot be replaced by zero. It seems desirable, however, to obtain similar results for cases when $r \neq 0$. So far as the direct theorems are concerned results for $a = 0$, $r = 1$ are easy to obtain, but I have been unable to solve even the problem $a = 1$, $r = 1$ for the converse theorems.

* Bosanquet, 5; Bosanquet and Hyslop, 7.

Let $f(x) = \frac{1}{2}x$. Then the Fourier series of $f(x)$ is $\Sigma(-1)^{n-1}n^{-1}\sin nx$ and its first derived series is $\Sigma(-1)^{n-1}\cos nx$, which is not summable $|C, 1|$ at $x = 0$. On the other hand, at $x = 0$,

$$g(t) - \frac{f(t) - f(-t)}{2t} = \frac{1}{2},$$

which is of bounded variation in $(0, \pi)$. Thus the δ of Theorem 1 cannot be replaced by zero when $\alpha = 0$, $r = 1$.

Let
$$f(x) = x^2 \operatorname{sgn}(\tfrac{1}{2}\pi - |x|).$$

Then it is easy to verify that the first derived series of the allied series of $f(x)$ at $x = 0$ is

$$\Sigma a_{n,0} + \Sigma a_{n,1} + \Sigma a_{n,2} + \Sigma a_{n,3},$$

where
$$\begin{aligned} a_{n,0} &= -\pi \sin \tfrac{1}{2}n\pi, & a_{n,1} &= 8\pi^{-1}n^{-2} \sin \tfrac{1}{2}n\pi, \\ a_{n,2} &= 4n^{-1}(-1)^n, & a_{n,3} &= -4n^{-1} \cos \tfrac{1}{2}n\pi. \end{aligned}$$

Of these four series the last three are summable $|C, 1|$ while the first is not. Thus the first derived series of the allied series of $f(x)$ at $x = 0$ is not summable $|C, 1|$. On the other hand, at $x = 0$,

$$h(t) = \frac{f(t) + f(-t)}{2t} = t \operatorname{sgn}(\tfrac{1}{2}\pi - |t|),$$

so that $h(t)$ is of bounded variation and $t^{-1}|h(t)|$ is integrable in $(0, \pi)$. Thus the δ of Theorem 2B cannot be replaced by zero when $\alpha = 0$, $r = 1$.

5.3. An interesting point which arises out of the paper is this. Theorems 3 and 4 have been proved on the supposition that $\alpha \geq r$ and the proofs which are given here would break down for $\alpha < r$. The question therefore arises whether these theorems are still true for a certain range of values of α less than r and, if not, what kind of results can be obtained for such values of α .

In conclusion I should like to express my thanks to Dr. L. S. Bosanquet and to a Referee for many helpful suggestions and criticisms. One of these suggestions enabled me to "improve" considerably the conclusions of Theorems 3 and 4.

[*Added, 15th July, 1939.* Since this paper was written interesting generalizations of the analogues of Theorems 1 and 3 for ordinary Cesàro summability have been obtained by Bosanquet; see *Quart. J. of Math. (Oxford Series)*, 10 (1939), 67-74.]

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SOME GENERAL SOLUTIONS OF ST. VENANT'S FLEXURE AND TORSION PROBLEM. I

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[Received 29 November, 1938.—Read 15 December, 1938.]

1. Interest in the potential problems involved in the general analysis of the small flexure of prismatic or cylindrical beams has recently been revived by an important paper by A. C. Stevenson*, who succeeded in extending considerably the field of possible solutions of this problem, but still within somewhat restricted limits on the type of cross-section. At Mr. Stevenson's suggestion I have applied my new, and in some ways still more general, method in two-dimensional potential theory to the determination of the solution for the wider class of prismatic section which the method covers, and the present paper contains the first of the results obtained. In it a complete formal solution of the torsion and flexure problem is given for a prism whose normal cross-section is a curve defined with reference to rectangular axes (x, y) on the plane of the section by the parametric equation

$$(1.1) \quad z = x + iy = \sum_{n=0}^{\infty} a_n e^{ni\xi} \quad (0 < \xi < 2\pi),$$

the coefficients in which are, in general, complex constants. The method involves treating the section as the curve $\eta = 0$ in the net defined by the transformation

$$(1.2) \quad z = \sum_{n=0}^{\infty} a_n e^{ni\zeta} \quad (\zeta = \xi + i\eta),$$

* "Flexure with shear and associated torsion in prisms of uni-axial and asymmetric cross-sections", *Phil. Trans. Royal Soc. (A)*, 237 (1939), 161. I have to thank Mr. Stevenson for drawing my attention to this problem, and for information about the contents of his paper before publication.

and it is restricted to the cases where this transformation is conformal inside the curve except possibly at the point $z = a_0$ ($\eta = \infty$). The problem of determining an equation of type (1.1) for a given section is the same as that of determining the conformal transformation mapping the area inside the curve in the z -plane on to the interior of the circle $|w| = 1$ in the w -plane, where $w = \exp i\zeta$. Elsewhere* I have pointed out that this transformation is known for most of the cross-section curves for which solutions of any type of two-dimensional potential problem have been obtained, and also for many other and different classes of curves. Each such transformation, in fact, defines a whole series of curves, the equipotentials or level surfaces of the net defined by it, and each curve of the series has an equation of the type (1.1). The curve $\eta = \alpha$, for example, in the net defined by (1.2) has an equation

$$z = \sum_{n=0}^{\infty} a_n' e^{ni\xi},$$

wherein

$$a_n' = a_n \exp(-n\alpha).$$

2. The prism is referred as usual to a rectangular coordinate system with the axis† Oz along the line of centroids of the cross-sections in the unstrained state. The axes Ox , Oy are any perpendicular axes at the centroid of the cross-section at one end of the prism. The departure from the usual practice, of assuming these axes to be the principal axes of inertia of the area, was made after it was realized that, if the fundamental forms of the theory due to St. Venant are properly interpreted, they do not require the axes to be principal axes so long as the origin is at the centroid. Stevenson adopts the rather cumbersome procedure of starting with the same forms referred to the principal axes, and then subsequently transforming them to axes in the section quite generally disposed relative to these principal axes. The object of using general axes is, of course, to gain the freedom necessary to choose axes which permit of the expression of the equation of the boundary curve in the particular form required by his method; but Stevenson's procedure introduces considerable complications into the problem by making it cover, by the way, a determination also of the position of the centroid and of the principal axes of inertia of the

* In a paper in the *Mathematische Annalen*, 116 (1939), 374 on "The internal problems of two-dimensional potential theory." A more detailed list of the curves is given in §10 of the present paper.

† No confusion should arise from the use of z both as a real coordinate on the rare occasions when it is needed, and more often as the complex variable $x + iy$. Both notations are standard, and alteration of either would probably cause more confusion than it saves.

section. The interpretation of the familiar formulae as relating directly to general axes at the centroid introduces the moments and products of inertia of the area relative to these axes (from which the principal moments and axes can be derived), but these can always be determined by a simple and independent process.

If the boundary curve referred to any set of axes has the form (1.1), it is easy to prove that the area S , the coordinates (x_G, y_G) of the centroid, and the moments A , B , and product H of inertia of the area are given by the following contour integrals taken round the boundary of the cross-section in the complex $z (=x+iy)$ plane:

$$(2.1) \quad S = -\frac{1}{2}i \int \bar{z} dz = -\frac{1}{2}i \int_0^{2\pi} \bar{z}(\zeta) z'(\zeta) d\zeta \\ = \pi \sum_{n=1}^{\infty} n a_n \bar{a}_n;$$

$$(2.2) \quad Sz_G = S(x_G + iy_G) = -\frac{1}{2}i \int z \bar{z} dz \\ = \pi \sum_{n=1}^{\infty} n a_n \bar{b}_n;$$

$$(2.3) \quad B - A - 2iH = \frac{1}{2}i \int z^2 \bar{z} dz \\ = -\pi \sum_{n=1}^{\infty} \bar{b}_n \{n a_n a_0 + (n-1) a_{n-1} a_1 + \dots + a_1 a_{n-1}\};$$

$$(2.4) \quad A + B = -\frac{1}{4}i \int z \bar{z}^2 dz \\ = \frac{1}{2}\pi \left[\sum_{n=0}^{\infty} b_n \{\bar{a}_{n+1} a_1 + 2\bar{a}_{n+2} a_2 + \dots\} \right. \\ \left. + \sum_{n=1}^{\infty} \bar{b}_n \{n a_n \bar{a}_0 + (n+1) a_{n+1} \bar{a}_1 + \dots\} \right].$$

In these formulae $b_r = \sum_{n=r}^{\infty} a_{n+r} \bar{a}_n$, a bar over a symbol denotes the conjugate complex number and a dash denotes a first derivative with respect to the parameter ζ . Writing

$$(2.5) \quad B - A - 2iH = K e^{-2i\epsilon},$$

where K is real, we see that ϵ is the angle made by the principal axes at the origin with the chosen axes.

Knowing these results we could, of course, follow the usual procedure and use the principal axes at the centroid as axes of reference, because in

our form of the analysis the equation of a curve is of the same form in any axes. In fact, if we first transfer the axes to the centroid and then carry out as above the determination of ϵ for the principal axes at the centroid, the equation (1.1), when referred to these principal axes, becomes

$$z = \sum_{n=0}^{\infty} a_n' e^{ni\xi},$$

wherein $a_0' = (a_0 - z_G) e^{-i\epsilon}$, $a_n' = a_n e^{-i\epsilon}$ ($n > 0$).

There is, however, some advantage in using the particular axes in which the majority of the coefficients a_n assume their simplest forms. Since, however, Stevenson's analysis shows that a good deal of complexity is introduced by using an origin other than the centroid, we always carry out the first step in this transformation, so that the origin for the equation of the section becomes the centroid. This affects only the first coefficient a_0 . Having done this, we can always assume in our analysis that the coefficients in the equation of type (1.1) always satisfy the condition

$$(2.6) \quad \sum_{n=1}^{\infty} n a_n \bar{b}_n = 0,$$

which we shall require later.

3. The curved surface of the prism is free from surface tractions, but the end $z = l$ is assumed to be subject to forces which are equivalent to a single force with components $(F_1, F_2, 0)$ through the *load point* (x_τ, y_τ, z) . The resultant of the stresses across any other cross section must then be equivalent to an equal force $(F_1, F_2, 0)$ at the corresponding point (x_τ, y_τ, z) of this section, together with a certain bending moment about an axis transverse to the z -axis.

We use E for Young's modulus; and, following Love, we write σ for Poisson's ratio, and μ for $\frac{1}{2}E/(1+\sigma)$. The fundamental stress components which satisfy the conditions of the semi-inverse method are, in Stevenson's form,

$$(3.1) \quad X_x = Y_y = X_y = 0, \quad Z_z = -E(l-z)(K_1 x + K_2 y),$$

and

$$(3.2) \quad Z_x = \mu K_1 \left\{ \frac{\partial \phi_1}{\partial x} - x^2 - \sigma(x^2 - y^2) \right\} + \mu K_2 \frac{\partial \phi_2}{\partial x} + \mu \tau \left(\frac{\partial \phi_3}{\partial x} - y \right),$$

$$(3.3) \quad Z_y = \mu K_1 \left\{ \frac{\partial \phi_1}{\partial y} + \mu K_2 \left(\frac{\partial \phi_2}{\partial y} - y^2 + \sigma(x^2 - y^2) \right) \right\} + \mu \tau \left(\frac{\partial \phi_3}{\partial y} + x \right).$$

In these formulae ϕ_1, ϕ_2, ϕ_3 are plane potential functions satisfying the following conditions on the boundary of the section:

$$(3.4) \quad \frac{\partial \phi_1}{\partial n} = l\{x^2 + \sigma(x^2 - y^2)\},$$

$$(3.5) \quad \frac{\partial \phi_2}{\partial n} = m\{y^2 - \sigma(x^2 - y^2)\},$$

$$(3.6) \quad \frac{\partial \phi_3}{\partial n} = ly - mx,$$

wherein n denotes the outward drawn normal to the boundary of the section, with direction cosines $(l, m, 0)$.

The constants K_1, K_2 are determined in terms of the force components F_1, F_2 producing the flexure by the conditions

$$(3.7) \quad \iint Z_x dx dy = F_1, \quad \iint Z_y dx dy = F_2,$$

the integrals both being taken over the area of the cross-section. Remembering that, if ϕ is harmonic,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} + x \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial}{\partial x} \left(x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(x \frac{\partial \phi}{\partial y} \right),$$

and using the boundary conditions (3.4) and (3.5), we deduce at once that

$$(3.8) \quad \begin{cases} \iint Z_x dx dy = E(K_1 A + K_2 H), \\ \iint Z_y dx dy = E(K_1 H + K_2 B), \end{cases}$$

wherein A, B, H are respectively the moments and product of inertia of the section area for the chosen axes of reference. Thus

$$(3.9) \quad \begin{cases} EK_1 = \frac{F_1 B - F_2 H}{AB - H^2}, \\ EK_2 = \frac{F_2 A - F_1 H}{AB - H^2}. \end{cases}$$

The above solution also has to satisfy the further condition that

$$(3.10) \quad \iint (xZ_y - yZ_x) dx dy = x_\tau F_2 - y_\tau F_1,$$

and the fact that this must be true independently of the values of F_1 and F_2 provides a condition which determines the coordinates (x_τ, y_τ) of the *load point* corresponding to the particular type of stress assumed.

4. At this stage a subdivision of the problem is introduced. Defining (x_0, y_0, l) as the *centre of flexure*, that is the position of the load point (x_τ, y_τ, l) when $\tau = 0$, we notice, with Stevenson, that the solution for the general case corresponds to the superposition upon the solution for this simpler case (with $\tau = 0$) of a simple torsion solution for the cross-section with a twist τ under a couple of moment $F_2(x_\tau - x_0) - F_1(y_\tau - y_0)$. We first solve, therefore, the simpler flexure problem with $\tau = 0$ in (3.2) and (3.3), determining by equation (3.10) the centre of flexure (x_0, y_0, l) , as the load point corresponding to that particular type of stress. Following this we sketch briefly a solution of the simpler torsion problem obtained by taking $K_1 = K_2 = 0$ in (3.2) and (3.3).

The simple flexure problem rests on the determination of two potential functions ϕ_1, ϕ_2 satisfying the boundary conditions (3.4) and (3.5). Following Stevenson closely, but using a slightly different notation, we now write

$$(4.1) \quad \phi_1 = (1 + \sigma)\phi_{11} - \sigma\phi_{21}, \quad \phi_2 = (1 + \sigma)\phi_{22} + \sigma\phi_{12}.$$

The boundary conditions (3.4) and (3.5) can then be written in the form

$$(1 + \sigma) \frac{d\phi_{11}}{dn} - \sigma \frac{d\phi_{21}}{dn} = (1 + \sigma)x^2 \frac{dx}{dn} - \sigma y^2 \frac{dy}{ds},$$

$$(1 + \sigma) \frac{d\phi_{22}}{dn} + \sigma \frac{d\phi_{12}}{dn} = (1 + \sigma)y^2 \frac{dy}{dn} + \sigma x^2 \frac{dx}{ds},$$

which suggest taking $\phi_{11}, \phi_{12}, \phi_{22}, \phi_{21}$ as four harmonic functions satisfying the following boundary conditions:

$$(4.2) \quad \frac{d\phi_{11}}{dn} = x^2 \frac{dx}{dn}, \quad \frac{d\phi_{22}}{dn} = y^2 \frac{dy}{dn},$$

$$(4.3) \quad \frac{d\phi_{12}}{dn} = x^2 \frac{dx}{ds}, \quad \frac{d\phi_{21}}{dn} = y^2 \frac{dy}{ds}.$$

The conditions (4.2) can be written as

$$(4.4) \quad \frac{d}{dn} (\phi_{11} - \tfrac{1}{3}x^3) = 0, \quad \frac{d}{dn} (\phi_{22} - \tfrac{1}{3}y^3) = 0,$$

while those in (4.3), interpreted in terms of the conjugate functions ψ_{12} and ψ_{21} , are equivalent to

$$(4.5) \quad \psi_{12} = \tfrac{1}{3}x^3 + \text{const.}, \quad \psi_{21} = \tfrac{1}{3}y^3 + \text{const.}$$

These are the boundary conditions which define the first four of Stevenson's six canonical flexure functions and, following him still, we proceed to solve them separately for the particular type of section that we are considering.

5. If we assume that the boundary curve is $\eta = 0$ in the net defined by $z = z(\xi)$, the boundary condition (4.5) implies that, on $\eta = 0$,

$$\psi_{12} = \frac{1}{24} \{z(\xi) + \bar{z}(\xi)\}^3 + \text{const.}$$

Obviously, if we write

$$(5.1) \quad \Omega_{12} = \frac{i}{24} \{z(\zeta) + \bar{z}(\zeta)\}^3 + F_{12}(\zeta),$$

where F_{12} is a real function of its argument ζ , we have a complex potential function whose imaginary part does in fact satisfy the required boundary condition. If, then, we can choose $F_{12}(\zeta)$ so that Ω_{12} satisfies the usual regularity conditions, we shall have determined not only the required ψ_{12} but also the conjugate function ϕ_{12} .

In a similar way we see that ψ_{21} , which satisfies the second condition (4.5), can be obtained as the imaginary part of the complex potential function

$$(5.2) \quad \Omega_{21} = -\frac{1}{24} \{z(\zeta) - \bar{z}(\zeta)\}^3 + F_{21}(\zeta),$$

where F_{21} is a real function of its argument to be chosen to ensure that Ω_{21} satisfies the usual regularity conditions.

The other two functions are slightly more complex in form. Consider first ϕ_{11} ; it is a potential function satisfying on $\eta = 0$ the first boundary condition of (4.4). In our notation this condition is equivalent to the condition that

$$\frac{d}{d\eta} \left[\phi_{11} - \frac{1}{24} \{z(\xi) + \bar{z}(\xi)\}^3 \right]_{\eta=0} = 0,$$

which is, of course,

$$\frac{d\phi_{11}}{d\eta} = \frac{i}{8} \{z(\xi) + \bar{z}(\xi)\}^2 \{z'(\xi) - \bar{z}'(\xi)\},$$

dashes denoting differentiation with respect to ξ . It is not difficult now to see that this condition is satisfied by the real part of the complex potential function

$$(5.3) \quad \Omega_{11} = \frac{1}{8} \int^{\xi} \{z(t) + \bar{z}(t)\}^2 \{z'(t) - \bar{z}'(t)\} dt + F_{11}(\zeta),$$

$F_{11}(\zeta)$ being again a real function of its argument to be chosen to secure regularity. A slight transformation of the integral enables this to be written also in the form

$$(5.4) \quad \Omega_{11} = \frac{1}{24} \{z(\zeta) - \bar{z}(\zeta)\}^3 + \frac{1}{2} \int^{\xi} z(t) \bar{z}(t) \{z'(t) - \bar{z}'(t)\} dt + F_{11}(\zeta).$$

In an exactly similar manner we can see that the potential function ϕ_{22} , satisfying the second boundary condition (4.4), is the real part of the complex potential

$$(5.5) \quad \Omega_{22} = \frac{i}{24} \{z(\zeta) + \bar{z}(\zeta)\}^3 - \frac{i}{2} \int^{\zeta} z(t) \bar{z}(t) \{z'(t) + \bar{z}'(t)\} dt + F_{22}(\zeta),$$

F_{22} being subject to the same restrictions as before.

The determination of the potentials is thus reduced to the determination of four real functions $F(\zeta)$, whose sole purpose is to secure the regularity of the corresponding complete potentials. Since, however, there is considerable repetition of certain forms in the individual potentials, some advantage is to be gained at this stage in putting them together and returning therefore to the original potentials defined by (4.1). We see then that ϕ_1 is the real part of

$$(5.6) \quad \Omega_1 = \frac{1}{24}(1+2\sigma)\{z(\zeta) - \bar{z}(\zeta)\}^3 + \frac{1}{2}(1+\sigma) \int^{\zeta} z(t) \bar{z}(t) \{z'(t) - \bar{z}'(t)\} dt + F_1(\zeta),$$

while ϕ_2 is the real part of

$$(5.7) \quad \Omega_2 = \frac{i}{24}(1+2\sigma)\{z(\zeta) + \bar{z}(\zeta)\}^3 - \frac{i}{2}(1+\sigma) \int^{\zeta} z(t) \bar{z}(t) \{z'(t) + \bar{z}'(t)\} dt + F_2(\zeta),$$

the functions $F_1(\zeta)$ and $F_2(\zeta)$, defined by

$$F_1(\zeta) = (1+\sigma)F_{11} - \sigma F_{21},$$

$$F_2(\zeta) = (1+\sigma)F_{22} + \sigma F_{12},$$

being still real.

6. To determine the functions $F_1(\zeta)$ and $F_2(\zeta)$ in (5.6) and (5.7) we proceed as follows. After expressing the known first terms in Ω_1 and Ω_2 in terms of ζ by (1.2), we divide the new terms into two sets, one containing all the exponentials of positive multiples of $i\zeta$ which converge at $\eta = \infty$, and the other containing all the exponentials of negative multiples which diverge there. To secure regularity at $\eta = \infty$ we must choose the corresponding F function so that it contains the negatives of all terms in the latter set. This makes these terms disappear from Ω , and then, if we include also the conjugates of all the terms in the former set (which converge at $\eta = \infty$) to secure reality, we have completed the determination of F and therefore of Ω .

Before proceeding to the general problem we notice that both Ω 's contain a term in $\bar{z}^3$ which is, of course, $(\sum_{n=0}^{\infty} \bar{a}_n e^{-ni\zeta})^3$ and therefore diverges at $\eta = \infty$. The corresponding F function must therefore remove this term and replace it by its conjugate (with the appropriate sign) which is simply z^3 . Again the terms in z^3 in each potential thus modified remain unaffected by the choice of the remaining F functions, since they give rise to no divergent terms. These terms can, therefore, be absorbed in Ω_1 and Ω_2 . We accordingly modify our problem slightly to the determination of the two complex potentials Ω_1' and Ω_2' defined by the relations

$$(6.1) \quad \Omega_1' = \Omega_1 - \frac{1}{12}(1+2\sigma)z^3 \\ = \frac{1}{2}(1+\sigma) \int_{\zeta}^{\zeta} z(t) \bar{z}(t) \{z'(t) - \bar{z}'(t)\} dt \\ - \frac{1}{8}(1+2\sigma)[z(\zeta) \bar{z}(\zeta) \{z(\zeta) - \bar{z}(\zeta)\}] + F_1'(\zeta),$$

and

$$(6.2) \quad \Omega_2' = \Omega_2 - \frac{i}{12}(1+2\sigma)z^3 \\ = -\frac{i}{2}(1+\sigma) \int_{\zeta}^{\zeta} z(t) \bar{z}(t) \{z'(t) + \bar{z}'(t)\} dt \\ + \frac{i}{8}(1+2\sigma)[z(\zeta) \bar{z}(\zeta) \{z(\zeta) + \bar{z}(\zeta)\}] + F_2'(\zeta).$$

Using now a notation already introduced in §3, we have

$$z\bar{z} = \left(\sum_{n=0}^{\infty} a_n e^{ni\zeta} \right) \left(\sum_{n=0}^{\infty} \bar{a}_n e^{-ni\zeta} \right) \\ = b_0 + \sum_{n=1}^{\infty} (b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta}),$$

and then

$$(6.3) \quad z^2 \bar{z} = \left\{ b_0 + \sum_{n=1}^{\infty} (b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta}) \right\} \left\{ \sum_{n=0}^{\infty} a_n e^{ni\zeta} \right\} \\ = \sum_{r=-\infty}^{+\infty} f_r e^{ri\zeta},$$

wherein

$$f_0 = a_0 \bar{b}_0 + a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots,$$

$$f_r = a_0 \bar{b}_r + a_1 \bar{b}_{r-1} + \dots + a_r \bar{b}_0 + a_{r+1} \bar{b}_1 + a_{r+2} \bar{b}_2 + \dots, \quad (r > 0),$$

$$f_{-r} = a_0 \bar{b}_r + a_1 \bar{b}_{r+1} + \dots, \quad (r > 0).$$

With this notation.

$$z^2 \bar{z} \pm \bar{z}^2 z = \sum_{r=-\infty}^{+\infty} (f_r \pm \bar{f}_{-r}) e^{ri\zeta}.$$

Again we have

$$(6.4) \quad \begin{aligned} z \bar{z} z' &= i \left\{ b_0 + \sum_{n=1}^{\infty} (b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta}) \right\} \left\{ \sum_{n=1}^{\infty} n \alpha_n e^{ni\zeta} \right\} \\ &= i \left\{ g_0 + \sum_{r=-\infty}^{+\infty} r g_r e^{ri\zeta} \right\}, \end{aligned}$$

where

$$\begin{aligned} g_0 &= a_1 \bar{b}_1 + 2a_2 \bar{b}_2 + 3a_3 \bar{b}_3 + \dots, \\ r g_r &= a_1 b_{r-1} + 2a_2 b_{r-2} + \dots + r a_r b_0 + (r+1) a_{r+1} \bar{b}_1 + \dots, \quad (r > 0), \\ -r g_{-r} &= a_1 \bar{b}_{r+1} + 2a_2 \bar{b}_{r+2} + 3a_3 \bar{b}_{r+3} + \dots, \quad (r > 0). \end{aligned}$$

Thus, noticing that, on account of our choice of origin, $g_0 = \bar{g}_0 = 0$, we have

$$\int_{\zeta}^{\zeta} z(t) \bar{z}(t) \{z'(t) \pm \bar{z}'(t)\} dt = \sum_{r=-\infty}^{+\infty} (g_r \pm \bar{g}_{-r}) e^{ri\zeta},$$

the term with $r = 0$ being omitted.

Inserting these values and choosing the F functions to cancel the divergent terms as indicated, we deduce at once that

$$(6.5) \quad \Omega_1' = \sum_{r=0}^{\infty} \Omega_{1r} e^{ri\zeta}, \quad \Omega_2' = \sum_{r=0}^{\infty} \Omega_{2r} e^{ri\zeta},$$

where

$$(6.6) \quad \Omega_{1r} = (1 + \sigma)(g_r - \bar{g}_{-r}) - \frac{1}{4}(1 + 2\sigma)(f_r - \bar{f}_{-r}),$$

$$(6.7) \quad \Omega_{2r} = \frac{i}{4}(1 + 2\sigma)(f_r + \bar{f}_{-r}) - i(1 + \sigma)(g_r + \bar{g}_{-r});$$

and these are formal solutions for the potential functions of our problem.

7. All that now remains to complete the solution of the particular flexure problem is to calculate the stresses, and the moment integral (3.10), and thence to derive the coordinates of the centre of flexure. First, however, we must interpret the stress components in terms of the modified potentials introduced by (6.1) and (6.2). Picking out the real parts of the potentials defined by these relations, we have

$$(7.1) \quad \begin{cases} \phi_1' = \phi_1' + \frac{1}{12}(1 + 2\sigma)(x^3 - 3xy^2), \\ \phi_2 = \phi_2' - \frac{1}{12}(1 + 2\sigma)(3x^2y - y^3). \end{cases}$$

Inserting these in (3.2) and (3.3) and omitting for the present the torsion terms, we find that

$$(7.2) \quad Z_x = \mu K_1 \left\{ \frac{\partial \phi_1'}{\partial x} + \frac{1}{4}(1-2\sigma)(x^2-y^2)-x^2 \right\} + \mu K_2 \left\{ \frac{\partial \phi_2'}{\partial x} - \frac{1}{2}(1+2\sigma)xy \right\}.$$

$$(7.3) \quad Z_y = \mu K_1 \left\{ \frac{\partial \phi_1'}{\partial y} - \frac{1}{2}(1+2\sigma)xy \right\} + \mu K_2 \left\{ \frac{\partial \phi_2'}{\partial y} - \frac{1}{4}(1-2\sigma)(x^2-y^2)-y^2 \right\}.$$

Inserting these in the integral for the moment, namely

$$\iint (x Z_y - y Z_x) dx dy,$$

and converting the integrals that arise from various terms into contour integrals round the boundary of the cross-section, we find that this moment integral is the real part of

$$(7.4) \quad \frac{1}{2}\mu \int z\bar{z}(K_1 d\Omega_1' + K_2 d\Omega_2') - \frac{\mu}{16}(K_1 - iK_2)(1-2\sigma) \int z^2 \bar{z}^2 dz.$$

When we use the values of Ω_1' and Ω_2' given by (6.3), the first of these integrals proves to be

$$\pi\mu i \sum_{n=1}^{\infty} n \bar{b}_n (K_1 \Omega'_{1n} + K_2 \Omega'_{2n}),$$

while, using (6.4) and

$$z\bar{z} = b_0 + \sum_{n=1}^{\infty} (b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta}),$$

the second proves to be

$$\frac{1}{8}\pi\mu i \left\{ b_0 g_0 + \sum_{n=1}^{\infty} n (\bar{b}_n g_n - b_n g_{-n}) \right\},$$

apart from the factor $(K_1 - iK_2)(1-2\sigma)$. We conclude therefore that $F_2 x_0 - F_1 y_0$ is the real part of

$$(7.5) \quad \pi\mu i \sum_{n=1}^{\infty} n \bar{b}_n (K_1 \Omega'_{1n} + K_2 \Omega'_{2n}) - \frac{1}{8}\pi\mu i (K_1 - iK_2)(1-2\sigma) \sum_{n=1}^{\infty} n (\bar{b}_n g_n - b_n g_{-n}),$$

remembering that $g_0 = 0$. Inserting in this expression the values of K_1 and K_2 given by (3.9), we derive therefrom the values of x_0 and y_0 as the

real parts of the coefficients of F_2 and $-F_1$, that is, as the real parts of

$$(7.6) \quad \pi\mu i E^{-1}(AB-H^2)^{-1} \sum_{n=1}^{\infty} n\bar{b}_n(A\Omega'_{2n}-H\Omega'_{1n}) \\ - \frac{1}{8}\pi\mu E^{-1}(AB-H^2)^{-1}(A-iH)(1-2\sigma) \sum_{n=1}^{\infty} n(\bar{b}_n g_n - b_n g_{-n})$$

and

$$(7.7) \quad -\pi\mu i E^{-1}(AB-H^2)^{-1} \sum_{n=1}^{\infty} n\bar{b}_n(B\Omega'_{1n}-H\Omega'_{2n}) \\ + \frac{1}{8}\pi\mu i E^{-1}(AB-H^2)^{-1}(B-iH)(1-2\sigma) \sum_{n=1}^{\infty} n(\bar{b}_n g_n - b_n g_{-n}),$$

respectively; and our solution is thus formally complete.

If, as frequently happens, the axes used are actually the principal axes of inertia of the curve of section, then $H=0$ and these formulae reduce considerably. We have then

$$EK_1 = F_1/A, \quad EK_2 = F_2/B,$$

while x_0 is the real part of

$$(7.8) \quad \pi\mu i E^{-1} B^{-1} \sum_{n=1}^{\infty} n\bar{b}_n \Omega'_{2n} - \frac{1}{8}\pi\mu E^{-1} B^{-1}(1-2\sigma) \sum_{n=1}^{\infty} n(\bar{b}_n g_n - b_n g_{-n}),$$

and y_0 is the real part of

$$(7.9) \quad -\pi\mu i E^{-1} A^{-1} \sum_{n=1}^{\infty} n\bar{b}_n \Omega'_{1n} + \frac{1}{8}\pi\mu i E^{-1} A^{-1}(1-2\sigma) \sum_{n=1}^{\infty} n(\bar{b}_n g_n - b_n g_{-n}).$$

8. Having thus solved the simple flexure problem, all that now remains to complete the solution of the more general problem is the solution of the torsion problem. The basis of this is the determination of the potential function ϕ_3 which satisfies the boundary condition (3.6). This condition interpreted in the usual way in terms of the conjugate function ψ_3 is simply that

$$(8.1) \quad \psi_3 = \frac{1}{2}(x^2 + y^2) + \text{const.}$$

Pursuing methods exactly analogous to those developed above in §4, and given in detail elsewhere*, we find that ψ_3 can be determined as the

* *loc. cit.*, page 384.

imaginary part of the complex potential function $\Omega_3 = \phi_3 + i\psi_3$, which assumes the form

$$(8.2) \quad \Omega_3 = \frac{1}{2}iz(\zeta)\bar{z}(\zeta) + F_3(\zeta),$$

where, as usual, $F_3(\zeta)$ is a real function of its argument introduced to secure regularity of Ω_3 and its derivatives in the space considered corresponding to $0 < \zeta < 2\pi$, $0 < \eta < \infty$. In the special case which we have considered above, where

$$z(\zeta)\bar{z}(\zeta) = b_0 + \sum_{n=1}^{\infty} (b_n e^{ni\zeta} + \bar{b}_n e^{-ni\zeta})$$

the appropriate form for $F_3(\zeta)$ proves to be

$$-\frac{1}{2}i \sum_{n=1}^{\infty} (\bar{b}_n e^{-ni\zeta} - b_n e^{ni\zeta}),$$

so that

$$(8.3) \quad \Omega_3 = i \sum_{n=0}^{\infty} b_n e^{ni\zeta}.$$

The torsion problem is now completed by determining the relation between the twist τ and the couple Γ . This is derived from

$$\Gamma = \iint (xZ_y - yZ_x) dx dy,$$

in the special case when

$$Z_x = \mu\tau \left(\frac{\partial \phi_3}{\partial x} - y \right), \quad Z_y = \mu\tau \left(\frac{\partial \phi_3}{\partial y} + x \right).$$

Inserting these stress components in the integral and interpreting the result in terms of the complex potential Ω_3 and contour integrals round the boundary of the section, we find that Γ is the real part of the integral

$$(8.4) \quad \frac{1}{4}\tau\mu \int \left(\frac{2d\Omega}{dz} - i\bar{z} \right) z\bar{z} dz,$$

i.e. of the expression

$$(8.5) \quad \frac{1}{2}\pi T\mu \left(c_0 b_0 + \sum_{n=1}^{\infty} (c_n \bar{b}_n + c_{-n} b_n) - 2 \sum_{n=1}^{\infty} n b_n \bar{b}_n \right),$$

wherein, for $n > 0$,

$$c_n = \sum_{r=0}^{\infty} (n+r) a_{n+r} \bar{a}_r, \quad c_{-n} = \sum_{r=0}^{\infty} r \bar{a}_{n+r} a_r.$$

9. We shall illustrate our results by obtaining the form of the solution for one particular section for which the results have only recently been obtained (Shepherd* and Stevenson†). When the cross-section curve is the cardioid whose polar equation is $r = 2c(1 - \cos \theta)$, a suitable transformation is given by

$$z = c(1 - e^{i\zeta})^2 = c(1 - 2e^{i\zeta} + e^{2i\zeta}),$$

so that $a_0 = c$, $a_1 = -2c$, $a_2 = c$, and all the other a 's are zero. The axes are not yet, however, the principal axes at the centroid. Using the relations of § 2 and noticing that with this form

$$b_0 = 6c^2, \quad b_1 = -4c^2, \quad b_2 = c^2; \quad b_n = 0, \quad (n > 2),$$

we have

$$S = 6\pi c^2, \quad Sz_G = 10\pi c^3,$$

so that

$$z_G = 5c/3.$$

When we change the origin to z_G , the equation becomes

$$z = -\frac{2c}{3} - 2ce^{i\zeta} + ce^{2i\zeta},$$

so that now

$$a_0 = -\frac{2c}{3}, \quad a_1 = -2c, \quad a_2 = c, \quad n > 2, \quad a_n = 0;$$

$$b_0 = \frac{49c^2}{9}, \quad b_1 = -\frac{2c^2}{3}, \quad b_2 = -\frac{2c^2}{3}, \quad n > 2, \quad b_n = 0.$$

With these values we have

$$B - A - 2iH = -\pi[\bar{b}_1(a_1 a_0) + \bar{b}_2(2a_2 a_0 + a_1^2)],$$

which is equal to $8\pi c^4/3$.

Also

$$\begin{aligned} B + A &= \frac{1}{2}\pi[b_0(a_1 \bar{a}_1 + 2a_2 \bar{a}_2) + b_1(\bar{a}_2 a_1) + \bar{b}_1(a_1 \bar{a}_0 + 2a_1 \bar{a}_1) + \bar{b}_2(2a_2 \bar{a}_0)] \\ &= 55\pi c^4/3. \end{aligned}$$

It follows that $H = 0$, so that the axes are principal axes, and then

$$A = \frac{47\pi c^4}{6}, \quad B = \frac{21\pi c^4}{2}.$$

* *Proc. Royal Soc., A*, 154 (1936), 500.

† *Loc. cit.*, p. 196.

The additional coefficients in the analysis are

$$f_0 = -\frac{80c^3}{27}, \quad f_1 = -\frac{100c^3}{9}, \quad f_2 = \frac{65c^3}{9}, \quad f_3 = \frac{2c^3}{3}, \quad f_4 = -\frac{2c^3}{3},$$

$$f_{-1} = \frac{16c^3}{3}, \quad f_{-2} = \frac{4c^3}{9}, \quad n > 4, \quad f_n = 0; \quad n > 2, \quad f_{-n} = 0;$$

$$g_0 = 0, \quad g_1 = -\frac{110c^3}{9}, \quad g_2 = \frac{55c^3}{9}, \quad g_3 = 0, \quad g_4 = -\frac{c^3}{3},$$

$$g_{-1} = -\frac{4c^3}{3}; \quad g_n = 0, \quad (n > 4); \quad g_{-n} = 0, \quad (n > 1).$$

The axes being principal, we have seen that x_0 is the real part of (7.8), whilst y_0 is the real part of (7.9).

These formulae give immediately

$$x_0 = \frac{2c(3+4\sigma)}{63(1-\sigma)}, \quad y_0 = 0,$$

and, since $(1-\eta)(1+\sigma) = 1$, these can be expressed in the form

$$x_0 = 2c(3+\eta)/63, \quad y_0 = 0,$$

which is analogous to the result given by Stevenson.

The torsional couple in this case is

$$17\pi\mu\tau c^4,$$

the value found previously.

10. To sum up, the preceding analysis demonstrates the possibility of including explicitly in one general solution all the details of the torsion and flexure problem for any prism belonging to the general and inclusive class which is characterised by the fact that the equation of its cross-section boundary can be thrown into the form (1.1). The curves which are known to fall into this category include, beside the cardioid examined above, also the following:

(i) The circle $z = ae^{i\zeta}.$

(ii) The more general cylinder treated by Wrinch

$$z = nae^{i\zeta} + be^{ni\zeta}.$$

(iii) The lemniscate

$$z = c(1 - e^{i\zeta})^{\frac{1}{2}} = -c \sum_{n=0}^{\infty} \frac{1 \cdot 3 \dots (2n-3)}{2^n n!} e^{ni\zeta}.$$

(iv) The circular sector; in this case the general transformation is

$$z = ae^{\frac{1}{2}i(\alpha+\beta)} [\sqrt{(1 - \tan^2 \frac{1}{2}\zeta)} - i \tan \frac{1}{2}\zeta]^{(\beta-\alpha)/\pi},$$

where its radii are inclined to the x -axis at angles α and β . This is somewhat difficult to express in the required form but the difficulties are analytical rather than fundamental. The transformation in the case of the semi-circular section which is, of course, a particular case of the above, is

$$z = a[\sqrt{(1 - \tan^2 \frac{1}{2}\zeta)} - i \tan \frac{1}{2}\zeta],$$

or

$$z = a[\sqrt{\{2(1 + e^{2i\zeta})\}} + (1 - e^{i\zeta})]/(1 + e^{i\zeta}),$$

which can be expressed in the required series form (1.1) wherein

$$a_0 = a(\sqrt{2} + 1), \quad a_1 = -a(2 + \sqrt{2}), \quad a_2 = a\left(\frac{3\sqrt{2}}{2} + 2\right), \quad a_3 = -a\left(\frac{3\sqrt{2}}{2} + 2\right),$$

$$a_n = -\left\{\frac{3\sqrt{2}}{2} + 2 + \sqrt{2} \sum_{r=2}^{(n-1)/2} (-)^{r+1} \frac{1 \cdot 3 \dots (2r-3)}{2^r r!}\right\}, \quad n \text{ odd},$$

$$a_n = \left\{\frac{3\sqrt{2}}{2} + 2 + \sqrt{2} \sum_{r=2}^{n/2} (-)^{r+1} \frac{1 \cdot 3 \dots (2r-3)}{2^r r!}\right\}, \quad n \text{ even}.$$

(v) The annulus between non-concentric circles

$$z = c \tan \frac{1}{2}\zeta = ci \left\{1 + 2 \sum_{n=1}^{\infty} (-)^n e^{ni\zeta}\right\}.$$

(vi) The polygon; in this case the general transformation is

$$z = A \int_{\zeta}^{\zeta} e^{i\zeta} \prod_{r=1}^n \{1 - e^{i(\zeta - \zeta_r)}\}^{-(\alpha_r/\pi - 1)} d\zeta,$$

where the α_r 's are the external angles of the polygon so that $\sum \alpha_r = (n+2)\pi$ and the ζ_r 's are constants. These can be expanded in terms of $e^{i\zeta}$ so long as $|e^{i\zeta}| < 1$, so that the expansion is generally valid only up to the cylinder and not actually on it. As a matter of fact they are also valid on the cylinder except in the neighbourhood of the corners where the singularities occur. In practice we replace $e^{i\zeta}$ in the integrand by

$$e^{i(\zeta + ia)} = e^{i\zeta - a},$$

obtaining thereby more general transformation formulae which tend to those specified when $\alpha \rightarrow 0$.

Particular cases are given by

(a) The rectangle

$$z = A \int^{\zeta} \{\cos 2(\zeta + ia) - \cos 2\beta\}^{-\frac{1}{2}} d\zeta,$$

$$= e^{i\zeta} \sum_{n=0}^{\infty} a_{2n} e^{2ni\zeta} + B,$$

where B is a constant of integration and

$$a_0 = A \sqrt{2} e^{-\alpha} \quad (n > 0), \quad a_{2n} = \frac{A \sqrt{2} e^{-(2n+1)\alpha}}{(2n+1)} P_n(\cos 2\beta),$$

$P_n(\cos 2\beta)$ denoting Legendre's polynomial of degree n .

(b) The regular n -sided polygon

$$z = A \int^{\zeta} e^{i\zeta} \{1 - e^{ni(\zeta + ia)}\}^{-2/n} d\zeta$$

$$= \sum_{m=0}^{\infty} a_n e^{mi\zeta},$$

where

$$a_0 = B,$$

$$a_1 = -Ai,$$

$$a_2 = a_3 \dots = a_n = 0,$$

$$a_{n+1} = -\frac{2Ai e^{-n\alpha}}{n(n+1)},$$

$$a_{rn+1} = -\frac{Ai 2(n+2) \dots \{(r-1)n+2\}}{n^r r! (rn+1)} e^{-rn\alpha} \quad (r > 1).$$

The equilateral triangle is, of course, the special case of this when $n = 3$.

(vii) Stevenson also has given a solution relating to the half of the lemniscate loop (iii) which lies above the real axis. This is a particular case of the solution relating to either section of the curve

$$z = \sum_{n=1}^{\infty} a_n e^{ni\theta} = \sum_{n=1}^{\infty} |a_n| e^{ni\theta + i\epsilon_n}$$

into which it is divided by the chord joining the points $\theta = 0$ and $\theta = \pi$, when the axes are turned so that this section is the portion of the curve

above the real axis. In fact, the parametric equation of this section boundary, including the chord along the real axis, assumes the form

$$z = \sum a_n' e^{ni\theta},$$

where the coefficients a_n' , determined by Fourier's theorem to make the value of y assume its appropriate values, are given by the relations

$$2\pi a_0' = -\pi i |a_0| \sin \epsilon - 2i \sum_{n=0}^{\infty} \frac{|a_{2n+1}|}{2n+1} \cos \epsilon_{2n+1},$$

$$2\pi a_n' = \pi \bar{a}_n - 2^2 \sum_{m=0}^{\infty} \frac{|a_n|}{n^2 - m^2} [n \sin \epsilon_m \sin^2 \{(n+m)\frac{1}{2}\pi\} - im \cos \epsilon_m \cos^2 \{(n+m)\frac{1}{2}\pi\}],$$

where the term with $m=n$ is omitted from the sum.

There are a few well-known solutions which are not covered by the above analysis for the reason that the prisms to which they relate have cross-sections whose equations cannot be expressed simply in the form (1.1). These outstanding cases which at first seemed to defy all attempts to bring them within the scope of the methods used here, have at last been found to be included in another and important general type of solution, of which details will be given in a later communication.

I have also applied my methods to obtain the solution for another general problem of some interest and practical importance, viz. that relating to the hollow prism whose interior and exterior surfaces are level surfaces in the net defined by the "internal" transformation of type (1.2) or by the corresponding "external" transformation

$$z = e^{-i\zeta} \sum_{n=0}^{\infty} a_n e^{ni\zeta}.$$

Again, however, details are reserved for a future publication.

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THE INDUCTION OF CURRENTS IN INFINITE PLANE CURRENT-SHEETS. I

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[Received 7 December, 1938.—Read 15 December, 1938.]

1. In 1872 Maxwell* obtained an approximate solution of the problem of a magnetic system moving towards an imperfectly conducting infinite plane current-sheet, the motion being supposed to have been going on for an infinite time before the instant considered. An exact solution of this problem is given in this paper in the case when the magnetic system moves with constant velocity; it is shown that Maxwell's solution requires correction in the case when the system moves obliquely to the sheet.

In a later paper an approximate solution will be given of the same problem for the case when the electrons in the sheet are free to move in any manner in the plane of the sheet. This problem is of some interest in the theory of geo-magnetic storms and also in the theory of super-conductivity in metals.

We begin by determining the vector potential for a plane current-sheet.

2. *The vector-potential for an infinite plane current-sheet.*

Let the plane of the sheet be the plane $z = 0$ of a system S of Cartesian coordinates (x, y, z) ; let σ_x and σ_y be the components of the current-density σ at any point in the sheet, parallel to the coordinate axes, of a system of currents flowing in the sheet. The vector potential $A^{(s)}$ of the

* *Proc. Royal Soc.*, 20 (1872), 160, and *Treatise on electricity and magnetism*, 2 (3rd ed., 1892), 286, *et seq.*

electro-magnetic field due to this system of currents at any point $(\mathbf{r}, t)$ is taken to be

$$(1) \quad \mathbf{A}^{(s)} = \iint \frac{[\sigma] dS}{|\mathbf{r} - \mathbf{r}_1|},$$

where $\mathbf{r}_1$ is the position vector of the point in the sheet at which the value of the current-density is σ , and $[\sigma]$ denotes the value of σ when t is replaced by $t - |\mathbf{r} - \mathbf{r}_1|/c$; since $\sigma_z = 0$, $A_z^{(s)}$ vanishes everywhere.

The magnetic field is given by

$$(2) \quad \mathbf{H}^{(s)} = \text{curl } \mathbf{A}^{(s)},$$

i.e.

$$(3) \quad H_x^{(s)} = -\partial A_y^{(s)} / \partial z,$$

$$(4) \quad H_y^{(s)} = \partial A_x^{(s)} / \partial z,$$

$$(5) \quad H_z^{(s)} = (\partial A_y^{(s)} / \partial x) - (\partial A_x^{(s)} / \partial y).$$

Since $|\mathbf{r} - \mathbf{r}_1|$ is an even function of z , $A_x^{(s)}$ and $A_y^{(s)}$ are also even functions of z and hence $H_x^{(s)}$ and $H_y^{(s)}$ are odd functions of z . The magnetic field due to the currents in the sheet on one side of it is therefore the mirror image of that on the other side.

3.1. The equations of the field.

In the regions unoccupied by the sheet and external magnetic systems, the electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ are derived from the scalar and vector potentials ϕ and $\mathbf{A}$ by means of the equations

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi, \quad \mathbf{H} = \text{curl } \mathbf{A};$$

the four-vector $(\mathbf{A}, \phi)$ forms a four-dimensional contravariant vector which satisfies the equations

$$(6) \quad \text{div } \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0,$$

and

$$(7) \quad Df \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0,$$

f being put in turn equal to A_x , A_y , A_z , ϕ .

3.2. *Boundary conditions.*

The boundary conditions have been given by Levi-Civita*, who has shown that the discontinuities in the tangential components of the magnetic field are given by

$$(8) \quad (H_x)_+ - (H_x)_- = 4\pi\sigma_y \quad (z = 0),$$

$$(9) \quad (H_y)_+ - (H_y)_- = -4\pi\sigma_x \quad (z = 0),$$

where H_+ and H_- refer to the values of H on the positive and negative surfaces of the sheet. If the current sheet is a conductor whose resistance per unit area of the sheet is R (e.m.u.), then (8) and (9) become

$$(10) \quad (H_x)_+ - (H_x)_- = (4\pi c/R) E_y \quad (z = 0),$$

$$(11) \quad (H_y)_+ - (H_y)_- = -(4\pi c/R) E_x \quad (z = 0),$$

where E_x and E_y denote the tangential components of the electric field. The normal component of the magnetic field is continuous but, in general, the normal component of the electric field is not.

It will be convenient to denote by $f^{(s)}$ and $f^{(i)}$ respectively the contributions to any field variable f from the sheet and from the inducing systems; then we have

$$(12) \quad \phi = \phi^{(s)} + \phi^{(i)},$$

$$(13) \quad \mathbf{A} = \mathbf{A}^{(s)} + \mathbf{A}^{(i)},$$

and $\phi^{(s)}$, $\phi^{(i)}$, $\mathbf{A}^{(s)}$, $\mathbf{A}^{(i)}$ satisfy the equations (6) and (7) also. The discontinuities in the tangential components of the magnetic field in crossing the sheet are clearly equal to the discontinuities of the tangential components of the field $\mathbf{H}^{(s)}$ due to the current-sheet. Thus

$$(H_x)_+ - (H_x)_- = (H_x^s)_+ - (H_x^s)_-,$$

with a similar equation for the y -component. Also, from what has been said in §2, we must have

$$(H_x^s)_+ = -\partial A_y^s / \partial |z| \quad (z = 0)$$

* *Rend. Accad. Lincei* (5), 11 (1902), 75-82.

and

$$(H_x^s) = \partial A_y^s / \partial |z| \quad (z = 0),$$

with similar relations for $(H_y^s)_+$ and $(H_y^s)_-$.

Hence (10) and (11) may be replaced by the single equation

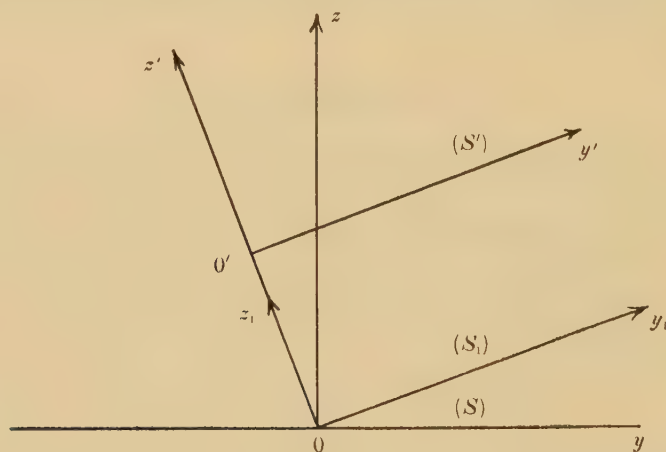
$$(14) \quad \frac{\partial A_a^s}{\partial |z|} = \frac{2\pi}{R} \left(\frac{\partial A_a}{\partial t} + c \frac{\partial \phi}{\partial a} \right) \quad (z = 0),$$

a being put in turn equal to x or y .

3.3. Let the region in which the supposed uniform motion of the magnetic system takes place be that for which z is positive. The direction of motion may, without loss of generality, be taken to be parallel to the (y, z) plane; let the constant velocity be U .

Consider a reference system S' moving with the magnetic system and a reference system S_1 fixed in the sheet, and let the z -axes of the two systems coincide and be parallel to the direction of the relative velocity of the two systems. Let the origins of the system S_1 and of the system S of § 2 also coincide.

The equations connecting the coordinates in the systems S_1 and S are of the form



$$(15) \quad y = ly_1 - mz_1, \quad z = my_1 + lz_1,$$

$$(16) \quad y_1 = ly + mz, \quad z_1 = -my + lz,$$

where $l^2 + m^2 = 1$; we may also suppose without loss of generality that both l and m are non-negative,

The Lorentz transformations connecting the system S_1 and S' are

$$(17) \quad x_1 = x', \quad y_1 = y', \quad z_1 = \kappa(z' - Ut'), \quad t_1 = \kappa(t' - Uz'/c),$$

$$(18) \quad x' = x_1, \quad y' = y_1, \quad z' = \kappa(z_1 + Ut), \quad t' = \kappa(t_1 + Uz_1/c),$$

where $\kappa = (1 - U^2/c^2)^{-\frac{1}{2}}$.

The 4-vector (A_x, A_y, A_z, ϕ) transforms contravariantly, so that we have

$$(19) \quad A_{1x} = A_x', \quad A_{1y} = A_y', \quad A_{1z} = \kappa(A_z' - U\phi'/c), \quad \phi_1 = \kappa(\phi' - UA_z'/c),$$

$$(20) \quad A_x' = A_{1x}, \quad A_y' = A_{1y}, \quad A_z' = \kappa(A_{1z} + U\phi_1/c), \quad \phi' = \kappa(\phi_1 + UA_{1z}/c).$$

4. *The scalar and vector potential of the moving system.*

Since $\phi^{(i)'} = 0$, we deduce from (20) that

$$(21) \quad \phi_1^{(i)} = -UA_{1z}^{(i)}/c,$$

and that

$$(22) \quad A_x^{(i)'} = A_{1x}^{(i)}, \quad A_y^{(i)'} = A_{1y}^{(i)}, \quad A_z^{(i)'} = \kappa^{-1} A_{1z}^{(i)}.$$

The four-vector $(\mathbf{A}, \phi)$, where $\mathbf{A}$ and ϕ are the vector and scalar potentials, is arbitrary to the extent of the four-vector

$$-\partial f/\partial x, \quad -\partial f/\partial y, \quad -\partial f/\partial z, \quad \partial f/\partial t,$$

where f is any function satisfying the wave equation

$$Df = 0.$$

It is, therefore, always possible to choose a four-vector $(\mathbf{A}, \phi)$ in which one of the components is zero.

For instance, if $(\mathbf{A}, \phi)$ is the 4-vector for some electro-magnetic field, then another 4-vector $(\mathbf{A}', \phi')$ can be chosen for it such that $\mathbf{A}'$ possesses a zero α -component, where α is x , y , or z ; in this case we choose f to be the function

$$\int A_\alpha d\alpha.$$

This procedure, in general, introduces singularities into the function $(\mathbf{A}', \phi')$ other than those belonging to the 4-vector $(\mathbf{A}, \phi)$.

Since the z -component of the vector-potential $\mathbf{A}^{(s)}$ is zero, it is natural to choose the vector potential $\mathbf{A}^{(i)}$ so that it also possesses a zero z -component.

It then follows from (16) that

$$(23) \quad A_{1x}^{(i)} = A_x^{(i)}, \quad A_{1y}^{(i)} = lA_y^{(i)}, \quad A_{1z}^{(i)} = -mA_y^{(i)}, \quad \phi_x^{(i)} = mUA_y^{(i)}/c$$

and hence, from (22),

$$(24) \quad A_x^{(i)'} = A_{1x}^{(i)}, \quad A_y^{(i)'} = lA_{1y}^{(i)}, \quad A_z^{(i)'} = -\kappa^{-1}mA_y^{(i)}, \quad \phi^{(i)'} = 0.$$

Since $\text{div } A^{(i)} + (1/c) \partial \phi^{(i)} / \partial t = 0$, we must have

$$\frac{\partial A_x^{(i)}}{\partial x} + \frac{\partial A_y^{(i)}}{\partial y} + \frac{1}{c} \frac{\partial \phi^{(i)}}{\partial t} = 0,$$

i.e.

$$(25) \quad \frac{\partial A_x^{(i)}}{\partial x} + \left(\frac{\partial}{\partial y} + \frac{mU}{c^2} \frac{\partial}{\partial t} \right) A_y^{(i)} = 0.$$

It then follows that there exists a function B (which we suppose to possess continuous second derivatives) of x, y, z, t such that

$$(26) \quad A_x^{(i)} = \frac{\partial B}{\partial y} + \frac{mU}{c^2} \frac{\partial B}{\partial t}; \quad A_y^{(i)} = -\frac{\partial B}{\partial x}.$$

If B' denotes the function B when x, y, z, t are expressed as functions of x', y', z', t' from (15) and (17), we have, from (20) and (16),

$$(27) \quad A_x^{(i)'} = l \frac{\partial B'}{\partial y'} - m\kappa^{-1} \frac{\partial B'}{\partial z'}, \quad A_y^{(i)'} = -l \frac{\partial B'}{\partial x'}, \quad A_z^{(i)'} = +m\kappa^{-1} \frac{\partial B'}{\partial x'},$$

and the condition $\text{div}' A^{(i)'} = 0$ is satisfied if

$$(28) \quad \partial B' / \partial t' = 0,$$

so that B' is independent of t' . Since the components of $A^{(i)'}$ separately satisfy Laplace's equation in the S' -system, it follows from (27) that

$$(29) \quad \nabla'^2 B' = 0.$$

Let Ω be the magnetic potential due to the magnetic system in the system S' ; then the components of (27) are consistent with the equation

$$\text{curl}' A'^{(i)} = -\text{grad}' \Omega$$

if B' is chosen so as to satisfy the equation

$$(30) \quad m\kappa^{-1} \frac{\partial B'}{\partial y'} + l \frac{\partial B'}{\partial z'} = -\Omega.$$

When this equation is solved for B' , (27) gives the components of $A'^{(i)}$ and from (19) we then derive those of $A_1^{(i)}$ and so those of $A^{(i)}$.

5. The differential equations of the problem.

We now substitute for (14) an equation which is valid throughout all space instead of being true only for $z = 0$. The function B' possesses, in addition to the singularities at points occupied by the magnetic system, singularities arising through making $A_z^{(i)}$ vanish everywhere. As will be shown in § 7, these singularities may be restricted to lie in a region ω of space for which $z > 0$. Let $\bar{B}$ denote the function which B becomes when $-|z|$ is substituted for z ; further let

$$(31) \quad \bar{A}_x^{(i)} = \frac{\partial \bar{B}}{\partial y} + \frac{mU}{c^2} \frac{\partial \bar{B}}{\partial t}; \quad \bar{A}_y^{(i)} = -\frac{\partial \bar{B}}{\partial z};$$

and

$$(32) \quad \bar{\phi}^{(i)} = mU \bar{A}_y^{(i)} / c.$$

Then the functions $\bar{B}$ and $\bar{A}^{(i)}$ and $\bar{\phi}^{(i)}$ possess no singularity in any region of space; also, if

$$r^2 = x^2 + y^2 + z^2,$$

then, as $r \rightarrow \infty$, $\bar{A}_a^{(i)} \rightarrow 0$ like $1/r^2$ at least. The function u defined by

$$(33) \quad u = \frac{\partial A_a^{(s)}}{\partial |z|} - \frac{2\pi}{R} \left(\frac{\partial A_a^{(s)}}{\partial t} + \frac{\partial \bar{A}_a^{(i)}}{\partial t} + c \frac{\partial \phi^{(s)}}{\partial a} + c \frac{\partial \bar{\phi}^{(i)}}{\partial a} \right),$$

where $a = x$ or $a = y$, satisfies the wave equation (7), possesses no singularity in the region $z > 0$ and, by (14), vanishes, for all values of t , over the plane $z = 0$; it also vanishes for $t = -\infty$. Since we seek a solution for the potentials (A, ϕ) which vanishes at infinity, let us suppose that u does so by an order higher than $1/r$. It then follows* that u must vanish throughout the region $z \geq 0$, and hence that

$$(34) \quad \frac{\partial A_a^{(s)}}{\partial |z|} = \frac{2\pi}{R} \left(\frac{\partial A_a^{(s)}}{\partial t} + \frac{\partial \bar{A}_a^{(i)}}{\partial t} + c \frac{\partial \phi^{(s)}}{\partial a} + c \frac{\partial \bar{\phi}^{(i)}}{\partial a} \right),$$

where $a = x$ or $a = y$.

From this equation we now derive another giving the scalar potential $\phi^{(s)}$. Differentiating (34) partially with respect to x (or y) when $a = x$ (or y),

* A. E. H. Love, *Proc. London Math. Soc.* (2), 1 (1903), 37, 42.

we obtain the equation

$$(35) \quad \left(\frac{\partial}{\partial |z|} - \frac{2\pi}{R} \frac{\partial}{\partial t} \right) \left(\frac{\partial A_x^{(s)}}{\partial x} + \frac{\partial A_y^{(s)}}{\partial y} \right) - \frac{2\pi}{R} \frac{\partial}{\partial t} \left(\frac{\partial \bar{A}_x^{(i)}}{\partial x} + \frac{\partial \bar{A}_y^{(i)}}{\partial y} \right) + \frac{2\pi c}{R} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\phi^{(s)} + \bar{\phi}^{(i)}).$$

Now $\phi^{(s)}$ and $\phi^{(i)}$ satisfy the wave equation, and $A_\alpha^{(i)}$ satisfies (8); further (27) is still true if $\bar{A}_\alpha^{(i)}$ is substituted for $A_\alpha^{(i)}$. When we use these results, and the fact that $\bar{\phi}^{(i)} = mU\bar{A}_y^{(i)}/c$, (35) becomes

$$(36) \quad -\frac{1}{c} \frac{\partial}{\partial |z|} \frac{\partial \phi^{(s)}}{\partial t} + \frac{2\pi c}{R} \frac{\partial^2 \phi^{(s)}}{\partial |z|^2} + \frac{2\pi mU}{R} \frac{\partial^2 \bar{A}_y^{(i)}}{\partial |z|^2} = 0.$$

Integrating with respect to $|z|$ from ∞ up to $|z|$, we get

$$(37) \quad -\frac{\partial \phi^{(s)}}{c \partial t} + \frac{2\pi c}{R} \frac{\partial \phi^{(s)}}{\partial |z|} + \frac{2\pi mU}{R} \frac{\partial \bar{A}_y^{(i)}}{\partial |z|} = -\frac{1}{c} \frac{\partial \phi^{(s)}}{\partial t} + \frac{2\pi c}{R} \frac{\partial \bar{\phi}}{\partial |z|} = 0,$$

where $\bar{\phi} = \phi^{(s)} + \bar{\phi}^{(i)}$.

The differential equations of the problem thus reduce to three, namely (30), which determines B' and hence $\bar{B}$, and (34) and (37), determining respectively the potentials $A^{(s)}$ and $\phi^{(s)}$.

6. Case of a perfectly conducting sheet.

In this case $R = 0$; and (34) and (37) reduce to

$$(38) \quad \frac{\partial A_\alpha^{(s)}}{\partial t} + \frac{\partial \bar{A}_\alpha^{(i)}}{\partial t} + c \frac{\partial \bar{\phi}}{\partial \alpha} = 0,$$

$$(39) \quad \frac{\partial \bar{\phi}}{\partial |z|} = 0.$$

The only solution of this equation which satisfies the field equations and vanishes at infinity is a constant; we may therefore put

$$(40) \quad \phi^{(s)} = -\bar{\phi}^{(i)},$$

and (38) then reduces to

$$\frac{\partial A_\alpha^{(s)}}{\partial t} + \frac{\partial \bar{A}_\alpha^{(i)}}{\partial t} = 0.$$

Integrating this equation with respect to t , and remembering that $\mathbf{A}^{(s)}$ and $\bar{\mathbf{A}}^{(i)}$ both vanish for $t = -\infty$, we get

$$A_a^{(s)} = -\bar{A}_a^{(i)}.$$

Thus the current-sheet reflects completely the electro-magnetic field of the moving magnetic system.

7. Solution of the equations.

We introduce two new variables η , ζ defined by the equations

$$(41) \quad \eta = \frac{ly' - m\kappa^{-1}z'}{(l^2 + m^2\kappa^{-2})^{\frac{1}{2}}}; \quad \zeta = \frac{m\kappa^{-1}y' + lz'}{(l^2 + m^2\kappa^{-2})^{\frac{1}{2}}}.$$

This amounts to a rotation of the system S' about the x' -axis; since we suppose l and m to be non-negative (§ 3.3), the positive part of the ζ -axis lies in the first quadrant of the system S' . From (27) we then find

$$(42) \quad A_x^{(i)'} = (l^2 + m^2\kappa^{-2}) \frac{\partial B'}{\partial \eta}, \quad A_\eta^{(i)'} = -(l^2 + m^2\kappa^{-2}) \frac{\partial B'}{\partial x'}, \quad A_\zeta^{(i)'} = 0,$$

and (30) becomes

$$(43) \quad (l^2 + m^2\kappa^{-2}) \frac{\partial B'}{\partial \zeta} = -\Omega.$$

Since B' is harmonic, the general solution of this equation is

$$(44) \quad (l^2 + m^2\kappa^{-2})B' = -\int \Omega d\zeta + F(x', \eta),$$

where $F(x', \eta)$ is any solution of the equation

$$(45) \quad \frac{\partial^2 F}{\partial x'^2} + \frac{\partial^2 F}{\partial \eta^2} = 0.$$

The values of the components of the magnetic field $\mathbf{H}^{(i)'}$ are derived from the equation

$$(46) \quad \mathbf{H}^{(i)'} = \text{curl}_{x', \eta, \zeta} \mathbf{A}^{(i)'},$$

and it is easily verified from (42) and (46) that the terms containing the function F on the right-hand side of this equation vanish identically. We may, therefore, put $F(x', \eta)$ equal to zero without loss of generality.

It is sufficient to derive the solution for a single magnetic pole, since the solution for any other system can be deduced therefrom. We take the pole strength to be unity. Then

$$(47) \quad \Omega = 1/r',$$

where

$$(48) \quad r'^2 = x'^2 + y'^2 + z'^2 = x'^2 + \eta^2 + \zeta^2.$$

Putting $F(x', \eta)$ equal to zero in (44) we have

$$(49) \quad (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} B' = \log \{-\zeta + \sqrt{(x'^2 + \eta^2 + \zeta^2)}\};$$

this function possesses the line singularity*

$$(50) \quad x = \eta = 0, \quad \zeta \geq 0,$$

which lies entirely in the region defined by $z > 0$.

Let the coordinates of the pole in the system S_1 at time $t = 0$ be $(0, 0, z_0)$, and let $\bar{B}$ be the function derived from B' by expressing η and ζ in terms of y, z and t and then replacing z by $-|z|$; the latter procedure amounts to substituting for the coordinates of any point in the S -system those of its mirror image in the sheet. If we now write

$$(51) \quad \alpha = (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} x,$$

$$(52) \quad \beta = y - mz_0 - mUt,$$

$$(53) \quad \gamma = \kappa[(lmU^2/c^2)(y - mz_0) + (l^2 + m^2 \kappa^{-2})(|z| + lz_0) - lUt],$$

we find that $\bar{B}$ is given by

$$(54) \quad (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} \bar{B} = \log \{\gamma + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)}\},$$

a function possessing no singularities in the region $z \geq 0$, since the mirror image of the line singularity (50) lies in the region $z < 0$.

From (31) and (32) we then find

$$(55) \quad \bar{A}_x^{(i)} = (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} \frac{\partial}{\partial \beta} \log \{\gamma + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)}\},$$

$$(56) \quad \bar{A}_y^{(i)} = -\frac{\partial}{\partial \alpha} \log \{\gamma + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)}\},$$

$$(57) \quad \bar{\phi}^{(i)} = -\frac{mU}{c} \frac{\partial}{\partial \alpha} \log \{\gamma + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)}\}.$$

* The electric and magnetic fields in the system S' do not, of course, possess this line singularity but only the singularity at the pole: this can easily be verified by considering (46) and (42).

7.1. Determination of $\phi^{(s)}$.

The equation for $\phi^{(s)}$ is

$$\frac{\partial \phi^{(s)}}{\partial |z|} - \frac{R}{2\pi c^2} \frac{\partial \phi^{(s)}}{\partial t} = -\frac{mU}{c} \frac{\partial \bar{A}_y^{(i)}}{\partial |z|},$$

i.e., since $\partial/\partial |z| = \kappa(l^2 + m^2 \kappa^{-2})(\partial/\partial \gamma)$,

$$(58) \quad \frac{\partial \phi^{(s)}}{\partial |z|} - \frac{R}{2\pi c^2} \frac{\partial \phi^{(s)}}{\partial t} = \frac{\kappa m U}{c} (l^2 + m^2 \kappa^{-2}) \frac{\partial}{\partial \alpha} (\alpha^2 + \beta^2 + \gamma^2)^{-\frac{1}{2}}.$$

Let β_τ and γ_τ denote the functions obtained by replacing $|z|$ by $|z| + \tau$ and t by $t - R\tau/2\pi c^2$ in β and γ respectively; then a solution of (58) is

$$(59) \quad \phi^{(s)} = -(\kappa m U/c)(l^2 + m^2 \kappa^{-2}) \int_0^\infty \frac{\partial}{\partial \alpha} (\alpha^2 + \beta_\tau^2 + \gamma_\tau^2)^{-\frac{1}{2}} d\tau.$$

The general solution differs from this only by the solution of the equation

$$\frac{\partial \phi^{(s)}}{\partial |z|} = \frac{R}{2\pi c^2} \frac{\partial \phi^{(s)}}{\partial t}$$

which satisfies also the wave equation $Df = 0$ and vanishes for $|z| = \infty$. Since such a solution is identically zero, (59) is the complete solution.

7.2. Determination of $A_x^{(s)}$ and $A_y^{(s)}$.

The equation for $A_y^{(s)}$ is

$$(60) \quad \frac{\partial A_y^{(s)}}{\partial t} - \frac{R}{2\pi} \frac{\partial A_y^{(s)}}{\partial |z|} = -\left(\frac{\partial \bar{A}_y^{(i)}}{\partial t} + c \frac{\partial \bar{\phi}^{(i)}}{\partial y} + c \frac{\partial \phi^{(s)}}{\partial y} \right).$$

If we use (55) and (57) and the equations of transformation (50)–(52), a solution of (60) is

$$(61) \quad A_y^{(s)} = \kappa l U (l^2 + m^2 \kappa^{-2}) \int_0^\infty \frac{\partial}{\partial \alpha} (\alpha^2 + \beta_\sigma^2 + \gamma_\sigma^2)^{-\frac{1}{2}} d\sigma \\ + \kappa m U (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} \int_0^\infty d\sigma \int_0^\infty \frac{\partial^2}{\partial \alpha \partial \beta_{\tau\sigma}} (\alpha^2 + \beta_{\tau\sigma}^2 + \gamma_{\tau\sigma}^2) d\tau,$$

where β_σ and γ_σ are the functions obtained by replacing $|z|$ by $|z| + R\sigma/2\pi$ and t by $t - \sigma$, in β and γ respectively, and $\beta_{\tau\sigma}$, $\gamma_{\tau\sigma}$ the functions obtained from β_τ and γ_τ in the same way. An argument similar to the one used in § 7.1 shows that (61) is the required solution.

We find likewise

$$(62) \quad A_x^{(s)} = -U(l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} \int_0^\infty \left(\kappa l \frac{\partial}{\partial \beta_\sigma} + m \frac{\partial}{\partial \gamma_\sigma} \right) (a^2 + \beta_\sigma^2 + \gamma_\sigma^2)^{-\frac{1}{2}} d\sigma \\ + \kappa m U (l^2 + m^2 \kappa^{-2})^{\frac{1}{2}} \int_0^\infty d\sigma \int_0^\infty \frac{\partial^2}{\partial \alpha^2} (a^2 + \beta_{\tau\sigma}^2 + \gamma_{\tau\sigma}^2)^{-\frac{1}{2}} d\tau.$$

8. Special case when $l = 1$, $m = 0$.

In this case the magnetic pole moves perpendicularly to the sheet; it follows from (59) that $\phi^{(s)} = 0$, so that there is no electric polarization of the sheet, and the magnetic vector potential is given by

$$(63) \quad \mathbf{A}^{(s)} = -\frac{U}{U + (2\pi)^{-1}R} \bar{\mathbf{A}}^{(i)};$$

this differs from Maxwell's result only in that the Lorentz transformation, and not the Newtonian one, is used in passing from the moving system to the system fixed in the sheet.

As in the case of a perfectly conducting sheet, the electro-magnetic field is reflected by the sheet, but in this case the reflected field is reduced in the ratio $U : (U + R/2\pi)$.

The magnetic field is given by

$$(64) \quad H_x^{(s)} = \pm \frac{1}{\kappa^2} \frac{U}{U + (2\pi)^{-1}R} \frac{x}{\tilde{r}^3},$$

$$(65) \quad H_y^{(s)} = \pm \frac{1}{\kappa^2} \frac{U}{U + (2\pi)^{-1}R} \frac{y}{\tilde{r}^3},$$

$$(66) \quad H_z^{(s)} = \pm \frac{1}{\kappa^2} \frac{U}{U + R(2\pi)^{-1}} \frac{|z| + z_0 - Ut}{\tilde{r}^3},$$

where

$$\tilde{r}^2 = \kappa^{-2}(x^2 + y^2) + (|z| + z_0 - Ut)^2,$$

and the $+$ or $-$ sign is taken according as z is positive or negative.

If $c \rightarrow \infty$, $\tilde{r}^2 \rightarrow x^2 + y^2 + (|z| + z_0 - Ut)^2$ and $\kappa \rightarrow 1$; the result then becomes identical with the approximate one given by Maxwell.

If $U \rightarrow c$, $\kappa^{-1} \rightarrow 0$, $\tilde{r} \rightarrow |z| + z_0 - Ut$ and the induced field, as well as the inducing field, becomes vanishingly small.

8.1. The case considered in § 8 is unique in that there is no electric polarization; the reason is that there the electro-motive force is proportional to $\mathbf{v} \wedge \mathbf{H}$ (where $\mathbf{v}$ denotes the velocity of the pole) and, since $\mathbf{v}$ is perpendicular to the plane of the sheet, this force is directed along the plane

of the sheet. In every other case this force possesses a component normal to the sheet whereby this will be polarized. For a perfectly conducting sheet the normal component is balanced by the normal component of the polarization electric field. Except in the case of perfect conduction or in the case considered in § 8, the induced field cannot be represented by a single image.

9. Conclusion.

In the case in which the motion of the magnetic system is not uniform, it can be shown that equations (34) and (37), viz.

$$\frac{\partial A_a^{(s)}}{\partial |z|} = \frac{2\pi}{R} \left(\frac{\partial A_a^{(s)}}{\partial t} + \frac{\partial \bar{A}_a^{(i)}}{\partial t} + c \frac{\partial \phi^{(s)}}{\partial a} + c \frac{\partial \bar{\phi}^{(i)}}{\partial a} \right),$$

and

$$\frac{\partial \phi^{(s)}}{\partial t} = \frac{2\pi c^2}{R} \left(\frac{\partial \phi^{(s)}}{\partial |z|} + \frac{\partial \bar{\phi}^{(i)}}{\partial |z|} \right),$$

are still valid, except that, of course, the scalar and vector potentials are no longer transformed according to the Lorentz transformation. In Maxwell's approximate theory only the first of these equations, with $\phi^{(s)}$ and $\bar{\phi}^{(i)}$ absent, is given, so that Maxwell's theory fails to give the polarization of the sheet.

The equations, however, must be replaced by others when the sheet ceases to conduct electric currents in accordance with Ohm's law

$$\sigma = (c/R) \mathbf{E}.$$

Some other relation must be found between the current density σ and the electro-magnetic field vectors. If the electrons in the sheet are free to move in any manner, this relation is contained in the equation of motion of an electron, viz.

$$m(d\mathbf{v}/dt) = -e(\mathbf{E} + \mathbf{v} \wedge \mathbf{H}/c),$$

where $-e$ (e.s.u.) and m denote the charge and mass of the electron. It has been stated, and it seems incorrectly, by F. and H. London* that the relation should be

$$\frac{d\sigma}{dt} = \frac{ne^2}{m} \mathbf{E},$$

* *Proc. Royal Soc. (A)*, 149 (1935), 71.

n being the electron number density; this seems to neglect the action of the magnetic field on the moving electrons. In the case in which the field is initially zero in the sheet, it can easily be shown that the relation required is

$$\sigma = -\frac{ne^2}{mc} \mathbf{A},$$

or the equivalent relation

$$\text{curl } \sigma = -\frac{ne^2}{mc} \mathbf{H}$$

adopted in the theory of super-conductors. This question will be considered more fully in a later paper.

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RELATIVISTIC MECHANICS

I. *The world-model, and the description of physical objects**By* A. G. WALKER.

[Received 15 November, 1938.—Read 17 November, 1938.]

Introduction.

1. Before turning to the scheme of mechanics considered in the present paper, it may be of some interest to examine briefly two other schemes, classical (Newtonian) mechanics and the mechanics of the general theory of relativity.

The scheme proposed by Newton consisted of concepts, definitions, and laws, by means of which it is possible to describe approximately what appears to be the behaviour of a real material system. This was, however, replaced later by a structure of concepts, definitions, and postulates, giving rules governing the behaviour of isolated models. Contact with reality is made through experiment, where the observed behaviour of a material system as nearly isolated as possible is compared with that of a model made to resemble the system. This structure is non-relativistic in that the quantities involved are supposed measurable by any observer whatever his position and motion. It thus gives rise to a number of *physical objects*, objects scalar, vector, or tensor in character, which cannot be "transformed away" by changing the observer or the frame of reference. Such objects are the time and distance intervals between two events, velocity, acceleration, mass, momentum, force, angular momentum and force, work, energy, etc., and these are all described in terms of Newtonian time, kept by gravitational clocks, and of Euclidean space.

In the General Theory of Relativity we again have definitions and postulates giving a set of rules governing the behaviour of isolated models. Each model is associated with a Riemannian 4-space, with a (1, 1) corre-

spondence between space-time events in the model and points in the space, and it is assumed that there may be an observer moving in any way at each event. The rules then relate observations made by each observer and geometrical quantities derived from the space. We now have a relativistic mechanics in which many of the classical objects, such as time and distance intervals, velocity, force, etc., are no longer strictly physical objects owing to the abundance of observers, although each observer can still measure quantities corresponding to the classical objects. The few remaining objects, such as the momentum vector and the energy tensor, are identified with geometrical objects given by the space alone, such as the tangent to a curve and the Riemannian curvature tensor.

The great disadvantage of general relativity, apart from whether the model gives a good description of reality or not, is that the form of the space depends upon the material system to be considered, and the problem of finding this form, and hence of discussing the behaviour of the model, is too complicated for any but the most simple systems. We no longer have a set of rules which can handle such complex systems as it was possible to consider with classical mechanics, so that the advantages which may be derived from closer similarity between model and reality are in most cases nullified.

Turning now to the point of view expressed by Milne and adopted in the present paper, we propose a set of principles and definitions giving a scheme of mechanics which closely resembles the classical scheme but which is also similar to general relativity in that objects are described in relation to a Riemannian space. In the first place we have Mach's Principle, which states that a system cannot be isolated from the rest of the universe. It follows that the model of a particular system can exist only as part of a larger model representing the whole universe, and so the problem of world structure comes first. This problem was discussed by Milne [4][†], who constructed a world-model from a system of fundamental particles. All observations are made by observers having temporal experience who are attached to these particles, and the system is constructed to satisfy the Cosmological Principle and the Principle of Symmetry in relation to the observers. Milne also proposed an important postulate which states that no observer is aware of any constant having physical dimensions. This postulate is severely restrictive, and is violated in general relativity, where there occurs a function $R(t)$ of time t , implying the existence of an absolute unit of time.

[†] Numerals in square brackets refer to the list of references at the end of the present paper.

The problem of world structure in accordance with the principles adopted by Milne was considered again by the author [10], and results more general than, but including, those given by Milne were derived by the application of group properties.

In relation to the world-model we can now construct another scheme of mechanics by means of further definitions, in accordance with the principles and postulates already stated. Since in this work it is sufficient to take into account only the fundamental observers, we need not consider all possible observers as in general relativity, and so we find that the physical objects in classical mechanics reappear and exist for all fundamental observers and in all frames of reference. In a series of papers [5, 6, 7] Milne considered possible definitions of these objects, chosen to agree as closely as possible with the classical forms, and described them in terms of particular coordinates and with reference to the flat 4-space map adopted in [4]. It has already [10] been shown, however, that there are other Riemannian spaces and coordinate systems giving convenient maps for the world-model, and the problem that arises is to define and describe physical objects in some invariant form which can at once be applied to any desired map. Further, it has been shown that there are other world-models, differing from that considered by Milne, which are consistent with the adopted principles, and our scheme of mechanics should also apply to these models. We shall, therefore, formulate definitions and obtain results from first principles, and it will be found later that certain results so obtained agree with those derived by Milne. It should be noted that although we refer to a Riemannian space, this space describes only the world-model, and is not, as in general relativity, deformed so that its intrinsic properties describe systems within the world-model.

The first part of the present paper is concerned with certain Riemannian spaces which serve as maps for the world-model, and with the way in which these spaces display the relationships between the different fundamental observers. It is then shown how physical objects may be described by tensors in each map. Several results found in a previous paper [10] are quoted here, and others found there occur again as examples of general theorems. Many of the equations connected with continuous groups of transformations given before are again required, and these are stated without detailed explanations.

In Part II the objects occurring in classical mechanics are defined and discussed. These objects and the equations connecting them are at first described in maps of three, and not four, dimensions, the procedure thus differing in this respect from that followed by Milne. This description is found to agree more closely with the classical form, Euclidean 3-space

being replaced by a Riemannian 3-space of constant curvature, and although this work is relativistic in character, it is not found necessary to refer to 4-space maps until a later stage. It is then seen to be a simple matter to pass from the representation in 3-space to that in 4-space.

The World-Model.

2. It will be remembered that the material system considered as a world-model consists of fundamental particles one of which occupies each event in space-time. Observers are supposed attached to these particles, observations are made only by these observers, and the system of particles is constructed so that, in distribution and motion, the system as seen by the observers satisfies the Cosmological Principle and the Principle of Symmetry. The general structure of such a system has already been discussed [10], and it was found that it can be conveniently described in terms of coordinates[†] t, z^i on a map consisting of a Riemannian 4-space S_4 , with metric

$$ds^2 = dt^2 - t^2 k_{ij} dz^i dz^j. \quad (1)$$

The coefficients k_{ij} do not involve t , and the 3-space S_3 with positive definite metric

$$de^2 = k_{ij} dz^i dz^j \quad (2)$$

is a space of constant curvature K , a dimensionless number having the value -1 in the system considered by Milne. The properties of the map are as follows.

(i) The curves $z^i = \text{constant}$ are the world lines of the fundamental particles.

(ii) For any value of t , the number of fundamental particles within the range $(z, z+dz)$ is

$$dN = B |k|^{\frac{1}{2}} dz^1 dz^2 dz^3, \quad (3)$$

where $|k|$ is $|k_{ij}|$ evaluated at (z) and B is a dimensionless number.

[†] All suffixes take the values 1, 2, 3, and summation over this range is implied when a suffix is repeated, unless otherwise stated. The notation used in the present paper differs from that given in [10], but is fully explained as we proceed.

(iii) The null geodesics in S_4 are the light paths.

(iv) There is one event in the history of each particle which lies on a surface $t = \text{constant}$, and at these events, for the same value of t , the observers see similar world-pictures.

If, therefore, the coordinate t at an event E is the time recorded by the observer at E , then the clocks of all the observers are similar and synchronised in the sense defined by Milne. This time t is natural kinematic time, and has been identified with the time kept by atomic, as distinct from gravitational, clocks. It is "natural" time in that, starting with quite arbitrary clocks, the observers can[†], from observations on their own and on each others clocks, regraduate them to read time t without being aware of any constant having the dimension of time, neglecting trivial changes of scale. It will be seen that t is measured from the event when all the fundamental particles were together, which event is thus given by $t = 0$ in the history of each particle. It must be remembered that we are here concerned only with the times assigned by each observer to events in his own history and not to the times assigned to distant events. The clocks recording kinematic time t are similar to those used by the observers in Milne's system, but we shall use different conventions for discussing distant events, and this difference must be appreciated to avoid confusion. The distinction is that whereas the clock-reading t is here the same as Milne's clock-reading t , the coordinate t used in this paper is not Milne's coordinate t but corresponds, as will be seen in §7, to Milne's invariant $X^{\frac{1}{2}} = (t^2 - \mathbf{P}^2/c^2)^{\frac{1}{2}}$.

(v) Angles are measured by the tensor k_{ij} , so that the angle at an event (t, z) subtended by two neighbouring particles whose world-lines are given by $z^i + d_1 z^i$, $z^i + d_2 z^i$ is θ , where

$$\cos \theta = \frac{k_{ij}(z) d_1^i d_2^j}{(k_{ij} d_1^i d_1^j)^{\frac{1}{2}} (k_{ij} d_2^i d_2^j)^{\frac{1}{2}}}.$$

From the above properties we see that a model is specified by two pure constants K , B . The constant K is significant in that it distinguishes between totally different models and not merely between different descriptions of the same model. It is reasonable to expect that it will be possible to determine a value of K to the exclusion of all others as a consequence of

[†] The problem of regraduation, and the derivation of kinematic time, is fully discussed in [8].

some postulate. It is also possible to find the value of K giving the best model by correlating in theory such astronomical observables as red-shift, distance from apparent brightness or size, and number density of distribution of the fundamental particles, and comparing with observations, the fundamental particles being regarded as the extra-galactic nebulae.

The constant B governs the distribution but, unlike K , it is not an absolute constant since the size of the particles is not specified. If m_0 is taken to be the mass of each particle, then the quantity which enters into calculations is $m_0 B$, and the value assigned to this quantity may be regarded as fixing the unit of mass. An appeal to observation is necessary if we want the unit of mass given in this way to agree with that used in physics, and a value is eventually given to B by taking m_0 to be the average mass of a nebula. The important point is that a different value of B does not give an essentially different material system, since there is no absolute unit of mass. The new system would appear to us more or less dense but its kinematic and dynamic properties would be unaltered. For this reason B does not enter into any of the expressions and equations in the scheme of mechanics.

(vi) The material system considered above, with the distribution (3), is called the *sub-stratum*, and its "gravitational field" is explored by considering the motions of free test-particles in accordance with the Cosmological Principle. The equations of motion of such a free particle have been found [4, 11] to be

$$\frac{d^2 z^i}{dt^2} + \Gamma_{jk}^i \frac{dz^j}{dt} \frac{dz^k}{dt} + \frac{1}{t} \frac{dz^i}{dt} = 0, \quad (4)$$

where Γ_{jk}^i are the Christoffel symbols calculated from (2). The paths in S_4 given by these equations are called the *free-paths*.

(vii) It has been shown [4, 10] that a model having the above properties can be formed by a statistical system of particles, in which certain singularities in the distribution correspond to the fundamental particles in the sub-stratum. This system satisfies the Cosmological Principle, and the statistical distribution is given by

$$dN = \bar{\psi}(\xi) t^3 |k| dz^1 dz^2 dz^3 d\dot{z}^1 d\dot{z}^2 d\dot{z}^3, \quad (5)$$

where $\dot{z}^i = dz^i/dt$ and $\xi = (dt/ds)^2$ from (1). The equations of motion of

a member of this system are

$$\frac{d^2 z^i}{dt^2} + \Gamma_{jk}^i \frac{dz^j}{dt} \frac{dz^k}{dt} = \frac{1}{t} \bar{G}(\xi) \frac{dz^i}{dt}, \quad (6)$$

where

$$\bar{G}(\xi) = -1 + \left(\frac{\xi}{\xi-1} \right)^{\frac{3}{2}} \frac{C}{\bar{\psi}(\xi)}. \quad (7)$$

The functions $\bar{\psi}(\xi)$, $\bar{G}(\xi)$ are related to Milne's functions $\psi(\xi)$, $G(\xi)$ by

$$\bar{\psi}(\xi) = \xi^{\frac{3}{2}} \psi(\xi), \quad \bar{G}(\xi) = -1 - \xi^{-1} (1 + G(\xi)), \quad (8)$$

and the constant C is the same as that given by Milne, who has discussed these quantities and their significance.

3. We return to the question of clocks; any regraduation, except a change of scale, must involve a constant having the dimension of time. The only such constant of which we are aware is the present time t_0 , and the only significant regraduation will give clocks measuring time T , where

$$t = t_0 \Psi(T/t_0), \quad \Psi(1) = \Psi'(1) = 1, \quad (9)$$

Ψ being a monotonic increasing function. The initial conditions are imposed in order that $T = t_0$ and $dT = (dt)_0$ when $t = t_0$, so that the new clock will agree and run with the old for a short interval at the present time. One such regraduation is given by

$$t = t_0 e^{-1+\tau/t_0}, \quad (10)$$

τ here being the *dynamic time* discussed by Milne and identified with gravitational time for reasons seen later. Substituting in (1) and neglecting the factor $(t/t_0)^2$, we have as our map the space Σ_4 with metric

$$d\sigma^2 = d\tau^2 - t_0^2 de^2. \quad (11)$$

The factor $(t/t_0)^2$ can be conveniently neglected since a space conformal to S_4 will serve equally well as a map, the properties (i)-(vii), and particularly (iii), being unaltered in the transition.

In (1) and (11) we see that the element of time measured by an observer is the element of arc along his world line in the map corresponding to whatever clock is being used.

4. We have so far been concerned only with each observer's time measurements at himself, but we can now discuss observations on distant

events. When the observers keep kinematic time the map is S_4 with metric (1) and the coordinates t, z^i may be regarded as the coordinates assigned to the event $E(t, z)$ by the observer A_0 at the "origin" $z^i = 0$. These coordinates may be defined by A_0 in terms of his own clock readings by means of light signals as follows.

In the space S_3 , metric de^2 , let O be the origin $z^i = 0$, P the point (z) , and let $e_P = \int_0^P de$ taken along the geodesic from O to P . Then, if a light signal from A_0 to E leaves A_0 at time t_1 , and if another signal leaves E and reaches A_0 at time t_2 , we have, since $ds = 0$ along each light path and since from (1) a null geodesic in S_4 corresponds to a geodesic in S_3 ,

$$\int_{t_1}^t \frac{dt}{t} = \int_t^{t_2} \frac{dt}{t} = \int_0^P de,$$

whence

$$t = (t_1 t_2)^{\frac{1}{2}}, \quad e_P = \frac{1}{2} \log (t_2/t_1). \quad (12)$$

The first formula gives the law that A_0 must adopt for assigning coordinate t to the event E in terms of his own recordings t_1, t_2 , and the second gives the geodesic distance from O to P in the space S_3 , where P in S_3 corresponds to E in S_4 . The actual determination of the coordinates z^i depends upon the form of the metric (2), which can vary subject to the conditions that it shall be positive definite and of constant curvature K , and upon the orientation of "axes" adopted by A_0 which again can vary owing to the symmetry of S_3 about O . When these variable factors have been fixed, the coordinates z^i of P , and hence of E , are determined by angles and by the geodesic distance e_P , whence they are expressible in terms of t_1, t_2 , and angle measurements made by A_0 .

With dynamic time τ , for which the map is (11), it is seen that the relations corresponding to (12) are

$$\tau = \frac{1}{2}(\tau_2 + \tau_1), \quad e_P = \frac{1}{2t_0} (\tau_2 - \tau_1). \quad (13)$$

5. In the discussion of observations on distant events Milne defined what may be called the "distance" of such an event, measured by an observer with a clock recording time T , as being $\frac{1}{2}c(T_2 - T_1)$, where T_1, T_2 are the times of emission and reception of a light signal reflected at the event and c is a constant chosen to be the same by all observers. There are other definitions of distance, as for example from apparent brightness or size, but in this work it is necessary, so far as actual distance is concerned, to consider only distances of events near an observer. For such an event

it is an experimental fact, given by the "constancy of the velocity of light", that the (small) distance as defined by Milne agrees with distance as measured by "rigid scales", c then being the velocity of light in the units adopted. Thus small distances from an observer may be regarded as immediately measurable by him.

If E is an event near an observer A at time t and if de is the geodesic distance in S_3 from A to E , then from (12), remembering that de is small and that t_1, t_2 are very nearly equal to t , we find that $\frac{1}{2}(t_2 - t_1) = t de$. Hence the distance of E measured by A is

$$dl = ct de. \quad (14)$$

Similarly we see at once from (13) that when dynamic time τ is used the distance is

$$dl' = ct_0 de. \quad (15)$$

Comparing these two measures, we have $dl/dl' = t/t_0$, showing that they agree at and only at the present time, and that a scale which is rigid with reference to one clock varies with reference to the other.

For a fundamental particle near A , de is constant and the kinematic distance is proportional to t . The particle is thus receding with velocity $c de$ and this velocity varies with different particles, the relation between velocity v and distance l at time t being $v/l = 1/t$. From (15) however we see that dynamic distance is constant, so that neighbouring particles are relatively stationary.

From (1) and (14) and again from (11) and (15) we see that, whichever clock is used, the element of distance is given in the appropriate map by c times the interval to the event from the observer's world-line, in a direction orthogonal to the world-line. This property of the map, together with (i), (iii), and (v) above, and the property that time is measured by interval along a world-line, were also possessed in general relativity by the Lemaitre space-time, which is similar in form to (1). It follows that the discussion [2] of observables in general relativity will also apply to the models considered in the present paper.

6. We now consider a particular coordinate system z^i found to be useful in describing S_3 . It is known [1, § 27] that a space of constant curvature K is conformal to flat space and that the general form exhibiting this property is

$$de^2 = \frac{(dz^1)^2 + (dz^2)^2 + (dz^3)^2}{(a + 2\beta_i z^i + \gamma |z|^2)^2},$$

where

$$|z|^2 = \Sigma (z^i)^2, \quad a\gamma - \Sigma (\beta_i)^2 = \frac{1}{4}K.$$

If the constants β_i are not all zero, the coordinates giving this form depend upon a preferential direction at $O(z^i = 0)$, and if therefore we wish the coordinates to be symmetric about O , they must be chosen so that $\beta_i = 0$. A simple transformation now gives $\alpha = 1$, and the metric takes the well-known form

$$de^2 = \frac{(dz^1)^2 + (dz^2)^2 + (dz^3)^2}{(1 + \frac{1}{4}K|z|^2)^2}, \quad |z|^2 = \Sigma(z^i)^2. \quad (16)$$

This choice of coordinates will be seen to have many advantages. In the first place there is geometrical simplicity, since the coordinates are symmetric about O , and the parametric directions are orthogonal at each point. Further, from the property (v) and from (14) and (15), when $|z|$ is small the Cartesian coordinates referring to the observer at O are ctz^i at kinematic time t , and are ct_0z^i at all times when dynamic time is used. Later it will be seen that at *any* distance the vector playing the part of the position vector in taking moments of momentum and force about O has components ctz^i for kinematic time and ct_0z^i for dynamic time.

With dynamic time it is found convenient to refer to coordinates ζ^i , where

$$\zeta^i = ct_0z^i, \quad (17)$$

so that, from the above remarks, the ζ 's, when small, are the Cartesian coordinates for the observer at O . From (14) and (16) we see that the z 's have zero physical dimensions and the ζ 's have the dimensions of length. With these coordinates it is better to take as a three-dimensional map the space Σ_3 with metric $d\epsilon^2$, where

$$d\epsilon = ct_0de, \quad (18)$$

so that, from (16),

$$d\epsilon^2 = \frac{(d\zeta^1)^2 + (d\zeta^2)^2 + (d\zeta^3)^2}{\left(1 + K|\zeta|^2/(4c^2t_0^2)\right)^2}, \quad [|\zeta|^2 = \Sigma(\zeta^i)^2]. \quad (19)$$

From (11) the space Σ_4 now has metric

$$d\sigma^2 = d\tau^2 - \frac{1}{c^2}d\epsilon^2. \quad (20)$$

7. The above coordinates t, z^i differ from the coordinates t', x, y, z referred to in Milne's work, where an observer recording time t assigns to an event

E an epoch t' and a distance $|P|$ given by

$$t' = \frac{1}{2}(t_2 + t_1), \quad |P| = \frac{1}{2}c(t_2 - t_1),$$

t_1, t_2 being the times of emission and reception of a light signal reflected at E . If the event is (t, z) and if P is the point (z) in S_3 , it follows from (12) that the quantities $t', |P|$ measured by the observer A_0 at the origin O are given by

$$t' = t \cosh e_P, \quad |P| = ct \sinh e_P.$$

Angles are measured in the same way in both systems, whence, if the observer A_0 refers to the same orthogonal axes in measuring $\mathbf{P} = (x, y, z)$ and $\mathbf{z} = (z', z'', z''')$ in (16), then

$$t' = t \cosh e_P, \quad \mathbf{P} = ct \sinh e_P \frac{\mathbf{z}}{|z|}, \quad e_P = \int_0^{|z|} \frac{d\lambda}{1 + \frac{1}{4}K\lambda^2}.$$

Substituting in (1) and (16), we get the metric of the space-time map used by A_0 referring to coordinates $t', \mathbf{P}$.

To get Milne's form we must take $K = -1$, whence

$$t' = t \frac{1 + \frac{1}{4}|z|^2}{1 - \frac{1}{4}|z|^2}, \quad \mathbf{P} = \frac{ct \mathbf{z}}{1 - \frac{1}{4}|z|^2}, \quad (21)$$

and substituting in (1) and (16) we find

$$ds^2 = dt'^2 - \frac{1}{c^2} d\mathbf{P}^2. \quad (22)$$

A fundamental world-line is given by $\mathbf{z} = \text{constant}$, so that in Milne's map it is given by $\mathbf{P} = t' \mathbf{V}$, where

$$\mathbf{V} = \left(\frac{d\mathbf{P}}{dt'} \right)_z = \frac{c\mathbf{z}}{1 + \frac{1}{4}|z|^2}. \quad (23)$$

The fundamental particles are thus receding with constant velocities, and each particle corresponds to a velocity $\mathbf{V}$ which is zero for A_0 .

8. The derivation of the metric (1) followed in [10] a discussion of certain transformations which are of great importance in this work since they relate observations made by different fundamental observers. These transformations arise as follows.

We return to a general coordinate system z^i in S_3 , and regard t, z^i as the coordinate system adopted by the observer A_0 at the origin $z^i = 0$;

then each observer A can adopt a similar system for his own description of events, so that, if there is an event (t, z) relative to A_0 , then there is an event (t, z) relative to A . If now A_0 and A observe the same event, A_0 will assign to it coordinates t, z^i and A will assign to it other coordinates t', z'^i , and, if all events are observed in this way, then t', z'^i as dependent variables can be expressed in terms of t, z^i as independent variables. These expressions give the transformation from one coordinate system to the other, and this is the *transformation from A_0 to A* . From the way in which t was defined it follows that both observers will assign the same epoch to an event, whence $t' = t$. Also, since z^i (and z'^i) may be regarded simply as coordinates attached to the fundamental particles, it follows that z'^i depends only upon the z 's and does not involve t . The transformation from A_0 to A is thus of the form

$$t' = t, \quad z'^i = f^i(z),$$

and there is a transformation of this form from A_0 to each other observer.

Keeping fixed the coordinate system z^i possessed by A_0 , we now note that, since each observer is at a centre of spherical symmetry, he may choose his similar system in a variety of ways, corresponding to an arbitrary orientation of his axes. The transformation from A_0 to any other observer A therefore involves six parameters, three to fix the position of A relative to A_0 , and three more to fix the orientation of A 's axes, and the complete set of transformations in the z 's is of the form

$$z'^i = f^i(z, a), \tag{24}$$

where a^θ , $\theta = 1, 2, \dots, 6$, are the parameters. As was shown in [10], this set forms a group G_6 , since the transformations from any one observer to the remaining observers are of the same form.

Since the orientation of each observer's axes is arbitrary, the group G_6 admits a sub-group R_3 , the group of rotations leaving the origin unaltered. The group G_6 also admits another sub-group G_3 which is transitive, *i.e.* it gives transformations relating all the observers, their axes being oriented in some particular way. This last point is important, for it must be remembered that, if each observer chooses his axes arbitrarily, then the three-parameter set of transformations from A_0 to the other observers do not necessarily form a group, although they are members of G_6 . For these transformations to form a group the various orientations must follow some law; this problem of orientation has been solved completely and it is hoped that the results will be published later.

The parameters a^θ may be chosen so that the identity $z'^i = z^i$ is given by $a^\theta = 0$, and so that the two sub-groups G_3 , R_3 are separated, a^α , $\alpha = 1, 2, 3$, being the parameters for G_3 and a^λ , $\lambda = 4, 5, 6$ the parameters for R_3 . The group G_3 is then given by (24) with $a^\lambda = 0$, and R_3 is given by (24) with $a^\alpha = 0$. When the a 's are small we have the infinitesimal transformations of G_6 , which are of the form

$$z'^i = z^i + \xi_\alpha^i(z) a^\alpha + \xi_\lambda^i(z) a^\lambda, \quad (25)$$

where $\xi_\alpha^i(z)$ are the group-vectors for G_3 and $\xi_\lambda^i(z)$ the vectors for R_3 . The notation in dealing with these groups will be as in [10], with z^i in place of x^i .

It has been shown that an important property of G_6 is that all the transformations leave invariant† a differential quadratic $k_{ij} dz^i dz^j$ which is the metric of a space of constant curvature. This quadratic gives the metric de^2 of the space S_3 , and G_6 is the complete group of motions in this space. It follows from (1) that the metric ds^2 of S_4 is also invariant in form under the transformations relating any two observers, since t is an invariant‡.

When $K = -1$, and referring to Milne's coordinates t' , $\mathbf{P}$, the transformation from A_0 ($\mathbf{V} = 0$) to the observer A having velocity $\mathbf{V}$ depends upon the orientation of A 's axes. If these are chosen to be "parallel" to those of A_0 in the sense used by Lorentz in discussing the velocity $(V, 0, 0)$, then the transformation from t' , $\mathbf{P}$ to t'' , $\mathbf{P}'$ is the Lorentz transformation

$$t'' = \beta \left(t' - \frac{\mathbf{P} \cdot \mathbf{V}}{c^2} \right) = -t' + \frac{\beta + 1}{\beta} \chi, \quad (26)$$

$$\mathbf{P}' = \mathbf{P} - \chi \mathbf{V},$$

where
$$\beta = (1 - \mathbf{V}^2/c^2)^{-\frac{1}{2}}, \quad \chi = \beta t' - \frac{\beta^2}{\beta + 1} \frac{\mathbf{P} \cdot \mathbf{V}}{c^2}.$$

These transformations do not satisfy the group property, for the transformation from A_0 to A_1 combined with the similar transformation from A_1 to A_2 gives the transformation direct from A_0 to A_2 only when the three observers are in line. The axes given by the above coordinate

† By invariant in form, we mean that the transform is a similar expression with z'^i in place of z^i .

‡ Our procedure here differs essentially from that followed by McVittie [3], who takes as the 4-space map any Riemannian space, regardless of whether it is related in any simple way to the transformations connecting the different observers.

systems are oriented in a certain way relative to A_0 and are not oriented in the same way relative to any other observer. These particular axes are, however, related to the coordinate system z^i for which the metric of S_3 is (16), for it can be verified that at any point of S_3 the axes of the observer at that point are simply the parametric directions.

9. We have now discussed various maps for describing a model; there are S_3 and S_4 with kinematic time t , and Σ_3 and Σ_4 with dynamic time τ . When referring to coordinates t, z^i , or τ, ζ^i , we may call t and τ *public times*, since the same epoch is assigned to an event by all observers. Similarly S_3 and Σ_3 are *public spaces*, since they are defined in the same way by all observers, being given by $t = \text{constant}$ and $\tau = \text{constant}$ for each observer. An observer corresponds to one point of S_3 (and of Σ_3), and we may talk of the observer at this point. The transformation from one observer to another is merely a change in the descriptions of S_3 and Σ_3 , independent of public time, and in this sense these spaces play the part of Euclidean space in classical mechanics. It will be found that τ and Σ_3 correspond more closely to classical time and space (whence the name “dynamic time”), but certain results will be first derived in terms of t and S_3 since we may then apply the dimensional postulate, avoiding the introduction of dimensional constants.

Milne's coordinates t', P refer to the same space S_4 , but they are private coordinates, since two observers do not in general assign the same epoch t' to an event; and the four coordinates are involved implicitly in the transformation from one observer to another. For this reason we cannot conveniently separate time and space and refer to a three-dimensional map. To describe an object in terms of such coordinates, we first describe the object in S_3 and then in tensor form in S_4 , the latter form being applicable to any coordinate system in S_4 and in particular to Milne's coordinates when $K = -1$.

The representation of physical objects.

10. We are here concerned in describing in a given situation certain characteristics apparent to all observers, as for example the “apartness” of two events, and the motion of a particle relative to the fundamental particles. Such characteristics have real existence in the model, and are called *physical objects*. Each situation can be analysed into characteristics in a number of ways, and the problem is to find some convenient method for carrying out this analysis and for describing the objects so given.

Since we wish to describe situations in relation to the geometrical map, we are led to consider the geometrical configuration, consisting of points, curves, etc., which corresponds to the situation in the model. A physical object will, from its concept, correspond to an invariant characteristic of this configuration, and will therefore best be described by a geometrical tensor, since this has the required invariance.

In this process we can refer to any one of the maps considered above, since we can find the appropriate configuration in each map. For convenience, therefore, we refer first to the 3-space map S_3 , and physical objects are described by each observer by means of scalars, vectors, and tensors in this map. Referring to S_3 there is an observer at each point, and the event $E(t, z)$ corresponds to the point $P(z)$, the observer at E being the observer at P . An object at E , describing the interval between E and a neighbouring event, or the motion of a particle at E , etc., will now be described by a tensor at P in accordance with certain rules to be determined.

Later we refer to Σ_3 , and in another paper (Part III of the present series) we consider descriptions in S_4 and Σ_4 .

11. If an object at an event E is scalar, it may be regarded as measured by the observer at E , who then communicates its numerical magnitude to all other observers, so that all observers assign the same magnitude to the object. It is now necessary to consider only vector objects, for a more general tensor object can be expressed in terms of scalars and vectors.

If E is the event (t, z) relative to A_0 ($z^i = 0$), and P is the corresponding point (z) of S_3 , then a vector object at E has a scalar magnitude which is communicated to A_0 as above, and a direction determined by a sequence of events which can be observed by A_0 . These events correspond to a sequence of points in S_3 directed away from P and this sequence is known to A_0 , who can describe it by the ratios of increments dz^i of the coordinates z^i . Hence, referring to the map with metric (2), A_0 can describe the object by components λ^i , such that the scalar magnitude is given by

$$\left(k_{ij}(z) \lambda^i \lambda^j \right)^{\frac{1}{2}} \quad (27)$$

and the direction by

$$\frac{dz^1}{\lambda^1} = \frac{dz^2}{\lambda^2} = \frac{dz^3}{\lambda^3}.$$

The object is now described by A_0 by the geometrical vector λ^i at P .

Any other observer A assigns to P coordinates z'^i , where $z'^i = f^i(z)$ is the transformation from A_0 to A . A can observe the same sequence of points in S_3 giving the direction of the vector object, and this direction is described by the ratios of increments dz'^i , where $dz'^i = (\partial f^i / \partial z^j) dz^j$. Hence A , following the same rules as A_0 , describes the vector by components λ'^i proportional to $\lambda^j \partial f^i / \partial z^j$. The quantities $k_{ij}(z')$, by means of which A defines magnitude at P , are related to $k_{ij}(z)$ by

$$k_{ij}(z) = k_{lm}(z') \frac{\partial f^l}{\partial z^i} \frac{\partial f^m}{\partial z^j},$$

since the quadratic $k_{ij} dz^i dz^j$ is invariant under the transformation from A_0 to A . It follows at once that, since A_0 and A assign the same scalar magnitude to the vector,

$$\lambda'^i = \lambda^j \frac{\partial f^i}{\partial z^j}.$$

Thus, the object is described by a geometric vector which transforms as a vector under the transformation from one observer to another.

It appears that physical vectors are described naturally by contravariant vectors in S_3 , but we can derive covariant vectors in the usual way by means of the tensor k_{ij} . Such vectors transform as above, and it can easily be verified that the result can now be extended to contravariant, covariant, and mixed tensors. The general theorem is that *a physical object can be described by a tensor in S_3 which transforms as a tensor under the transformation from one observer to another.*

It is essential at this stage to distinguish between two kinds of tensors.

I. A tensor of the first kind relative to an observer is described by a set of quantities which transform as tensor components when the observer changes his coordinate system.

II. A tensor of the second kind represents an object common to the experience of all observers, and is described by a set of quantities which transform as tensor components under the transformation from one observer to another.

It is understood in II that, in representing an object, a tensor is describing characteristics which are independent of any particular coordinate system, whence a tensor of the second kind is also a tensor of the first kind. The previous theorem now states that a physical object is described in S_3 by a tensor of the second, and hence also of the first, kind.

12. We now consider the application of the Cosmological Principle, and for this purpose we restrict ourselves to the group G_3 , since the principle need apply only to one observer attached to each fundamental particle. Certain properties of G_3 are given below, and a useful identity is proved.

The transformations of G_3 are $z'^i = f^i(z, a)$, where a^α , $\alpha = 1, 2, 3$, are the parameters. The group vectors are $\xi_a^i(z)$, the normalized cofactors in $|\xi_a^i|$ being $\xi_i^a(z)$, and these vectors are related to the transformations by equations of the form

$$\frac{\partial z'^i}{\partial a^\alpha} = \xi_\beta^i(z') A_\alpha^\beta(a).$$

Coefficients Λ_{jk}^i , which behave as connection coefficients, are defined by

$$\Lambda_{jk}^i = -\xi_{a \cdot j}^i \xi_k^a = \xi_a^i \xi_{k \cdot j}^a, \quad (28)$$

where a point denotes partial differentiation, and these are such that the equations

$$\frac{\partial x^i}{\partial z^k} + x^j \Lambda_{jk}^i = 0$$

for x^i are completely integrable. Three independent vector solutions to these equations are written $\zeta_a^i(z)$, $a = 1, 2, 3$, so that the general solution is of the form $x^a f_a^i(z)$, where x^a are constants. It can be seen from the form of the equations that any solution is a tensor of the first kind. The normalised cofactors in $|\zeta_a^i|$ are written $\zeta_i^a(z)$, and from the differential equation we have

$$\Lambda_{jk}^i = -\zeta_{a \cdot k}^i \zeta_j^a = \zeta_a^i \zeta_{j \cdot k}^a. \quad (29)$$

It can now be verified that, for each transformation of G_3 ,

$$\frac{\partial z'^i}{\partial z^j} = \zeta_a^i(z') \zeta_j^a(z). \quad (30)$$

In the first place we see that these equations hold for the identity transformation $z'^i = z^i$ given by $a^\alpha = 0$. Differentiating with respect to a^α and substituting from the fundamental group equations, we have on the left, from (28),

$$\begin{aligned} \frac{\partial^2 z'^i}{\partial z^j \partial a^\alpha} &= A_\alpha^\beta \frac{\partial}{\partial z^j} \xi_\beta^i(z') = A_\alpha^\beta \xi_{\beta \cdot k}^i(z') \frac{\partial z'^k}{\partial z^j} \\ &= -A_\alpha^\beta \Lambda_{kl}^i(z') \xi_\beta^l(z') \frac{\partial z'^k}{\partial z^j}, \end{aligned}$$

and on the right, from (29),

$$\begin{aligned}\zeta_j^a(z) \frac{\partial}{\partial a^a} \zeta_a^i(z') &= \zeta_j^a(z) \zeta_{a,l}^i(z') \xi_{\beta}^l(z') A_{\bullet}^{\beta} \\ &= -A_{\bullet}^{\beta} \Lambda_{kl}^i(z') \xi_{\beta}^l(z') \zeta_a^k(z') \zeta_j^a(z).\end{aligned}$$

The derived equations are thus seen to hold for values of a^a for which the original equation is true. It follows that, since (30) holds for $a^a = 0$, it is true generally.

13. Consider now a field of objects satisfying the Cosmological Principle. These objects are described by A_0 by a tensor field $T(z)$, where $T_{j,\dots}^{i,\dots}(z)$ is the tensor at the point (z) , and each member of the field is a tensor of the second kind. Hence the tensor at (z) relative to A_0 is described by another observer A as being the tensor $T'_{j,\dots}{}^{i,\dots}$ at the point (z') relative to A , where $z'^i = f^i(z)$ is the transformation from A_0 to A , and

$$T'_{j,\dots}{}^{i,\dots} = T_{m,\dots}^{l,\dots}(z) \frac{\partial z'^i}{\partial z^l} \dots \frac{\partial z^m}{\partial z'^j} \dots$$

If the field satisfies the Cosmological Principle, it must appear the same to all observers, whence the tensor at the point (z') relative to A is $T_{j,\dots}^{i,\dots}(z')$. Replacing $T'_{j,\dots}{}^{i,\dots}$ by $T_{j,\dots}^{i,\dots}(z')$, and substituting for $\partial z'^i/(\partial z^l)$, $\partial z^m/(\partial z'^j)$ from (30), we find that the field must satisfy identically the equations

$$T_{j,\dots}^{i,\dots}(z') \zeta_i^a(z') \dots \zeta_b^j(z') \dots = T_{j,\dots}^{i,\dots}(z) \zeta_i^a(z) \dots \zeta_b^j(z) \dots$$

Since any point (z') can be related to (z) by some transformation of G_3 , it follows that the expression on the right must be independent of z^i and so is a constant $T_{b,\dots}^{a,\dots}$. Hence a necessary and sufficient condition to be satisfied by $T(z)$ is that it is of the form

$$T_{j,\dots}^{i,\dots}(z) = T_{b,\dots}^{a,\dots} \zeta_a^i(z) \dots \zeta_j^b(z) \dots, \quad (31)$$

where $T_{b,\dots}^{a,\dots}$ are constants.

A field of this kind is known in geometry as a field given by *distant parallelism* [9], the fundamental vectors of the parallelism being $\zeta_a^i(z)$. The vectors at a point P are regarded as forming a local frame of reference, or base, and the constants $T_{b,\dots}^{a,\dots}$ are the components of T at P referred to the local base. Thus each member of the field is described locally by the same components. A property easily verified is that the covariant derivative of T for the connection Λ_{jk}^i is zero, this condition being also sufficient to give a parallel tensor field.

We see from (31) that a field $T(z)$ is determined completely by the member at any one point. This is to be expected, since the field has been constructed in such a way that it may be regarded as describing relative to one observer the observations on one particular object made by all the observers. If $\bar{T}$ is a particular tensor object at the point $\bar{P}$, of coordinates $\bar{z}^i$ relative to A_0 , and if P is the point of coordinates z^i relative to A_0 , then there is an observer A such that $\bar{P}$ has coordinates z^i relative to A . Then A measures $\bar{T}$ at $\bar{P}$ as being a tensor T at the point (z) relative to A , T being given in terms of $\bar{T}$ by the transformation from A_0 to A , since it is a tensor of the second kind. If now A_0 sets up the tensor T at the point $P(z)$, and if this is repeated for each point P , we finally have the complete field relative to A_0 , and, from its construction, it satisfies the Cosmological Principle.

14. An object of the kind considered here is the frame of reference possessed by an observer within the group G_3 . Such a frame is determined by three vectors, the "axes", at the observer, whence these vectors must satisfy (31) and are therefore linear combinations of the vectors ζ_a^i . Starting with the axes of A_0 given by vectors h_a^i at O , then from the differential equations for ζ_a^i we can define the fields $\zeta_a^i(z)$ uniquely by initial conditions $\zeta_a^i(0) = h_a^i$. It then follows from (31) that the axes belonging to the observer at the point $P(z)$ are simply the vectors $\zeta_a^i(z)$.

Another object is the tensor measuring angle, *i.e.* k_{ij} , and we see from (27) that $k_{ij}(z)$ is a field of tensors of the kind considered above. Hence, or by direct verification, $k_{ij}(z) \zeta_a^i(z) \zeta_b^j(z)$ are constants, as we should expect since these expressions give the relations between the axes.

Other fields of tensors of the first kind associated with G_3 are given by the vectors ξ_a^i and by tensors Ω_{jk}^i , V_{jk}^i , where

$$\Omega_{jk}^i = \frac{1}{2}(\Lambda_{jk}^i - \Lambda_{kj}^i), \quad (32)$$

$$V_{jk}^i = \Lambda_{jk}^i - \Gamma_{jk}^i = \Omega_{jk}^i + k^{il}(k_{jm} \Omega_{kl}^m + k_{km} \Omega_{jl}^m). \quad (33)$$

Γ_{jk}^i being the Christoffel symbols calculated from k_{ij} . The latter form of V_{jk}^i is a consequence of the Killing equations relating k_{ij} and Λ_{jk}^i . Testing these fields in accordance with (31), we find, from (28) and (29), that the quantities $\xi_a^i \zeta_a^i$ are constants only when $\Omega_{jk}^i = 0$, in which case it can be shown that S_3 is a flat space. In general, therefore, the vectors ξ_a^i cannot describe objects satisfying the Cosmological Principle. For Ω_{jk}^i however, it is known that $\Omega_{jk}^i \zeta_a^j \zeta_b^j \zeta_c^k$ is constant, so that Ω_{jk}^i , and hence V_{jk}^i , satisfy

(31). Thus Ω_{jk}^i and V_{jk}^i can describe objects satisfying the Cosmological Principle.

We have already mentioned that, if T satisfies (31), then its covariant derivative for the connection Λ_{jk}^i is zero. It can be seen to follow from the above property of Ω_{jk}^i and V_{jk}^i that the covariant derivative of T for either of the connections Γ_{jk}^i , $\bar{\Lambda}_{jk}^i = \Lambda_{kj}^i$ is also a tensor satisfying (31).

15. The above discussion concerns only tensors which depend upon position; consider now a tensor depending also upon a number of vectors. Let $\bar{\lambda}_{(r)}^i$, $r = 1, 2, \dots, n$, be n vectors at a particular point $\bar{P}$ relative to A_0 , and let $\bar{T}_{j\dots}^{i\dots}$ be a tensor at $\bar{P}$ observed by A_0 to depend upon the n vectors. Then we can write

$$\bar{T}_{j\dots}^{i\dots} = (\zeta_a^i \dots \zeta_j^b \dots)_P T_{b\dots}^{a\dots}(\lambda_{(r)}^e), \quad [\lambda_{(r)}^e = (\zeta_i^e)_P \bar{\lambda}_{(r)}^i],$$

since we are considering dependence at a particular point. A field of such tensors which satisfies the Cosmological Principle is generated by $\bar{T}_{j\dots}^{i\dots}$ and from (31) the field is described by A_0 by

$$T_{j\dots}^{i\dots} = \zeta_a^i(z) \dots \zeta_j^b(z) \dots T_{b\dots}^{a\dots}(\lambda_{(r)}^e). \quad (34)$$

The vectors $\bar{\lambda}_{(r)}^i$ also generate fields $\lambda_{(r)}^i(z)$ given by $\lambda_{(r)}^i(z) = \lambda_{(r)}^a \zeta_a^i(z)$, and we can write

$$\lambda_{(r)}^e = \zeta_i^e(z) \lambda_{(r)}^i(z). \quad (35)$$

The components of the tensor in (34) are now functions of z^i and $\lambda_{(r)}^i$, and equations (34) and (35) show how the tensor at any point depends upon the coordinates of the point and upon the n vectors at the point. We now have the general form of a tensor depending upon a number of vectors when the tensor is a member of a field satisfying the Cosmological Principle.

16. We have so far been concerned with fields of objects satisfying only the Cosmological Principle, but we now consider how these fields are further restricted if they are to satisfy the Principle of Symmetry. Each observer is now allowed to orient his frame of reference in any way, but since the objects considered already satisfy the Cosmological Principle, it is sufficient to consider orientations at one observer A_0 . The transformations giving different orientations are members of the rotation group R_3 , parameters a^λ , $\lambda = 4, 5, 6$, and each member can

be regarded as the transformation from A_0 to another observer A_0' also situated at O .

Consider now a field of position tensors $T(z)$ which satisfies both principles. Then, by the previous argument, T must satisfy

$$T_{j...}^{i...}(z') = T_{m...}^{l...}(z) \frac{\partial z'^i}{\partial z^l} \cdots \frac{\partial z^m}{\partial z'^j} \cdots \quad (36)$$

under all transformations of G_6 . The restrictions due to G_3 lead to (31) and, substituting in (36), we finally deduce that the constants $T_{b...}^{a...}$ must satisfy

$$T_{b...}^{a...} = T_{d...}^{c...} \eta_c^a \cdots \bar{\eta}_b^d \cdots, \quad (37)$$

where

$$\eta_c^a = \zeta_i^a(z') \frac{\partial z'^i}{\partial z^j} \zeta_c^j(z), \quad (38)$$

$\bar{\eta}_b^a$ is the normalized cofactor of η_d^b in $|\eta_i^a|$, and $z'^i = f^i(z)$ is any transformation of R_3

From its derivation in [10] we see that the tensor $k_{ij}(z)$ satisfies (36), whence η_j^a satisfy the relations

$$k_{ab} \eta_c^a \eta_d^b = k_{cd}, \quad (39)$$

where k_{ab} is the constant $k_{ij} \zeta_a^i \zeta_b^j$. The coefficients η_c^a may involve z^i and certainly involve a^λ , since from (38) we see that at $O(z^i = z'^i = 0)$ they are the coefficients relating the axes of A_0 and A_0' . It follows that, since (37) must hold for all rotations, η_c^a may be regarded as parameters which are arbitrary except for (39), and equation (37) must be a consequence of (39). Hence $T_{b...}^{a...}$ must be a combination of k_{ab} , k^{ab} , δ_b^a , whence $T_{j...}^{i...}(z)$ can be expressed in terms of the tensors $k_{ij}(z)$, $k^{ij}(z)$, δ_j^i . We can therefore state that the only irreducible position tensors satisfying both principles are k_{ij} , k^{ij} , δ_j^i , and that any solution of (36) is a combination of these tensors.

17. We turn to the more general case when a tensor depends upon a number of vectors. The Cosmological Principle alone leads to equations (34) and (35), which give the form of $T_{j...}^{i...}$ at the point P , with coordinates z^i relative to A_0 . The observer A_0' can also observe the tensor and vectors at P , and for the Principle of Symmetry to be satisfied the tensor must depend upon the observed position and vectors in the same way as for A_0 . Relative to A_0' the point P has coordinates z'^i , the n vectors are described by $\lambda_{(r)}^i = \lambda_{(r)}^j(z) (\partial z'^i / \partial z^j)$, and the tensor is described by

$$T_{j...}^{i...} = T_{m...}^{l...} \frac{\partial z'^i}{\partial z^l} \cdots \frac{\partial z^m}{\partial z'^j} \cdots$$

Hence, from (38), the quantities corresponding to $\lambda_{(r)}^e$ in (35) are

$$\lambda_{(r)}^{'e} = \zeta_i^e(z') \lambda_{(r)}^{'i} = \eta_f^e \lambda_{(r)}^f,$$

and from (34) and (38) we deduce that $T_{b...}^{a...}$ must satisfy

$$T_{b...}^{a...}(\lambda_{(r)}^{'e}) = \eta_c^a \dots \bar{\eta}_b^d \dots T_{d...}^{c...}(\lambda_{(r)}^e) \quad [\lambda_{(r)}^{'e} = \eta_f^e \lambda_{(r)}^f] \quad (40)$$

for all coefficients η_e^a satisfying (39).

We do not discuss the general solution of these equations, since in applications we are concerned for the present only with scalar and vector objects. If χ is a scalar having the above properties, then it must satisfy

$$\chi(\lambda_{(r)}^{'e}) = \chi(\lambda_{(r)}^e) \quad \lambda_{(r)}^{'e} = \eta_f^e \lambda_{(r)}^f,$$

and, as may be expected, it can be shown that χ must involve the coordinates and vectors through constants $\lambda_{(rs)}$ given by

$$\lambda_{(rs)} = k_{ab} \lambda_{(r)}^a \lambda_{(s)}^b = k_{ij} \lambda_{(r)}^i \lambda_{(s)}^j. \quad (41)$$

If T^i is a vector having the above properties, then $T^i = \zeta_a^i(z) T^a$, where

$$T^a(\lambda_{(r)}^{'e}) = \eta_c^a T^c(\lambda_{(r)}^e) \quad \lambda_{(r)}^{'e} = \eta_f^e \lambda_{(r)}^f.$$

If now λ^a are any quantities and $\lambda'^a = \eta_b^a \lambda^b$, then, from (39),

$$\chi = k_{ab} \lambda'^a T^b(\lambda_{(r)}^{'e}) = k_{ab} \lambda_a T^b(\lambda_{(r)}^e),$$

whence, from the previous result, χ involves only $\lambda_{(rs)}$, $k_{ab} \lambda^a \lambda_{(r)}^b$, $k_{ab} \lambda^a \lambda^b$, $r, s = 1, 2, \dots, n$. Since, however, χ is linear in the λ 's, which are arbitrary, we must have

$$k_{ab} \lambda^a T^b = \sum_r \chi_{(r)} k_{ab} \lambda^a \lambda_{(r)}^b$$

for all λ 's, where $\chi_{(r)}$ are any functions of $\lambda_{(rs)}$. Hence

$$T^a = \sum_r \chi_{(r)} \lambda_{(r)}^a,$$

and we finally have

$$T^i(z, \lambda_{(r)}^k) = \sum_r \chi_{(r)} \lambda_{(r)}^i(z), \quad (42)$$

where the χ 's are any functions of $\lambda_{(rs)}$.

We observe that the dependence of a tensor upon a number of vectors cannot now be arbitrary at any one point. This result is significant, for if, for example, force is to be expressed in terms of velocity and acceleration, the form of such an expression is restricted,

II. *Mechanics in public 3-space.**Kinematical objects with t-time.*

18. The foregoing discussion and results can at once be applied to the formulation of a scheme of mechanics, where certain physical objects are defined to agree as closely as possible with classical concepts. We first consider kinematical objects such as distance-displacement, time-interval, velocity, and acceleration; results given in [10] concerning statistical systems and free paths are derived again. We then consider dynamical objects such as mass, momentum, force, angular momentum and force, work and energy.

If E , E' are two neighbouring events, the interval between them can be described by a displacement vector in S_3 together with a time-interval, and these can be defined so that they are physical objects. If A is the observer at E , then the coordinates of E , E' relative to A are of the form $(t, 0)$, $(t+dt, dz_0)$ respectively, and, from (14), the event E' occurs at a distance $ct \left(k_{ij}(0) dz_0^i dz_0^j \right)^{\frac{1}{2}}$ from A in the direction given by the ratios of dz_0^i . The position of E' relative to A is therefore described by the *distance-displacement vector* $dl_0^i = ct dz_0^i$, whose scalar magnitude is the scalar distance. If now E , E' are the events (t, z) , $(t+dt, z+dz)$ relative to an observer A_0 , it follows at once from § 11 that the distance-displacement vector relative to A_0 is

$$dl^i = ct dz^i, \quad (43)$$

the scalar distance being given by

$$dl = (k_{ij}(z) dl^i dl^j)^{\frac{1}{2}} = ct de,$$

where, from (2), de is the arc interval in S_3 between the points corresponding to E , E' .

Relative to any observer the interval in kinematic time t between the two events is dt . For the sake of generality, however, we consider as the time differential in mechanics a *time-interval* dT which may depend upon t , z^i , dt , dz^i . We evidently require dT to be homogeneous and of the first degree in dt , dz^i , and to be an invariant, measured in the same way by all observers, and hence, since dt is an invariant, dT/dt must be an invariant function of t , z^i , and the vector dz^i/dt . From § 17 we see that the only such invariant is a function of t and $k_{ij}(z) (dz^i/dt) (dz^j/dt)$. Since, therefore, dT is required to have the dimensions of time, the general form of dT/dt , excluding constants having the dimensions of time, is a function of $t^2 k_{ij} (dz^i/dt) (dz^j/dt)$,

It will be found convenient in this work to refer to η , defined by

$$\eta = 1 - t^2 k_{ij}(z) \frac{dz^i}{dt} \frac{dz^j}{dt} = \left(\frac{ds}{dt} \right)^2, \quad (44)$$

from (1). Then from the above argument *any dimensionless invariant involving t , z^i , dz^i/dt must be a function of η alone*. Milne refers to an invariant parameter ξ which, as we shall see later, is $1/\eta$.

We now have for the most general possible time-interval an expression of the form

$$dT = dt/\theta(\eta). \quad (45)$$

If E , E' both occur in the history of one observer A , then $dz^i = 0$, and $\eta = 1$. In this case we evidently require dT to be the time interval dt recorded by A , whence $\theta(\eta)$ must be such that $\theta(1) = 1$. Subject to this restriction θ can be any function of η .

In his kinematical theory Milne takes ds to be the time differential, so that his time-interval is given by $\theta(\eta) \equiv \eta^{-\frac{1}{2}}$.

19. The velocity of a particle in the interval between two neighbouring events on its world line is defined as the distance-displacement divided by the time-interval. We therefore have, from (43) and (45), that velocity is a vector object described relative to A_0 by v^i at the point (z) , where

$$v^i = \frac{dl^i}{dT} = c\theta(\eta) \frac{dz^i}{dt}. \quad (46)$$

The scalar velocity is thus, from (44),

$$|v| = (k_{ij} v^i v^j)^{\frac{1}{2}} = c\theta(\eta)(1-\eta)^{\frac{1}{2}}, \quad (47)$$

and η may be regarded as a function of $|v|/c$, depending upon the form of $\theta(\eta)$. It will be remembered from [10] that the increments dt , dz^i along the path of a particle are restricted so that $t^2 k_{ij} dz^i dz^j \leq dt^2$, equality holding only along a light path*. Hence, from (44), $0 \leq \eta \leq 1$, and $\eta = 0$ for a particle moving with the velocity of light, $\eta = 1$ for a particle moving with a fundamental particle. Referring to velocity v^i , the velocity of light is $c\theta(0)$.

* This is a consequence of Fermat's principle, which was adopted in the construction of the world-model,

Suppose now that we have a vector object defined at points along a path, and that we wish to describe as an object the change in the vector passing from E to E' along the path. If the vector is λ^i at E and $\lambda^i + d\lambda^i$ at E' , then $d\lambda^i$ is not a vector of the first kind and so cannot describe an object. We therefore consider

$$\delta\lambda^i = d\lambda^i + K_{jk}^i(z) \lambda^j dz^k,$$

where K_{jk}^i are some connection coefficients, independent of t in order that they may not involve a dimensional constant.

If $z'^i = f^i(z)$ is the transformation from A_0 to any other observer A , *i.e.* a member of G_6 , then, relative to A , E has coordinates z'^i , E' has coordinates $z'^i + dz'^i$, where $dz'^i = (\partial z'^i / \partial z^j) dz^j$, the vector at E is

$$\lambda'^i = \frac{\partial z'^i}{\partial z^j} \lambda^j,$$

and the vector at E' is $\lambda'^i + d\lambda'^i$, where

$$d\lambda'^i = (\lambda^j + d\lambda^j) \left(\frac{\partial z'^i}{\partial z^j} \right)_{z+dz} - \lambda^j \frac{dz'^i}{dz^j} = \frac{\partial z'^i}{\partial z^j} d\lambda^j + \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \lambda^j dz^k.$$

The vector $\delta\lambda^i$ relative to A_0 is described by $\delta\lambda'^i = (\partial z'^i / \partial x^j) \delta\lambda^j$ relative to A , and since we want this vector to be calculated in the same way by all observers, we must have

$$\begin{aligned} \delta\lambda'^i &= \frac{\partial z'^i}{\partial z^l} (d\lambda^l + K_{jk}^l(z) \lambda^j dz^k) = d\lambda'^i + K_{jk}^i(z') \lambda'^j dz'^k \\ &= \frac{\partial z'^i}{\partial z^j} d\lambda^j + \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \lambda^j dz^k + K_{lm}^i(z') \frac{\partial z'^l}{\partial z^j} \frac{\partial z'^m}{\partial z^k} \lambda^j dz^k. \end{aligned}$$

If we write $K_{jk}^i = \Gamma_{jk}^i + T_{jk}^i$, where Γ_{jk}^i are the Christoffel symbols for k_{ij} , then from the invariance of $k_{ij}(z)$ it follows that

$$\frac{\partial z'^i}{\partial z^l} \Gamma_{jk}^l(z) = \frac{\partial^2 z'^i}{\partial z^j \partial z^k} + \Gamma_{lm}^i(z') \frac{\partial z'^l}{\partial z^j} \frac{\partial z'^m}{\partial z^k},$$

whence T_{jk}^i must satisfy

$$\frac{\partial z'^i}{\partial z^l} T_{jk}^l(z) = T_{lm}^i(z') \frac{\partial z'^l}{\partial z^j} \frac{\partial z'^m}{\partial z^k}$$

for all transformations of G_6 . Thus $T_{jk}^i(z)$ is a tensor field of the kind considered in § 16, and must be expressible in terms of k_{ij} , k^{ij} , δ_j^i . Hence,

since T_{jk}^i is of odd order, the only possible form is $T_{jk}^i = 0$, and the only vector object which can measure the change in λ^i is

$$\delta\lambda^i = d\lambda^i + \Gamma_{jk}^i(z) \lambda^j dz^k. \quad (48)$$

This vector satisfies the required conditions and is measured in the same way by all observers.

With time-interval dT , the rate of change of a vector λ^i is now measured by $\delta\lambda^i/dT$. The acceleration of a particle is defined as the rate of change of the velocity and is therefore the vector object described by $\dot{v}^i$, where, from (48) and (46),

$$\dot{v}^i = \frac{\delta v^i}{dT} = \theta(\eta) \frac{dv^i}{dt} + \frac{1}{ct} \Gamma_{jk}^i(z) v^j v^k. \quad (49)$$

Differentiating $k_{ij} v^i v^j$ along the path, we have at once, from (47) and (48),

$$k_{ij} v^i \dot{v}^j = \frac{1}{2} c^2 \frac{d}{dT} \{ \theta^2 (1 - \eta) \}, \quad (50)$$

giving the rate of change of η in terms of velocity and acceleration. Since $\theta(1) = 1$, it follows that $\theta(\eta)$ cannot be of the form $a(1 - \eta)^{-\frac{1}{2}}$, whence, if the acceleration is zero or orthogonal to the velocity, then η is constant along the path. The scalar velocity is then also constant, agreeing with the classical result.

Remembering that in classical kinematics the acceleration is in the direction of the velocity only for motion in a straight line, we can show that the geodesics in S_3 play the part of straight lines in Euclidean space. For, if $\dot{v}^i = \mu v^i$, then, from (46) and (49), $\delta(dz^i/de)$ is of the form νdz^i , where de is the element of arc in S_3 . It is well known, however, that the vector $\delta(dz^i/de)$ is orthogonal to dz^i , whence it must vanish, so that the path in S_3 is a geodesic. Conversely, if the path in S_3 is a geodesic, then the acceleration is in the direction of the velocity.

20. Results similar to those given in [10] for a statistical system of particles satisfying in distribution and motion the Cosmological and Symmetry Principles can now be obtained in terms of position coordinates z^i and velocity coordinates v^i . In such a system let dN be the number of particles at time t situated and moving in the range $(z, v; z + dz, v + dv)$ relative to A_0 . Then we can write

$$dN = \chi(t, z, v) dz^1 dz^2 dz^3 dv^1 dv^2 dv^3,$$

Relative to another observer we have $z'^i = f^i(z)$, $v'^i = (\partial z'^i / \partial z^j) v^j$, whence

$$\frac{dz'^1 dz'^2 dz'^3 dv'^1 dv'^2 dv'^3}{dz^1 dz^2 dz^3 dv^1 dv^2 dv^3} = \frac{\partial(z', v')}{\partial(z, v)} = \left| \frac{\partial z'^i}{\partial z^j} \right|^2.$$

Since dN is required to be invariant in form and since

$$|k_{ij}(z)| = |k_{ij}(z')| \left| \frac{\partial z'^i}{\partial z^j} \right|^2,$$

it follows that $\chi |k|^{-1}$ must be invariant in form, whence $\chi = |k| \chi(t, \eta)$, by a previous result. Also dN is a pure number, so that χ must have the dimensions (velocity)⁻³. Thus finally the distribution is of the form

$$dN = c^{-3} \chi(\eta) |k| dz^1 dz^2 dz^3 dv^1 dv^2 dv^3. \quad (51)$$

In such a system the path of a particle is determined by the position and velocity at any one instant, and the paths may therefore be described by an equation of the form $\dot{v}^i = F^i(t, z, v)$. The same system must be described in the same way by all observers whence, from § 17, F^i must be of the form $F(t, \eta) v^i$. Considering the dimensions involved, we finally have the path equations in the form

$$\dot{v}^i = t^{-1} F(\eta) v^i. \quad (52)$$

Since v^i is in the direction v^i , we see, from § 19, that the paths in S_3 are geodesics.

The paths (52) are to be consistent with the distribution (51), number being conserved, and the functions F , χ are therefore related by a Boltzmann equation. By the usual procedure we find that F is determined by χ save for a constant factor C , the relation being

$$F(\eta) = \{\theta(\eta)\}^{-2} (1-\eta)^{-3} \frac{C}{\chi(\eta)}. \quad (53)$$

With $\eta = 1/\xi$ and $\theta(\eta) \equiv 1$, equations (51), (52), (53) agree with (5), (6), (7), where $\chi(\eta) = \bar{\psi}(\xi)$, $F(\eta) = 1 + \bar{G}(\xi)$.

21. The argument leading to (52) also applies to free test particles in the presence of the sub-stratum of fundamental particles with distribution (3). The path of such a particle is determined by the position and velocity at any one instant and is determined in the same way by all observers, whence the free paths satisfy equations of the form (52). Following

Milne's argument, we expect the free particles to differ from those of a statistical system in that the distribution is indeterminate. From the analysis leading to (53) we find that this occurs when and only when $F(\eta) = 0$. It follows, then, that the free paths are given simply by

$$\dot{v}^i = 0. \quad (54)$$

From previous results we now see that a free particle moves so that the acceleration is zero, η is constant along the path, the scalar velocity is constant, and the path in S_3 is a geodesic. It is easily seen that the free paths in S_4 are determined completely by the properties that the paths in S_3 are geodesics and the scalar velocity is constant along each path. We note the important result that the free paths in S_4 are not the geodesics in this space. From (46) and (49), since η is constant, the free paths are, as we should expect, independent of the form of $\theta(\eta)$ and are given simply by the equations

$$\frac{\delta}{dt} \left(t \frac{dz^i}{dt} \right) = 0. \quad (55)$$

Milne has shown [7] that the free paths are given by a variational equation, and the general form of this equation is easily found. The most general invariant function of t, z^i, dt, dz^i , homogeneous of the first degree in the differentials, is $W(\eta) t^n dt$, excluding dimensional constants, and we therefore consider the variational equation

$$\delta \int W(\eta) t^n dt = 0. \quad (56)$$

This is equivalent to the system of differential equations

$$\frac{d}{dt} \left(t^n \frac{\partial W}{\partial \dot{z}^i} \right) - t^n \frac{\partial W}{\partial z^i} = 0, \quad \dot{z}^i = \frac{dz^i}{dt},$$

where $\eta = 1 - t^2 k_{ij} \dot{z}^i \dot{z}^j$, and we find by direct calculation that these again are equivalent to

$$\frac{\delta}{dt} \left(W' t^{n+2} \frac{dz^i}{dt} \right) = 0, \quad W' = \frac{dW}{d\eta}.$$

Comparing with (55), we take $n = -1$. If now we contract with $W' t k_{ij} \dot{z}^j$, we find that an integral is $W'(1 - \eta)^{\frac{1}{2}} = \text{constant}$. If W is of the form $\alpha(1 - \eta)^{\frac{1}{2}}$, then the variational equation becomes $\delta \int de = 0$, where de is the element of arc in S_3 , and this merely tells us that the paths in S_3 are geodesics, so that equations (55) are not reproduced completely.

Excluding, therefore, the form $W = a(1-\eta)^{\frac{1}{2}}$, we have $\eta = \text{constant}$, whence W' is constant along each path, and we now see that the variational equation is equivalent to (55). The free paths are therefore given by

$$\delta \int W(\eta) \frac{dt}{t} = 0 \quad (57)$$

for any form of $W(\eta)$ except $a(1-\eta)^{\frac{1}{2}}$. The form given by Milne is equivalent to (57) with $W(\eta) \equiv \eta$.

For any value of n other than -1 , $W(\eta)$ in (56) can be chosen so that this equation reproduces the paths (52) for a statistical system. This problem was considered in [11].

Dynamical objects with t -time.

22. Dynamics differs from kinematics in that dynamical objects such as momentum and force are chosen to depend not only upon the motion of a particle but also upon an inertial characteristic, the *mass*, which may vary for different particles. We are not here concerned with an analysis of the concept of mass. For our purpose it will be assumed that different elements can be recognised and that a mass number is attached by all observers to an atom of a known element. An invariant mass number m , the same for all observers, is now associated with any particle, whatever its position and motion, either by direct observation or from information received from an observer near the particle. This inertial mass will be called, as in General Relativity, the *proper mass* of the particle.

In recent theories it has been found convenient to consider not only the proper mass m of a particle but also a *relative mass* M , which depends upon m and also upon the velocity of the particle, and momentum and force are defined to depend upon M , just as in classical mechanics they depend upon m . The relative mass is defined to involve velocity in such a way that Mc^2 gives the observed energy of the particle, which again is related to relative mass through the ideas of force and work. The energy is required to agree with the classical form when the velocity is small compared with that of light, and in so doing it is found to approximate to $mc^2 + \frac{1}{2}m|v|^2$. The constant term mc^2 arising here is interpreted as an amount of energy possessed by the particle in virtue of its constitution, and agrees with modern theories of matter. Similar particles, whatever their motions, have the same proper mass m and the same latent energy, and for a particle moving with an observer the velocity is zero and the relative mass is the proper mass.

The relative mass M now depends upon m , t , z^i , v^i , and taking M to be an invariant physical object, we find from dimensional considerations that M/m is a function of η only, so that we can write

$$M = m\phi(\eta). \quad (58)$$

For a particle moving with an observer, $\eta = 1$ and M is to reduce to m , whence $\phi(1) = 1$. When the velocity is small compared with that of light, $\eta \sim 1 - |v|^2/c^2$ whatever the form of $\theta(\eta)$, and therefore

$$Mc^2 \sim mc^2 - m\phi'(1)|v|^2.$$

If, therefore, the energy is Mc^2 and is to approximate to the classical form $\frac{1}{2}m|v|^2$ when $|v|/c$ is small, it follows that $\phi'(1) = -\frac{1}{2}$.

In modern theories energy is defined in such a way that it is infinite for a particle moving with the speed of light. This property we may conveniently include in our definition of relative mass, and, since $\eta = 0$ for such a particle, we have $\phi(0) = \infty$. Thus $\phi(\eta)$ passes from 1 to ∞ as η decreases from 1 to 0. It is natural therefore to assume that $\phi(\eta)$ increases monotonically as η decreases, for otherwise there would be certain preferential velocities given by values of η for which $\phi(\eta)$ is critical. Our final result is that the relative mass is given by (58), where $\phi(\eta)$ is some function satisfying

$$\phi(1) = 1, \quad \phi'(1) = -\frac{1}{2}, \quad \phi(0) = \infty, \quad \phi'(\eta) < 0. \quad (59)$$

The energy of the particle is the scalar object given by

$$E = Mc^2 = mc^2 \phi(\eta). \quad (60)$$

We now define momentum as the physical vector object given by the product of relative mass and velocity. It is therefore described by the vector p^i , where

$$p^i = Mv^i = mc\phi\theta t \frac{dz^i}{dt}. \quad (61)$$

23. The concept of force is not essential to mechanics, for the motions which accompany any genuinely dynamical phenomenon do not depend in any way on the definition adopted for force. If the definition is changed, the description of phenomena in terms of force is changed, but the motions finally deduced are unaltered. The convenience and usefulness of the concept have, however, been clearly demonstrated in the development of classical mechanics, and it is natural, therefore, to consider in the present

scheme a vector object which we may call force and which possesses certain properties possessed by classical force. There is some freedom in our choice of definition, for we can use, to define the force on a particle, any vector which is a measure of the difference between the actual motion of the particle and the free motion of a particle moving with the same velocity through the same event. We could formally define force to be the rate of change of momentum, but it is interesting to find that this form arises from the following elementary properties.

(i) The force on a particle at an instant is a vector object depending upon the mass, position, velocity, and acceleration at the instant, the components of acceleration being involved up to the first degree.

(ii) The dimensions of force are $mass \times velocity \times (time)^{-1}$.

(iii) The force on a freely moving particle is zero.

If the force on a particle of proper mass m is F^i , then from (ii) F^i/m has dimensions $velocity/time$, and is from (i) a vector function of $t, z^i, v^i, \dot{v}^i$. Hence from (42), excluding dimensional constants other than m and c , F^i must be of the form

$$F^i = m\chi_1(\eta)\dot{v}^i + m\left\{\chi_2(\eta)\frac{1}{c^2}k_{ik}v^j\dot{v}^k + \frac{1}{t}\chi_3(\eta)\right\}v^i.$$

For a freely moving particle $\dot{v}^i = 0$ from (54). Hence, since (iii) is required to hold for all free particles, it follows that, for all values of η and v^i , F^i must vanish when $\dot{v}^i = 0$, whence $\chi_3(\eta) \equiv 0$. We see from (50) that $c^{-2}k_{ij}v^i\dot{v}^j$ is $d\eta/dT$ multiplied by a function of η . Hence, since χ_1 and χ_2 are arbitrary and since $\dot{v}^i = \delta v^i/dT$, F^i can be written in the form

$$F^i = A \frac{\delta}{dT} (mBv^i),$$

where A, B are arbitrary functions of η .

Thus we see that, as a consequence of the above conditions, force appears as the rate of change of a vector similar to the momentum vector. We have already taken time-interval and relative mass, and hence momentum, in their most general possible forms, and no purpose is served by introducing further arbitrary functions. To agree, therefore, with the classical form of force as rate of change of momentum, we take $A \equiv 1$, $B \equiv \phi$, whence

$$F^i = \frac{\delta}{dT} (p^i) = \theta \frac{\delta}{dt} \left(mc\phi\theta t \frac{dz^i}{dt} \right). \quad (62)$$

24. Having defined force, the next object for us to consider is the work done by the force on a particle in a small interval. Since distance-displacement and force have appeared as vectors in 3-space, it is natural to adopt a 3-space definition of work, and to define the work done in a small interval as the product of force and distance-displacement as in classical mechanics. If therefore dl^i is a distance-displacement along the path of the particle, we have for the work done

$$dW = k_{ij} F^i dl^j. \quad (63)$$

From (46), (61), and (62),

$$dW = k_{ij} \frac{\delta}{\delta T} (m\phi v^i) v^j dT = \frac{m}{2\phi} d(\phi^2 k_{ij} v^i v^j),$$

whence, from (47) and (58),

$$dW = \frac{1}{2} mc^2 \phi^{-1} d\{\phi^2 \theta^2 (1-\eta)\}.$$

Thus, since θ and ϕ are functions of η , dW is a true differential, and the work done in a finite interval (1, 2) is of the form $W(\eta_2) - W(\eta_1)$, where $W(\eta)$ is a function determined by θ and ϕ .

The energy of a particle has already been mentioned, and the problem now is to choose θ and ϕ so that, to agree with the classical result, the work done by the force on a particle in any interval is the increase in energy in this interval. From (60) we must therefore have

$$\frac{1}{2} \phi^{-1} d\{\phi^2 \theta^2 (1-\eta)\} = d\phi;$$

and this equation can be integrated at once to give

$$\phi^2 \theta^2 (1-\eta) = \phi^2 + \alpha,$$

where α is a constant. Since θ , ϕ are required to satisfy $\theta(1) = \phi(1) = 1$, we find, on putting $\eta = 1$, that $\alpha = -1$, whence

$$\theta^2 (1-\eta) = 1 - \phi^{-2}. \quad (64)$$

Also ϕ is required to satisfy $\phi(0) = \infty$, so that on putting $\eta = 0$ in (64) we find $\theta(0) = 1$. Thus $\theta(1) = \theta(0) = 1$, and assuming that $\theta(\eta)$ is monotonic to avoid preferential velocities, it follows that $\theta(\eta) \equiv 1$. Hence θ and ϕ are completely determined, and, from (64),

$$\theta(\eta) \equiv 1, \quad \phi(\eta) \equiv \eta^{-\frac{1}{2}}. \quad (65)$$

It is seen that the form of $\phi(\eta)$ agrees with (59).

With the above forms for θ and ϕ , the time-interval is simply dt , the element of kinematic time. The components v^i and the invariant η are now

$$v^i = \frac{dl^i}{dt} = ct \frac{dz^i}{dt}, \quad \eta = 1 - \frac{1}{c^2} k_{ij} v^i v^j = 1 - \frac{|v|^2}{c^2}, \quad (66)$$

and the relative mass is

$$M = m\eta^{-\frac{1}{2}} = m(1 - |v|^2/c^2)^{-\frac{1}{2}}, \quad (67)$$

agreeing with the form of mass in the Special Theory of Relativity. The force is given by

$$F^i = \frac{\delta}{dt} (Mv^i) = \frac{\delta}{dt} \left(mc\eta^{-\frac{1}{2}} t \frac{dz^i}{dt} \right); \quad (68)$$

and it can easily be verified that the covariant force vector can be expressed in the Lagrangean form

$$k_{ij} F^j = \frac{d}{dt} \left(\frac{\partial E^*}{\partial v^i} \right) - \frac{\partial E^*}{\partial t^i}, \quad (69)$$

where

$$t^i = ctz^i, \quad E^* = -mc^2\eta^{\frac{1}{2}}, \quad (70)$$

and E^* is regarded as a function of t, t^i, v^i . The invariant E^* here playing the part of energy is not the same as $E = mc^2\eta^{-\frac{1}{2}}$ given by work, but E^* gives $\frac{1}{2}m|v|^2$ approximately as before when $|v|/c$ is small.

The above development following a 3-space definition of work differs essentially from that given by Milne, who defines force and relative velocity as 4-vectors and gives a 4-space definition of work. Equations (63) and (64) do not hold in this case, and θ is taken at the outset to be $\eta^{-\frac{1}{2}}$, so that $dT = ds$, the element of arc in S_4 . Milne's definition of work, however, also yields $\phi(\eta) = \eta^{-\frac{1}{2}}$, and mass M and energy E have the same forms as above. Force as a 3-vector is given by (62) with $\theta = \phi = \eta^{-\frac{1}{2}}$, and it can be verified that the covariant force satisfies

$$k_{ij} F^j = \frac{d}{ds} \left(\eta^{\frac{1}{2}} \frac{\partial E}{\partial t^i} \right) - \frac{\partial E}{\partial t^i}, \quad (71)$$

where $t^i = ctz^i$, $\dot{t}^i = ct dz^i/dt$, and E is the energy $mc^2\eta^{-\frac{1}{2}}$ given by work.

25. Another problem arising from a knowledge of force is to find the relativistic form for the conservation of angular momentum and to relate

angular momentum and force. Here we must generalize the idea of moment about a point and find which vector, involving only t, z^i , plays the part of the position vector in Euclidean space. To do this we first find an integral of motion similar to classical angular momentum.

In classical mechanics the angular momentum of a particle about a point O is conserved if the line of action of the force on the particle passes through O . In space S_3 consider the origin O ($z^i = 0$). Then the lines through O corresponding to straight lines in Euclidean space are the geodesics, and if P is any point we should expect some vector at P tangent to the geodesic OP to correspond to the Euclidean position vector. Let us therefore consider the motion of a particle when the force at P is tangent to the geodesic OP .

In order to deal with the geodesics through O , we return to the particular coordinate system for which the metric of S_3 is given by (16). Then it is easily verified that at the point $P(z)$ a vector tangent to the geodesic OP is z^i , and we therefore consider the motion when $F^i = \mu z^i$ for any μ . It is convenient here to use the well-known tensor ϵ_{ijk} , which is $\pm |k|^{\frac{1}{2}}$ when i, j, k are 1, 2, 3 in even or odd order and zero when two suffixes are equal; this tensor gives what corresponds to the vector product in Cartesian vector analysis. The statement $F^i = \mu z^i$ is now seen to be equivalent to $\epsilon_{ijk} z^j F^k = 0$, and, returning to general θ and ϕ , we see that the motion satisfies

$$\epsilon_{ijk} z^j \frac{\delta p^k}{dT} = 0. \quad (72)$$

Regarding z^i as a vector at the point (z) in S_3 , we have, from (48),

$$\delta z^i = dz^i + \Gamma_{jk}^i(z) z^j dz^k,$$

where, from (16),

$$\Gamma_{jk}^i = -\frac{1}{2} K (1 + K |z|^2/4)^{-1} (\delta_j^i z^k + \delta_k^i z^j - \delta_k^j z^i), \quad (73)$$

whence

$$\frac{\delta z^i}{dT} = \frac{1 - K |z|^2/4}{1 + K |z|^2/4} \frac{dz^i}{dT},$$

i.e. $\delta z^i/dT$ is in the direction p^i . Also it is well known that the covariant derivative of ϵ_{ijk} for the connection Γ_{jk}^i is zero. Hence, identically,

$$\epsilon_{ijk} z^j \frac{\delta p^k}{dT} = \frac{\delta}{dT} (\epsilon_{ijk} z^j p^k), \quad (74)$$

and from (72) the vector $\epsilon_{ijk} z^j p^k$ is conserved in the sense that its tensor

derivative along the path is zero†. This, then, is the result corresponding to the classical conservation of angular momentum.

It appears from this example that the covariant vector $\epsilon_{ijk} z^j p^k$ corresponds to angular momentum, but we at once notice a significant point. The dimensions of this vector are those of momentum instead of angular momentum, arising from the fact that z^i is not a distance vector. Remembering that in the particular coordinate system now being used the quantities ctz^i are the Cartesian coordinates at time t when the z 's are small, we are led to consider as the position vector at time t not z^i but P^i , where

$$P^i = ctz^i \quad (75)$$

for all z 's. The vector corresponding to angular momentum is now $\epsilon_{ijk} P^j p^k$, and the vector which is conserved is not angular momentum but $t^{-1} \times$ angular momentum.

In consequence of the above discussion we now define the moment about O of a vector λ^i at $P(z)$ at time t as the covariant vector $m_{ij} \lambda^j$, where

$$m_{ij} = \epsilon_{ikj} P^k = -ct\epsilon_{ijk} z^k, \quad (76)$$

(z) being the coordinate system for which the metric of S_3 is (16). Here P^i is to be regarded as a vector defined by O and P , and this can be demonstrated by expressing z^i in terms of the geodesic distance OP and the unit vector at P tangent to the geodesic. In transforming to any other coordinate system or from one observer to another, P^i must transform as a vector of the second kind. With this meaning of P^i , we see that, although dependent upon a particular point O , i.e. a particular observer, m_{ij} describes a physical object except that the sense is not invariant. From (27) we find that, in transforming from one observer to another,

$$\epsilon_{ijk}(z) = \pm \epsilon_{lmn}(z') \frac{\partial z'^l}{\partial z^i} \frac{\partial z'^m}{\partial z^j} \frac{\partial z'^n}{\partial z^k},$$

and the ambiguity in sign arises from the fact that an observer can choose either a right-handed or left-handed set of axes, both of which are included in the rotation group of transformations.

With the above definition, the angular momentum about O is $m_{ij} p^j$ and the angular force is $m_{ij} F^j$. From (62), (74), and (76) we have at once

$$\frac{1}{t} m_{ij} F^j = \frac{\delta}{\delta T} \left(\frac{1}{t} m_{ij} p^j \right), \quad (77)$$

† Geometrically, the vector is displaced by Levi-Civita parallel transport along the path in S_3 .

relating angular momentum and force. When the force is zero or tangent to the geodesic from O , $m_{ij} F^j = 0$, whence

$$\frac{\delta}{dT} \left(\frac{1}{t} m_{ij} p^j \right) = 0. \quad (78)$$

Another interesting form of equation (77), easily verified by calculating the derivative on the right and using (73), is

$$\left(1 + \frac{K|z|^2}{4} \right) \frac{1}{t} m_{ij} F^j = \frac{d}{dT} \left\{ \left(1 + \frac{K|z|^2}{4} \right) \frac{1}{t} m_{ij} p^j \right\}. \quad (79)$$

Hence, when the force is zero or tangent to the geodesic from O ,

$$\left(1 + \frac{K|z|^2}{4} \right) \frac{1}{t} m_{ij} p^j = \text{constant}. \quad (80)$$

This is not an invariant statement, holding for all coordinate systems, but is true only in the particular system for which the metric of S_3 is (16).

Kinematical and dynamical objects with τ -time.

26. In the above discussion the time kept by the observers is public kinematic time t and objects are presented in the map S_3 , metric

$$de^2 = k_{ij}(z) dz^i dz^j.$$

If now the observers keep public dynamic time τ , related to t by (10), the appropriate map as mentioned in § 6 is Σ_3 with metric $d\epsilon^2 = c^2 t_0^2 de^2$, and corresponding to any coordinate system z^i in S_3 there is a convenient system ζ^i in Σ_3 , where $\zeta^i = ct_0 z^i$. The two maps are thus related by the equation

$$\tau = t_0 + t_0 \log(t/t_0), \quad \zeta^i = ct_0 z^i, \quad d\epsilon = ct_0 de, \quad (81)$$

and it is a simple matter to pass from the description of an object in S_3 with time t to that in Σ_3 with time τ . Results will be expressed in tensor form in Σ_3 and will therefore apply to all coordinate systems in this space.

From (81) we see that the metric of Σ_3 is

$$d\epsilon^2 = k_{ij}(\zeta) d\zeta^i d\zeta^j, \quad (82)$$

where $k_{ij}(\zeta)$ is not the same function as $k_{ij}(z)$ but is numerically equal to it because $\zeta^i = ct_0 z^i$. We use the same symbol k_{ij} to denote the fundamental tensors in S_3 and Σ_3 , remembering that the derivative with respect

to ζ^i is not the same as that with respect to z^i . For example, the Christoffel symbols in S_3 and Σ_3 are Γ_{jk}^i and $\Gamma_{jk}^{\prime i}$ respectively, where

$$\Gamma_{jk}^{\prime i} = \frac{1}{ct_0} \Gamma_{jk}^i. \quad (83)$$

The dimensionless invariant η given in (44) can now be expressed in terms of τ , ζ^i . Defining $d\sigma^2$, the metric of Σ_4 , as in § 3, we have

$$d\sigma = (t_0/t) ds,$$

where ds^2 is the metric of S_4 , and we can finally write

$$\eta = 1 - \frac{1}{c^2} k_{ij}(\zeta) \frac{d\zeta^i}{d\tau} \frac{d\zeta^j}{d\tau} = \left(\frac{d\sigma}{d\tau} \right)^2. \quad (84)$$

A physical object is now to be described by a geometrical object in Σ_3 by the rules given in § 11. If a vector object is described by components λ^i in S_3 and λ'^i in Σ_3 , then since $d\zeta^i$ are proportional to dz^i , the components λ'^i are proportional to λ^i , and, since scalar magnitude is given by the same tensor k_{ij} in both spaces, it follows that $\lambda'^i = \lambda^i$. This result can at once be extended to all kinds of tensors, and we deduce that, if an object is described by components $T_{j\dots}^i$ at the point z^i in S_3 , then the same object is described by the same components $T_{j\dots}^i$ at the point $\zeta^i = ct_0 z^i$ in Σ_3 .

If λ^i is a vector defined at points of a curve in S_3 , then the variation along the curve is measured by the vector $\delta\lambda^i$, given by (48). The vector λ^i has the same components in Σ_3 , but the variation is now measured by the vector $\delta'\lambda^i = \delta\lambda^i + \Gamma_{jk}^{\prime i} \lambda^j d\zeta^k$. From (81) and (83), however, we find that $\delta'\lambda^i = \delta\lambda^i$, so that the variation measured in Σ_3 is the same as that in S_3 . We can therefore use the same symbol δ in both cases, the calculation in Σ_3 being with the connection $\Gamma_{jk}^{\prime i}$ and the differentials $d\zeta^i$. For example, we can at once transform (55) to get the free path equations in Σ_3 . From (81), $dt/t = d\tau/t_0$, and the free paths are given by the equations

$$\frac{\delta}{d\tau} \left(\frac{d\zeta^i}{d\tau} \right) = 0. \quad (85)$$

From (57), we see that these paths are given by the variational equation

$$\delta \int W(\eta) d\tau = 0, \quad (86)$$

where η is given by (84).

Many objects have already been described in S_3 , and these same objects can, as we have seen, be described in Σ_3 . With dynamic time τ in place of

kinematic time t , however, time intervals, and hence distances, are measured differently and it is necessary to re-define kinematical and dynamical objects. We find, for example, that force, defined for time t , differs from force defined for time τ . Such differences are demonstrated by comparing the new object described in Σ_3 with the old object defined in S_3 and then described in Σ_3 .

If E, E' are two neighbouring events $(\tau, \zeta), (\tau+d\tau, \zeta+d\zeta)$ relative to the observer A_0 ($\zeta^i = 0$), then the distance of E' from the observer at E is, from (15), $dl' = ct_0 de = d\epsilon$, and, as in § 18, the distance-displacement vector is

$$dl'^i = d\zeta^i. \quad (87)$$

This is the vector for time τ , but from (43) the vector for time t is

$$dl^i = ctdz^i = (t/t_0)d\zeta^i.$$

The two displacement vectors, one for each time, are not therefore the same object but are related in the same space (S_3 , or Σ_3) by

$$dl'^i = (t_0/t)dl^i. \quad (88)$$

With time t the time-interval between E and E' is $dT = dt/\theta(\eta)$. With time τ , it is natural to take dT' as time-interval, where dT' is related to dT as $d\tau$ is related to dt , so that

$$dT' = d\tau/\theta(\eta) = (t_0/t)dT, \quad (89)$$

where η , the same invariant as before, is now given by (84).

The velocity for dynamic time is now defined as dl^i/dT' , but from (88) and (89) this is $dl^i/dT = v^i$, the velocity for kinematic time. Thus the same object, velocity, is given in the same way for each time, and we can use the one symbol v^i to denote velocity for both times and in both spaces.

Acceleration for time τ , however, is

$$\frac{\delta v^i}{dT'} = \frac{t}{t_0} \frac{\delta v^i}{dT} = \frac{t}{t_0} \dot{v}^i,$$

thus differing from $\dot{v}^i$, the acceleration for time t . It can be verified that the results given at the end of § 19 still hold in form, the geodesics in Σ_3 corresponding to straight lines in Euclidean space. From the relation $d\epsilon = ct_0 de$, it is at once seen that the geodesics in S_3 transform into the geodesics in Σ_3 .

28. From the discussion in § 22 and from the form of (58) we see that the proper and relative masses of a particle are not affected by a

reg graduation of the observers' clocks, so that m is still the proper mass, $M = m\phi(\eta)$ the relative mass, and $E = Mc^2$ the energy. The momentum is now Mv^i as before, whence this object is the same for both times. We can therefore use the same symbol p^i in both cases, and from (61) we deduce that, for dynamic time,

$$p^i = Mv^i = m\phi\theta \frac{d\zeta^i}{d\tau}. \quad (90)$$

With time τ the force F'^i , defined as rate of change of momentum, is

$$F'^i = \frac{\delta}{dT'} (p^i) = \theta \frac{\delta}{d\tau} \left(m\phi\theta \frac{d\zeta^i}{d\tau} \right). \quad (91)$$

Comparing with (62) we see from (89) that force for time τ is not the same object as force for time t , the relation being

$$F'^i = (t/t_0) F^i. \quad (92)$$

The work done by the force on a particle in a small interval is now given by $k_{ij} F'^i dl'^j$ and from (88) and (92) we find

$$dW = k_{ij} F^i dl^j = k_{ij} F'^i dl'^j,$$

showing that work, defined in this way, is the same for both times. Equating dW to $c^2 dM$ we find, as before, that $\theta/\eta \equiv 1$ and $\phi(\eta) \equiv \eta^{-\frac{1}{2}}$, so that from (89) the time interval is now $d\tau$. With these forms for θ and ϕ we find, as in (69), that the covariant force can be expressed in the Lagrangean form

$$k_{ij} F'^j = \frac{d}{d\tau} \left(\frac{\partial E^*}{\partial v^i} \right) - \frac{\partial E^*}{\partial \zeta^i}, \quad (93)$$

where $E^* = -mc^2\eta^{\frac{1}{2}}$ expressed as a function of ζ^i and $v^i = d\zeta^i/d\tau$ from (84).

In Milne's work, $\theta = \phi = \eta^{-\frac{1}{2}}$, and the dynamic time-interval is therefore $dT' = \eta^{\frac{1}{2}} d\tau = d\sigma$, the element of arc in Σ_4 . Substituting in the above equations we get formulae equivalent to those given by Milne, except in connection with work, which is defined differently.

29. In defining the moment of a vector about the point O , we turn to the particular coordinate system ζ^i for which the metric of Σ_3 takes the form (19). The relations between ζ^i and the coordinates z^i in § 25 are still $\zeta^i = ct_0 z^i$, and the identity (74) can now be written

$$\epsilon_{ijk} \zeta^j \frac{\delta p^k}{dT'} = \frac{\delta}{dT'} (\epsilon_{ijk} \zeta^j p^k), \quad (94)$$

where ϵ_{ijk} , the tensor defined in § 25 for S_3 , is also given in the same way for Σ_3 , since $|k|$ is the same for both spaces.

We are now led to consider $\epsilon_{ijk}\zeta^i\lambda^k$ as the moment of λ^i at the point $P(\zeta)$ about O , and here the dimensions are correct, since from (81) the ζ 's are lengths. Further the ζ 's, when small, are the actual Cartesian coordinates about O . We therefore take as the position vector at the point P relative to O the vector P'^i with components

$$P'^i = \zeta^i \quad (95)$$

in the particular coordinate system now considered, with the understanding that P'^i transforms as a vector of the second kind in passing to any other coordinate system or observer. As before, P'^i can be expressed in an invariant form in terms of the geodesic distance OP in Σ_3 and the unit tangent at P to the geodesic. The position vector P'^i for time τ differs from the position vector P^i for time t , the relation being, from (75),

$$P'^i = (t_0/t) P^i, \quad (96)$$

when both vectors are described in the same space.

The moment about O of a vector λ^i at $P(\zeta)$ is now defined to be the covariant vector $m'_{ij}\lambda^j$, where

$$m'_{ij} = \epsilon_{ikj} P'^k = -\epsilon_{ijk} \zeta^k = (t_0/t) m_{ij}. \quad (97)$$

The angular momentum is thus

$$m'_{ij} p^j = (t_0/t) m_{ij} p^j, \quad (98)$$

differing from the angular momentum $m_{ij} p^j$ for time t , and the angular force is

$$m'_{ij} F'^j = m_{ij} F^j, \quad (99)$$

from (92), *i.e.* is the same as for time t . From (94) the relation between angular momentum and force is

$$m'_{ij} F'^j = \frac{\delta}{\delta T'} (m'_{ij} p^j). \quad (100)$$

We now see that when the force is zero or tangent to the geodesic from O , $m'_{ij} F'^j = 0$ and the angular momentum is conserved in the sense that

$$\frac{\delta}{\delta T'} (m'_{ij} p^j) = 0. \quad (101)$$

Time does not enter here explicitly, as in (78), and it is this result with others which justifies the name "dynamic time" applied to τ . With time τ the coordinates ζ^i are lengths and the space Σ_3 plays the part of Euclidean space, the dynamical equations referred to this space and time being very similar to those in classical mechanics. It can be verified that no other clock regraduation of the form (9) will yield a system of mechanics in which the equations of dynamics so closely resemble the classical forms.

Corresponding to (79), equations (100) can be written in the form

$$\left(1 + \frac{K|\zeta|^2}{4c^2 t_0^2}\right) m'_{ij} F'^j = \frac{d}{dT'} \left\{ \left(1 + \frac{K|\zeta|^2}{4c^2 t_0^2}\right) m'_{ij} p^j \right\}, \quad (102)$$

and when the force is zero or tangent to the geodesic from O ,

$$\left(1 + \frac{K|\zeta|^2}{4c^2 t_0^2}\right) m'_{ij} p^j = \text{constant} \quad (103)$$

in the particular coordinate system for which the metric of Σ_3 is (19).

Summary.

30. This concludes the first two parts of the subject. In the first part we considered the various maps suitable for the description of physical objects. In the second part we defined and related, within the framework of three-dimensional public space, such objects as are found in classical kinematics and dynamics. There are two schemes of mechanics, one for kinematic time t and the other for dynamic time τ , and there is a 3-space map appropriate to each. In each scheme the time and 3-space play the parts of time and Euclidean space of classical mechanics, and the various physical objects discussed here closely resemble the classical concepts, particularly for dynamic time τ . We have only been concerned with the definitions and relations between these objects, but the equations given in this paper can be applied to many classical problems of kinematics and dynamics.

To complete this work it is necessary to describe the objects in four dimensional space-time so that they can be expressed in terms of any other coordinate systems in 4-space. This will be done in Part III, to be published later, where it is shown how the results derived here compare with those given by Milne. This part will also include a detailed discussion of potential and conservative fields of force, and it will be seen that our point of view differs essentially from that adopted by Milne.

In conclusion I wish to acknowledge my indebtedness to Prof. E. A. Milne for the help derived from his work and from many conversations, and for the valuable criticisms of a rough draft of the present paper.

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CANONICAL MATRIX ROOTS OF EQUATIONS

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[Received 2 February, 1939.—Read 16 February, 1939.]

1. Many matrices can be found to satisfy a given algebraic equation of degree n with coefficients in a field R . These matrices may be of different orders, and their elements may or may not be rational in R . Further, a set of n conjugate matrix solutions may be found which are commutative, are all of the same order (not necessarily n), and are such that their elementary symmetric functions are scalar matrices which are equal to the corresponding symmetric functions of the algebraic roots of the given equation.

A. R. Richardson* has shown that, if the elements of such a set of conjugate matrices are to be rational in R , then the smallest possible order S , of all matrices in the set, is that of the Galois group of the algebraic equation in R . These matrices are transformed into each other by a group of matrices, each of order S , simply isomorphic with the Galois group. In this note we choose canonical matrix forms for this transforming group and derive corresponding canonical matrix roots of certain types of algebraic equation. It is found that the irreducible representations of the Galois group give rise to simple canonical matrix roots, these being rational in the field R enlarged, when necessary, by adjoining the group characters of the Galois group.

The problem of determining these canonical forms* was suggested to me by Prof. A. R. Richardson, who has also been very helpful at various stages of the work, and to whom I wish to express my indebtedness.

2. *The quadratic equation.*

For the equation

$$x^2 - a_1x + a_2 \equiv (x - x_1)(x - x_2) = 0,$$

we note that

$$X_1 = \begin{bmatrix} x_1 & \cdot \\ \cdot & x_2 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} x_2 & \cdot \\ \cdot & x_1 \end{bmatrix}$$

may be taken as commutative matrix roots which are transformed into one another by the permutation matrix

$$P_{12} = \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix},$$

which, with its square, constitutes the Galois group of an irreducible quadratic equation. To find the rational roots, which by the general theory are known to exist, we transform X_1 and X_2 by the matrix which leaves P_{12} unaltered by transformation. This matrix is

$$\theta = \begin{bmatrix} a & \beta \\ \beta & a \end{bmatrix},$$

where $a \neq \beta$, and

$$\theta^{-1} = \frac{1}{a^2 - \beta^2} \begin{bmatrix} a & -\beta \\ -\beta & a \end{bmatrix},$$

giving

$$\theta X_1 \theta^{-1} = \frac{1}{a^2 - \beta^2} \begin{bmatrix} a^2 x_1 - \beta^2 x_2 & -a\beta(x_1 - x_2) \\ a\beta(x_1 - x_2) & a^2 x_2 - \beta^2 x_1 \end{bmatrix},$$

in which a and β have to be chosen so that the elements of the matrix are rational.

When $a = x_1$, $\beta = x_2$, then

$$\theta X_1 \theta^{-1} = \frac{1}{a_1} \begin{bmatrix} a_1^2 - a_2 & -a_2 \\ a_2 & a_2 \end{bmatrix}, \quad \theta X_2 \theta^{-1} = \frac{1}{a_1} \begin{bmatrix} a_2 & a_2 \\ -a_2 & a_1^2 - a_2 \end{bmatrix}.$$

These roots have to be modified when $a_1 = 0$. We can write $x^2 + a_2 = 0$ as $(y - a)^2 + a_2 = 0$, where a is a non-zero parameter, and we then find that the two matrix values of x are

$$\frac{1}{2a} \begin{bmatrix} a^2 - a_2 & -(a^2 + a_2) \\ a^2 + a_2 & -(a^2 - a_2) \end{bmatrix}, \quad \frac{1}{2a} \begin{bmatrix} -(a^2 - a_2) & a^2 + a_2 \\ -(a^2 + a_2) & a^2 - a_2 \end{bmatrix}.$$

3. We can obtain canonical roots corresponding to the irreducible representation

$$\begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix} = T_{12}$$

of the symmetric group of order two by transforming X_1 and X_2 by the general matrix which transforms P_{12} into T_{12} . This matrix is

$$\psi = \begin{bmatrix} a & a \\ \beta & -\beta \end{bmatrix} = \begin{bmatrix} a & . \\ . & \beta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

where

$$\psi^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha^{-1} & . \\ . & \beta^{-1} \end{bmatrix},$$

and a, β are non-zero.

The canonical roots are then

$$\psi X_1 \psi^{-1} = \frac{1}{2} \begin{bmatrix} a_1 & \frac{a}{\beta} \sqrt{\Delta} \\ \frac{\beta}{a} \sqrt{\Delta} & a_1 \end{bmatrix}, \quad \psi X_2 \psi^{-1} = \frac{1}{2} \begin{bmatrix} a_1 & -\frac{a}{\beta} \sqrt{\Delta} \\ -\frac{\beta}{a} \sqrt{\Delta} & a_1 \end{bmatrix},$$

where $\Delta = a_1^2 - 4a_2$. For $a/\beta = \sqrt{\Delta}$, these are rational and agree with those given by A. R. Richardson†.

4. The cyclic cubic equation.

The permutation group of the cyclic cubic equation is 1, (123), (132), and the corresponding matrix representation consists of the powers of

$$P_{123} = \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix}.$$

As commutative matrix roots of

$$x^3 - a_1 x^2 + a_2 x - a_3 \equiv (x - x_1)(x - x_2)(x - x_3) = 0$$

we may take

$$X_1 = \begin{bmatrix} x_1 & . & . \\ . & x_2 & . \\ . & . & x_3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 & . & . \\ . & x_3 & . \\ . & . & x_1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 & . & . \\ . & x_1 & . \\ . & . & x_2 \end{bmatrix}.$$

These are permuted cyclically when transformed by P_{123} . To find the rational solutions, which are known to be of order three when the Galois

† *Loc. cit.*, 268.

group is cyclical, we transform X_1, X_2, X_3 by the general matrix which leaves P_{123} unaltered by transformation. This matrix is

$$\theta = \begin{bmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix}, \quad \text{where} \quad \theta^{-1} = \frac{1}{|\theta|} \begin{bmatrix} \alpha^2 - \beta\gamma & \beta^2 - \alpha\gamma & \gamma^2 - \alpha\beta \\ \gamma^2 - \alpha\beta & \alpha^2 - \beta\gamma & \beta^2 - \alpha\gamma \\ \beta^2 - \alpha\gamma & \gamma^2 - \alpha\beta & \alpha^2 - \beta\gamma \end{bmatrix}$$

and $|\theta| = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma \neq 0$.

We thus obtain

$$\theta X_1 \theta^{-1} = \frac{1}{|\theta|} \begin{bmatrix} \alpha(\alpha^2 - \beta\gamma)x_1 + \gamma(\gamma^2 - \alpha\beta)x_2 + \beta(\beta^2 - \alpha\gamma)x_3 \\ \beta(\alpha^2 - \beta\gamma)x_1 + \alpha(\gamma^2 - \alpha\beta)x_2 + \gamma(\beta^2 - \alpha\gamma)x_3 \\ \gamma(\alpha^2 - \beta\gamma)x_1 + \beta(\gamma^2 - \alpha\beta)x_2 + \alpha(\beta^2 - \alpha\gamma)x_3 \\ \alpha(\beta^2 - \alpha\gamma)x_1 + \gamma(\alpha^2 - \beta\gamma)x_2 + \beta(\gamma^2 - \alpha\beta)x_3 \\ \beta(\beta^2 - \alpha\gamma)x_1 + \alpha(\alpha^2 - \beta\gamma)x_2 + \gamma(\gamma^2 - \alpha\beta)x_3 \\ \gamma(\beta^2 - \alpha\gamma)x_1 + \beta(\alpha^2 - \beta\gamma)x_2 + \alpha(\gamma^2 - \alpha\beta)x_3 \\ \alpha(\gamma^2 - \alpha\beta)x_1 + \gamma(\beta^2 - \alpha\gamma)x_2 + \beta(\alpha^2 - \beta\gamma)x_3 \\ \beta(\gamma^2 - \alpha\beta)x_1 + \alpha(\beta^2 - \alpha\gamma)x_2 + \gamma(\alpha^2 - \beta\gamma)x_3 \\ \gamma(\gamma^2 - \alpha\beta)x_1 + \beta(\beta^2 - \alpha\gamma)x_2 + \alpha(\alpha^2 - \beta\gamma)x_3 \end{bmatrix}.$$

We can obtain rational values by taking $\alpha = x_1, \beta = x_3, \gamma = x_2$. For these special values,

$$\theta X_1 \theta^{-1} = \frac{1}{a_1 H} \begin{bmatrix} (a_1^2 - a_2)H - H_2 & \frac{1}{2}(-a_2 H + H_2 + a_1 \sqrt{\Delta}) & \frac{1}{2}(-a_2 H + H_2 - a_1 \sqrt{\Delta}) \\ \frac{1}{2}(a_2 H + H_2 + a_1 \sqrt{\Delta}) & \frac{1}{2}(a_2 H + H_2 - a_1 \sqrt{\Delta}) & -H_2 \\ \frac{1}{2}(a_2 H + H_2 - a_1 \sqrt{\Delta}) & -H_2 & \frac{1}{2}(a_2 H + H_2 + a_1 \sqrt{\Delta}) \end{bmatrix},$$

where $H = a_1^2 - 3a_2, \quad H_2 = a_2^2 - 3a_1 a_3,$

and

$$\Delta = \{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)\}^2 = 18a_1 a_2 a_3 + a_1^2 a_2^2 - 4a_1^3 a_3 - 27a_3^2 - 4a_2^3.$$

When $a_1 = 0$ we obtain solutions from those of*

$$(y - a)^3 + a_2(y - a) - a_3 = 0,$$

* H cannot be zero for a cyclic cubic, since this makes $\Delta = -\frac{1}{27}(a_1^3 - 27a_3)^2$, which cannot be a square.

where a is a non-zero parameter. This method gives

$$\frac{1}{9aa_2} \begin{bmatrix} 6a^2a_2 - 2a_2^2 - 9aa_3 & \frac{1}{2}(-6a^2a_2 - 4a_2^2 + 9aa_3 - 3a\sqrt{\Delta}) & \frac{1}{2}(-6a^2a_2 - 4a_2^2 + 9aa_3 + 3a\sqrt{\Delta}) \\ \frac{1}{2}(12a^2a_2 + 2a_2^2 + 9aa_3 - 3a\sqrt{\Delta}) & \frac{1}{2}(-6a^2a_2 + 2a_2^2 + 9aa_3 + 3a\sqrt{\Delta}) & -3a^2a_2 + a_2^2 - 9aa_3 \\ \frac{1}{2}(12a^2a_2 + 2a_2^2 + 9aa_3 + 3a\sqrt{\Delta}) & -3a^2a_2 + a_2^2 - 9aa_3 & \frac{1}{2}(-6a^2a_2 + 2a_2^2 + 9aa_3 - 3a\sqrt{\Delta}) \end{bmatrix}$$

as one of the roots of $x^3 + a_2x - a_3 = 0$.

As a numerical example†, take $x^3 + x^2 - 2x - 1 = 0$. The roots are

$$\begin{bmatrix} -2 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

and for $27y^3 - 63y - 7 = 0$, obtained by substituting $x = y - \frac{1}{3}$, the roots are

$$\frac{1}{3} \begin{bmatrix} -5 & -3 & -6 \\ 3 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} 1 & 3 & 3 \\ 3 & 4 & 0 \\ -3 & -6 & -5 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} 4 & 0 & 3 \\ -6 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

5. The irreducible representation of P_{123} is

$$T_{123} = \begin{bmatrix} 1 & . & . \\ . & \omega & . \\ . & . & \omega^2 \end{bmatrix},$$

where ω and ω^2 are the complex cube roots of unity. To find a canonical set of roots corresponding to this representation we transform X_1, X_2, X_3 by the general matrix which transforms P_{123} into T_{123} . This is

$$\psi = \begin{bmatrix} a & a & a \\ \beta & \omega^2\beta & \omega\beta \\ \gamma & \omega\gamma & \omega^2\gamma \end{bmatrix} = \begin{bmatrix} a & . & . \\ . & \beta & . \\ . & . & \gamma \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix},$$

† *Loc. cit.*, 259.

where

$$\psi^{-1} = \frac{1}{3} \begin{bmatrix} \alpha^{-1} & \beta^{-1} & \gamma^{-1} \\ \alpha^{-1} & \omega\beta^{-1} & \omega^2\gamma^{-1} \\ \alpha^{-1} & \omega^2\beta^{-1} & \omega\gamma^{-1} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} \alpha^{-1} & . & . \\ . & \beta^{-1} & . \\ . & . & \gamma^{-1} \end{bmatrix},$$

and α, β, γ are non-zero. We can omit α, β, γ from our transformation, and reintroduce them later if required by multiplication and division of the appropriate rows and columns. We thus obtain the following set of canonical roots:—

$$\frac{1}{3} \begin{bmatrix} a_1 & \lambda_1 & \lambda_2 \\ \lambda_2 & a_1 & \lambda_1 \\ \lambda_1 & \lambda_2 & a_1 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} a_1 & \omega^2\lambda_1 & \omega\lambda_2 \\ \omega\lambda_2 & a_1 & \omega^2\lambda_1 \\ \omega^2\lambda_1 & \omega\lambda_2 & a_1 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} a_1 & \omega\lambda_1 & \omega^2\lambda_2 \\ \omega^2\lambda_2 & a_1 & \omega\lambda_1 \\ \omega\lambda_1 & \omega^2\lambda_2 & a_1 \end{bmatrix},$$

where $\lambda_1 = x_1 + \omega x_2 + \omega^2 x_3$, $\lambda_2 = x_1 + \omega^2 x_2 + \omega x_3$.

These may be transformed into matrices having their elements rational in the field formed by adjoining the group characteristic ω to the field of the coefficients. The transforming matrix is

$$\begin{bmatrix} 1 & . & . \\ . & \lambda_2^{-1} & . \\ . & . & \lambda_1^{-1} \end{bmatrix}$$

and the three roots are then

$$\frac{1}{3} \begin{bmatrix} a_1 & H & H \\ 1 & a_1 & \frac{1}{2H} \{G - \sqrt{(-27\Delta)}\} \\ 1 & \frac{1}{2H} \{G + \sqrt{(-27\Delta)}\} & a_1 \end{bmatrix},$$

$$\frac{1}{3} \begin{bmatrix} a_1 & \omega^2 H & \omega H \\ \omega & a_1 & \frac{\omega^2}{2H} \{G - \sqrt{(-27\Delta)}\} \\ \omega^2 & \frac{\omega}{2H} \{G + \sqrt{(-27\Delta)}\} & a_1 \end{bmatrix},$$

$$\frac{1}{3} \begin{bmatrix} a_1 & \omega H & \omega^2 H \\ \omega^2 & a_1 & \frac{\omega}{2H} \{G - \sqrt{(-27\Delta)}\} \\ \omega & \frac{\omega^2}{2H} \{G + \sqrt{(-27\Delta)}\} & a_1 \end{bmatrix},$$

where $G = 27a_3 - 9a_1a_2 + 2a_1^3$.

6. From the cases already considered, it appears that the matrix roots associated with the irreducible representations are of a more convenient form than are those pertaining to the permutation representations. In the following cases we shall deal only with the rational matrix roots associated with the irreducible representations. Whenever the group characters are irrational in the field of the coefficients of the equation, the field is enlarged by adjoining the characters.

7. *The symmetric cubic equation.*

The irreducible representations of the symmetric group on three letters may be obtained from those of the cyclic group by replacing the unit matrix of order three by the unit matrix of order six, replacing unity in T_{123} and T_{132} by the unit matrix of order two and finally giving ω and ω^2 their expressions as matrix roots of $\omega^2 + \omega + 1 = 0$. The three sixth order matrices so obtained are then combined with a matrix representation of T_{23} which consists of positive and negative units alternately in the diagonal. The group is thus generated by

$$T_{123} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & & & \\ & & & -\frac{1}{2} & -\frac{3}{2} & \\ & & & \frac{1}{2} & -\frac{1}{2} & \\ & & & & & -\frac{1}{2} & \frac{3}{2} \\ & & & & & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and $T_{23} = \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & 1 & \\ & & & & & & -1 \end{bmatrix}.$

The corresponding solution of the cubic is then written as

$$z_1 = \frac{1}{3} \begin{bmatrix} a_1 & . & \lambda_1 & . & \lambda_2 & . \\ . & a_1 & . & \lambda_1 & . & \lambda_2 \\ \lambda_2 & . & a_1 & . & \lambda_1 & . \\ . & \lambda_2 & . & a_1 & . & \lambda_1 \\ \lambda_1 & . & \lambda_2 & . & a_1 & . \\ . & \lambda_1 & . & \lambda_2 & . & a_1 \end{bmatrix}.$$

This can be expressed as a matrix rational in the field of the coefficients of the cubic, since the group characters are rational for the symmetric group. To find such a rational form we transform z_1 by the matrix which transforms each of the T 's into itself, viz.,

$$\phi = \begin{bmatrix} \alpha & & & & & \\ & \beta & & & & \\ & & \gamma & & \epsilon & \\ & & & \gamma & & -\epsilon \\ & & & & \zeta & \delta \\ & & & & -\zeta & \delta \end{bmatrix},$$

where $\phi^{-1} = \begin{bmatrix} \alpha^{-1} & & & & & \\ & \beta^{-1} & & & & \\ & & \delta M^{-1} & & -\epsilon M^{-1} & \\ & & & \delta M^{-1} & & \epsilon M^{-1} \\ & & -\zeta M^{-1} & & \gamma M^{-1} & \\ & & & \zeta M^{-1} & & \gamma M^{-1} \end{bmatrix},$

in which $M = \gamma\delta - \epsilon\zeta$, and, like α and β , is non-zero. We thus obtain

$$\phi z_1 \phi^{-1} =$$

$$\left[\begin{array}{ccc} a_1 & . & \frac{\alpha}{M} (-\zeta\lambda_2 + \delta\lambda_1) & . & \frac{\alpha}{M} (\gamma\lambda_2 - \epsilon\lambda_1) & . \\ . & a_1 & . & \frac{\beta}{M} (\zeta\lambda_2 + \delta\lambda_1) & . & \frac{\beta}{M} (\gamma\lambda_2 + \epsilon\lambda_1) \\ \hline \frac{1}{\alpha} (\gamma\lambda_2 + \epsilon\lambda_1) & . & a_1 + \frac{1}{M} (\delta\epsilon\lambda_2 - \gamma\zeta\lambda_1) & . & \frac{1}{M} (-\epsilon^2\lambda_2 + \gamma^2\lambda_1) & . \\ . & \frac{1}{\beta} (\gamma\lambda_2 - \epsilon\lambda_1) & . & a_1 - \frac{1}{M} (\delta\epsilon\lambda_2 - \gamma\zeta\lambda_1) & . & \frac{1}{M} (-\epsilon^2\lambda_2 + \gamma^2\lambda_1) \\ \hline \frac{1}{\alpha} (\zeta\lambda_2 + \delta\lambda_1) & . & \frac{1}{M} (\delta^2\lambda_2 - \zeta^2\lambda_1) & . & a_1 - \frac{1}{M} (\delta\epsilon\lambda_2 - \gamma\zeta\lambda_1) & . \\ . & \frac{1}{\beta} (-\zeta\lambda_2 + \delta\lambda_1) & . & \frac{1}{M} (\delta^2\lambda_2 - \zeta^2\lambda_1) & . & a_1 + \frac{1}{M} (\delta\epsilon\lambda_2 - \gamma\zeta\lambda_1) \end{array} \right],$$

in which $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ have to be given values such that every element of the matrix is rational. Suitable values are $\alpha = 1$, $\beta = \lambda_2^3 - \lambda_1^3$, $\gamma = \lambda_2^2$, $\delta = \lambda_2$, $\epsilon = \lambda_1^2$, $\zeta = \lambda_1$, giving $M = \lambda_2^3 - \lambda_1^3$, and

$$X_1 = \phi z_1 \phi^{-1} = \frac{1}{3} \left[\begin{array}{ccc} a_1 & . & 0 & . & 1 & . \\ . & a_1 & . & 2H & . & G \\ \hline G & . & a_1 & . & H & . \\ . & 1 & . & a_1 & . & H \\ \hline 2H & . & 1 & . & a_1 & . \\ . & 0 & . & 1 & . & a_1 \end{array} \right],$$

from which

$$X_2 = T_{12} X_1 T_{12}^{-1} = \frac{1}{3} \left[\begin{array}{ccc} a_1 & . & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \\ . & a_1 & -H & -H & \frac{1}{2}G & -\frac{1}{2}G \\ \hline -\frac{1}{2}G & -\frac{3}{2} & a_1 & . & -\frac{1}{2}H & \frac{3}{2}H \\ \frac{1}{2}G & -\frac{1}{2} & . & a_1 & -\frac{1}{2}H & -\frac{1}{2}H \\ \hline -H & 0 & -\frac{1}{2} & -\frac{3}{2} & a_1 & . \\ -H & 0 & \frac{1}{2} & -\frac{1}{2} & . & a_1 \end{array} \right]$$

and

$$X_3 = T_{23} X_2 T_{23}^{-1} = \frac{1}{3} \begin{bmatrix} a_1 & . & 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ . & a_1 & H & -H & -\frac{1}{2}G & -\frac{1}{2}G \\ -\frac{1}{2}G & \frac{3}{2} & a_1 & . & -\frac{1}{2}H & -\frac{3}{2}H \\ -\frac{1}{2}G & -\frac{1}{2} & . & a_1 & \frac{1}{2}H & -\frac{1}{2}H \\ -H & 0 & -\frac{1}{2} & \frac{3}{2} & a_1 & . \\ H & 0 & -\frac{1}{2} & -\frac{1}{2} & . & a_1 \end{bmatrix}.$$

Other rational solutions may be obtained from these by transforming with rational forms of ϕ , or with any other matrices which interchange the T 's among themselves.

8. Quartic equations.

There are five types of irreducible quartic equation, corresponding to the five transitive permutation groups on four letters. These give rise to two sets of conjugate matrix roots of order four and one set each of orders eight, twelve, and twenty-four. We consider here the three groups of smallest order.

9. The doubly quadratic equation.

When the equation

$$x^4 - a_1 x^3 + a_2 x^2 - a_3 x + a_4 = (x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0$$

has 1, (12)(34), (13)(24), (14)(23) as its Galois group, it breaks up into two quadratics in three different quadratic fields $[\sqrt{l}]$, $[\sqrt{m}]$, $[\sqrt{lm}]$. The three quantities l, m, lm , which are here taken to be free of square factors, are given by the three rational roots $\phi_1 = a^2 l$, $\phi_2 = b^2 m$, $\phi_3 = c^2 lm$ of Euler's cubic

$$\phi^3 + 3H\phi^2 + \frac{1}{4}(12H^2 - I)\phi - \frac{1}{4}G^2 = 0,$$

in which

$$H = \frac{1}{48}(8a_2 - 3a_1^2), \quad G = \frac{1}{32}(-8a_3 + 4a_1 a_2 - a_1^3),$$

$$I = \frac{1}{12}(12a_4 - 3a_1 a_3 + a_2^2).$$

The roots of the quartic are then*

$$x_1 = \frac{1}{4}a_1 + \sqrt{\phi_1} + \sqrt{\phi_2} + \sqrt{\phi_3}, \quad x_2 = \frac{1}{4}a_1 + \sqrt{\phi_1} - \sqrt{\phi_2} - \sqrt{\phi_3},$$

$$x_3 = \frac{1}{4}a_1 - \sqrt{\phi_1} + \sqrt{\phi_2} - \sqrt{\phi_3}, \quad x_4 = \frac{1}{4}a_1 - \sqrt{\phi_1} - \sqrt{\phi_2} + \sqrt{\phi_3}.$$

For the permutation matrix representation of the group, viz., 1, and

$$(12)(34) = \begin{bmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{bmatrix}, \quad (13)(24) = \begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{bmatrix}, \quad (14)(23) = \begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix},$$

we note that

$$\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4 \end{bmatrix}, \quad \begin{bmatrix} x_2 & & & \\ & x_1 & & \\ & & x_4 & \\ & & & x_3 \end{bmatrix}, \quad \begin{bmatrix} x_3 & & & \\ & x_4 & & \\ & & x_1 & \\ & & & x_2 \end{bmatrix}, \quad \begin{bmatrix} x_4 & & & \\ & x_3 & & \\ & & x_2 & \\ & & & x_1 \end{bmatrix}$$

constitute a conjugate set of roots. The matrix which transforms the permutation matrices into the irreducible forms

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$$

is

$$\psi = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & -\beta & -\beta \\ \gamma & -\gamma & \gamma & -\gamma \\ \delta & -\delta & -\delta & \delta \end{bmatrix} = \begin{bmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

* It must be noted here that $\sqrt{(\phi_1 \phi_2 \phi_3)} = -\frac{1}{2}G$.

here $\alpha, \beta, \gamma, \delta$ are non-zero, and

$$\psi^{-1} = \frac{1}{4} \begin{bmatrix} \alpha^{-1} & \beta^{-1} & \gamma^{-1} & \delta^{-1} \\ \alpha^{-1} & \beta^{-1} & -\gamma^{-1} & -\delta^{-1} \\ \alpha^{-1} & -\beta^{-1} & \gamma^{-1} & -\delta^{-1} \\ \alpha^{-1} & -\beta^{-1} & -\gamma^{-1} & \delta^{-1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^{-1} & & & \\ & \beta^{-1} & & \\ & & \gamma^{-1} & \\ & & & \delta^{-1} \end{bmatrix}.$$

We can omit $\alpha, \beta, \gamma, \delta$ from our transformation and reintroduce them when required by transforming by the matrix having $\alpha, \beta, \gamma, \delta$ in the diagonal and zeros elsewhere. By transforming the above roots by ψ we obtain the following set of roots associated with the irreducible representations:—

$$X_1 = \begin{bmatrix} \frac{1}{4}a_1 & \sqrt{\phi_1} & \sqrt{\phi_2} & \sqrt{\phi_3} \\ \sqrt{\phi_1} & \frac{1}{4}a_1 & \sqrt{\phi_3} & \sqrt{\phi_2} \\ \sqrt{\phi_2} & \sqrt{\phi_3} & \frac{1}{4}a_1 & \sqrt{\phi_1} \\ \sqrt{\phi_3} & \sqrt{\phi_2} & \sqrt{\phi_1} & \frac{1}{4}a_1 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} \frac{1}{4}a_1 & \sqrt{\phi_1} & -\sqrt{\phi_2} & -\sqrt{\phi_3} \\ \sqrt{\phi_1} & \frac{1}{4}a_1 & -\sqrt{\phi_3} & -\sqrt{\phi_2} \\ -\sqrt{\phi_2} & -\sqrt{\phi_3} & \frac{1}{4}a_1 & \sqrt{\phi_1} \\ -\sqrt{\phi_3} & -\sqrt{\phi_2} & \sqrt{\phi_1} & \frac{1}{4}a_1 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} \frac{1}{4}a_1 & -\sqrt{\phi_1} & \sqrt{\phi_2} & -\sqrt{\phi_3} \\ -\sqrt{\phi_1} & \frac{1}{4}a_1 & -\sqrt{\phi_3} & \sqrt{\phi_2} \\ \sqrt{\phi_2} & -\sqrt{\phi_3} & \frac{1}{4}a_1 & -\sqrt{\phi_1} \\ -\sqrt{\phi_3} & \sqrt{\phi_2} & -\sqrt{\phi_1} & \frac{1}{4}a_1 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} \frac{1}{4}a_1 & -\sqrt{\phi_1} & -\sqrt{\phi_2} & \sqrt{\phi_3} \\ -\sqrt{\phi_1} & \frac{1}{4}a_1 & \sqrt{\phi_3} & -\sqrt{\phi_2} \\ -\sqrt{\phi_2} & \sqrt{\phi_3} & \frac{1}{4}a_1 & -\sqrt{\phi_1} \\ \sqrt{\phi_3} & -\sqrt{\phi_2} & -\sqrt{\phi_1} & \frac{1}{4}a_1 \end{bmatrix}.$$

It is evident that the rational root corresponding to X_1 is

$$\begin{bmatrix} \frac{1}{4}a_1 & al & bm & clm \\ a & \frac{1}{4}a_1 & cm & bm \\ b & cl & \frac{1}{4}a_1 & al \\ c & b & a & \frac{1}{4}a_1 \end{bmatrix},$$

and the other three can be written down at once.

10. *The cyclic quartic equation.*

The cyclic quartic permutation group may be represented as the powers of

$$P_{1234} = \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix},$$

to which correspond the set of matrix roots with x_1, x_2, x_3, x_4 permuted cyclically in the principal diagonals. We transform these by means of

$$\psi = \begin{bmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix},$$

$$\psi^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} \alpha^{-1} & & & \\ & \beta^{-1} & & \\ & & \gamma^{-1} & \\ & & & \delta^{-1} \end{bmatrix},$$

which transform P_{1234} into the irreducible form

$$T_{1234} = \begin{bmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{bmatrix},$$

and so obtain the following set of conjugate roots corresponding to T_{1234} :

$$\frac{1}{4} \begin{bmatrix} a_1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_3 & a_1 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 & a_1 & \lambda_1 \\ \lambda_1 & \lambda_2 & \lambda_3 & a_1 \end{bmatrix}, \quad \frac{1}{4} \begin{bmatrix} a_1 & -i\lambda_1 & -\lambda_2 & i\lambda_3 \\ i\lambda_3 & a_1 & -i\lambda_1 & -\lambda_2 \\ -\lambda_2 & i\lambda_3 & a_1 & -i\lambda_1 \\ -i\lambda_1 & -\lambda_2 & i\lambda_3 & a_1 \end{bmatrix},$$

$$\frac{1}{4} \begin{bmatrix} a_1 & -\lambda_1 & \lambda_2 & -\lambda_3 \\ -\lambda_3 & a_1 & -\lambda_1 & \lambda_2 \\ \lambda_2 & -\lambda_3 & a_1 & -\lambda_1 \\ -\lambda_1 & \lambda_2 & -\lambda_3 & a_1 \end{bmatrix}, \quad \frac{1}{4} \begin{bmatrix} a_1 & i\lambda_1 & -\lambda_2 & -i\lambda_3 \\ -i\lambda_3 & a_1 & i\lambda_1 & -\lambda_2 \\ -\lambda_2 & -i\lambda_3 & a_1 & i\lambda_1 \\ i\lambda_1 & -\lambda_2 & -i\lambda_3 & a_1 \end{bmatrix},$$

where

$$\lambda_1 = x_1 + ix_2 - x_3 - ix_4,$$

$$\lambda_2 = x_1 - x_2 + x_3 - x_4 = 4\sqrt{\phi_2},$$

$$\lambda_3 = x_1 - ix_2 - x_3 + ix_4.$$

As in the previous case, we have omitted $\alpha, \beta, \gamma, \delta$.

To obtain a set of conjugate roots rational in $R[i]$, the field of the coefficients of the quartic extended to include the irrational group characteristic i , we note that certain simple combinations of $\lambda_1, \lambda_2, \lambda_3$ are rational in $R[i]$. Thus, taking the radical expressions for the roots to be*

$$x_1 = a + b\sqrt{m} + (c + d\sqrt{m})\sqrt{(m + e\sqrt{m})},$$

$$x_2 = a - b\sqrt{m} + (c - d\sqrt{m})\sqrt{(m - e\sqrt{m})},$$

$$x_3 = a + b\sqrt{m} - (c + d\sqrt{m})\sqrt{(m + e\sqrt{m})},$$

$$x_4 = a - b\sqrt{m} - (c - d\sqrt{m})\sqrt{(m - e\sqrt{m})},$$

where $m = e^2 + f^2$, we find that

$$\lambda_1\lambda_3 = 8(\phi_1 + \phi_3),$$

$$\lambda_2 = 4b\sqrt{m} = 4\sqrt{\phi_2},$$

$$\lambda_1^2 = 8\sqrt{m}[(c^2 + md^2)e + 2cdm + i(c^2 - md^2)f] = 8\sqrt{m}(P + iQ),$$

$$\lambda_3^2 = 8\sqrt{m}(P - iQ).$$

Alternatively λ_1^2 and λ_3^2 may be found as the roots of

$$\Lambda^2 - 32\sqrt{(\phi_1\phi_3)}\Lambda + 64(\phi_1 + \phi_3)^2 = 0.$$

Transforming by the matrix having $[\lambda_3, 1, \lambda_3\sqrt{m}^{-1}, \sqrt{m}^{-1}]$ in the diagonal and zeros elsewhere, we obtain

$$\frac{1}{4} \begin{bmatrix} a_1 & 8(\phi_1 + \phi_3) & 4bm & 8(P - iQ)m \\ 1 & a_1 & \frac{\phi_1 + \phi_3}{m(P - iQ)}m & 4bm \\ 4b & 8(P - iQ) & a_1 & 8(\phi_1 + \phi_3) \\ \frac{\phi_1 + \phi_3}{m(P - iQ)} & 4b & 1 & a_1 \end{bmatrix}$$

* G. B. Mathews, *Algebraic equations* (revised by W. E. H. Berwick, 1930), 52.

as a root of a cyclic quartic rational in $R[i]$, and the other roots can be written down immediately.

11. *The biquadratic equation.*

When the Galois group of a quartic equation is 1, (12)(34), (13)(24), (14)(23), (1234), (13), (1432), (24), the equation breaks up into two quadratics in $R[\sqrt{m}]$; and Euler's cubic, as in the cyclic case, has one, and only one, rational root $\phi_2 = b^2 m$. The irreducible representations of the biquadratic group can be obtained from those of the cyclic group by replacing unity by the unit matrix of order two, and i by its expression as a rational matrix root of $x^2 + 1 = 0$. The eighth order matrix so obtained is then combined with a matrix representation of T'_{13} in which positive and negative units alternate in the principal diagonal.

The group is thus generated by

$$T_{1234} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & . & -1 & & & & \\ & & 1 & . & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & . & 1 \\ & & & & & & -1 & . \end{bmatrix}$$

and

$$T_{13} = \begin{bmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{bmatrix}.$$

The corresponding root of the biquadratic is

$$z_1 = \frac{1}{4} \begin{bmatrix} a_1 & & \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a_1 & & \lambda_1 & & \lambda_2 & \lambda_3 \\ \lambda_3 & & a_1 & & \lambda_1 & & \lambda_2 \\ & \lambda_3 & & a_1 & & \lambda_1 & \lambda_2 \\ \lambda_2 & & \lambda_3 & & a_1 & & \lambda_1 \\ & \lambda_2 & & \lambda_3 & & a_1 & \lambda_1 \\ \lambda_1 & & \lambda_2 & & \lambda_3 & & a_1 \\ & \lambda_1 & & \lambda_2 & & \lambda_3 & a_1 \end{bmatrix}.$$

To find rational matrix roots, we transform z_1 by the matrix which transforms each matrix of the group into itself. This matrix is

$$\phi = \begin{bmatrix} \alpha & . & . & . & . & . & . & . & . \\ . & \beta & . & . & . & . & . & . & . \\ . & . & \gamma & . & . & . & \delta & . & . \\ . & . & . & \gamma & . & . & . & -\delta & . \\ . & . & . & . & . & \epsilon & . & . & . \\ . & . & . & . & . & . & \zeta & . & . \\ . & . & \theta & . & . & . & . & \eta & . \\ . & . & . & -\theta & . & . & . & . & \eta \end{bmatrix},$$

$$\text{where } \phi^{-1} = \begin{bmatrix} \alpha^{-1} & . & . & . & . & . & . & . & . \\ . & \beta^{-1} & . & . & . & . & . & . & . \\ . & . & \eta M^{-1} & . & . & . & -\delta M^{-1} & . & . \\ . & . & . & \eta M^{-1} & . & . & . & \delta M^{-1} & . \\ . & . & . & . & \epsilon^{-1} & . & . & . & . \\ . & . & . & . & . & \zeta^{-1} & . & . & . \\ . & . & -\theta M^{-1} & . & . & . & \gamma M^{-1} & . & . \\ . & . & . & \theta M^{-1} & . & . & . & \gamma M^{-1} & . \end{bmatrix},$$

in which $M = \gamma\eta - \delta\theta$, and, like $\alpha, \beta, \epsilon, \zeta$, is non-zero. This gives

$$\phi z_1 \phi^{-1} =$$

$$\frac{1}{4} \left[\begin{array}{ccc} a_1 & . & \frac{\alpha}{M} (-\theta\lambda_3 + \eta\lambda_1) \\ . & a_1 & \frac{\beta}{M} (\theta\lambda_3 + \eta\lambda_1) \\ \frac{1}{\alpha} (\gamma\lambda_3 + \delta\lambda_1) & . & a_1 + \frac{\lambda_2}{M} (\delta\eta - \gamma\theta) \\ . & \frac{1}{\beta} (\gamma\lambda_3 - \delta\lambda_1) & a_1 - \frac{\lambda_2}{M} (\delta\eta - \gamma\theta) \\ \frac{\epsilon}{\alpha} \lambda_2 & . & \frac{\epsilon}{M} (\eta\lambda_3 - \theta\lambda_1) \\ . & \frac{\zeta}{\beta} \lambda_2 & \frac{\zeta}{M} (\eta\lambda_3 + \theta\lambda_1) \\ \frac{1}{\alpha} (\theta\lambda_3 + \eta\lambda_1) & . & \frac{\lambda_2}{M} (\eta^2 - \theta^2) \\ . & \frac{1}{\beta} (-\theta\lambda_3 + \eta\lambda_1) & \frac{\lambda_2}{M} (\eta^2 - \theta^2) \\ \frac{\alpha}{\epsilon} \lambda_2 & . & \frac{\alpha}{M} (\gamma\lambda_3 - \delta\lambda_1) \\ . & \frac{\beta}{\zeta} \lambda_2 & \frac{\beta}{M} (\gamma\lambda_3 + \delta\lambda_1) \\ \frac{1}{\epsilon} (\delta\lambda_3 + \gamma\lambda_1) & . & \frac{\lambda_2}{M} (\gamma^2 - \delta^2) \\ . & \frac{1}{\zeta} (\delta\lambda_3 + \gamma\lambda_1) & \frac{\lambda_2}{M} (\gamma^2 - \delta^2) \\ a_1 & . & \frac{\epsilon}{M} (-\delta\lambda_3 + \gamma\lambda_1) \\ . & a_1 & \frac{\zeta}{M} (\delta\lambda_3 + \gamma\lambda_1) \\ \frac{1}{\epsilon} (\eta\lambda_3 + \theta\lambda_1) & . & a_1 - \frac{\lambda_2}{M} (\delta\eta - \gamma\theta) \\ . & \frac{1}{\zeta} (\eta\lambda_3 - \theta\lambda_1) & a_1 + \frac{\lambda_2}{M} (\delta\eta - \gamma\theta) \end{array} \right]$$

in which we have to choose $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ so that all the elements of this matrix are rational. Suitable values are $\alpha = 16/\lambda_2$, $\beta = \lambda_1^2 - \lambda_3^2$,

$\gamma = \lambda_1/\lambda_2$, $\delta = \lambda_3/\lambda_2$, $\epsilon = 16/\lambda_2^2$, $\zeta = (\lambda_1^2 - \lambda_3^2)/\lambda_2$, $\eta = \lambda_1/\lambda_2^2$, $\theta = \lambda_3/\lambda_2^2$, giving $M = (\lambda_1^2 - \lambda_3^2)/\lambda_2^3$ and

$$X_1 = \phi z_1 \phi^{-1} =$$

$$\frac{1}{4} \left[\begin{array}{cccc|cccc} a_1 & . & 16 & . & 16\phi_2 & . & 0 & . \\ . & a_1 & . & -64G & . & 16\phi_2 & . & 16(\phi_1 + \phi_3)16\phi_2 \\ \phi_1 + \phi_3 & . & a_1 & . & \left(-\frac{G}{4\phi_2}\right)16\phi_2 & . & 16\phi_2 & . \\ . & 0 & . & a_1 & . & \left(\frac{1}{16\phi_2}\right)16\phi_2 & . & 16\phi_2 \\ 1 & . & 0 & . & a_1 & . & 16 & . \\ . & 1 & . & 16(\phi_1 + \phi_3) & . & a_1 & . & -64G \\ -\frac{G}{4\phi_2} & . & 1 & . & \phi_1 + \phi_3 & . & a_1 & . \\ . & \frac{1}{16\phi_2} & . & 1 & . & 0 & . & a_1 \end{array} \right]$$

from which X_2 , X_3 , X_4 can be obtained by transforming with T_{1234} . A slight modification gives

$$X_1 =$$

$$\frac{1}{4} \left[\begin{array}{cccc|cccc} a_1 & . & 16 & . & 4bm & . & 0 & . \\ . & a_1 & . & 128\sqrt{(m\phi_1\phi_3)} & . & 4bm & . & 64(\phi_1 + \phi_3)m \\ \phi_1 + \phi_3 & . & a_1 & . & \left\{\frac{2\sqrt{(m\phi_1\phi_3)}}{m}\right\}m & . & 4bm & . \\ . & 0 & . & a_1 & . & \left(\frac{1}{4m}\right)m & . & 4bm \\ 4b & . & 0 & . & a_1 & . & 16 & . \\ . & 4b & . & 64(\phi_1 + \phi_3) & . & a_1 & . & 128\sqrt{(m\phi_1\phi_3)} \\ \frac{2\sqrt{(m\phi_1\phi_3)}}{m} & . & 4b & . & \phi_1 + \phi_3 & . & a_1 & . \\ . & \frac{1}{4m} & . & 4b & . & 0 & . & a_1 \end{array} \right]$$

which is applicable when b , and therefore ϕ_2 , is zero.

12. The methods that we have employed here could be used for certain types of equation of higher degree, in particular for cyclic equations. Also, by noting the matrix forms of $p+a\sqrt{l}$ and $p+a\sqrt{l}+b\sqrt{m}+c\sqrt{lm}$, we can write down matrix roots of any equation in which the roots are rational functions of a number of quadratic surds.

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DUALITY AND INTERSECTION IN GENERAL COMPLEXES

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[Received and read November 18, 1937.]

Introduction.

This paper is based on the conviction that interesting results in topology must fundamentally arise from geometric rather than algebraic considerations. Striking work has recently been done in the realms of duality and of multiplications of cycles, but it has been almost entirely algebraic in methods and results. Here I undertake a geometric investigation of these notions and a geometric interpretation of the results, where "geometric" refers to combinatorial rather than set-theoretic topology.

In the first part of the paper I analyse in some detail the ideas behind duality in a complex, and the conditions required for it; I reach a duality theorem which includes as special cases all the general duality theorems for complexes that are known to me. The second part is concerned with various types of intersections, emphasis being laid on the interpretation of recent work in the light of classical intersection theory. Also intersections are defined for certain homology classes on complexes much less restricted than manifolds.

Of the tools used in this paper, the most important are two different models for the dual complex. To define the first, we consider the complex as being embedded in an orientable manifold; for each simplex of the complex we define its dual in the manifold in the usual way, and so construct a dual complex which essentially comprises its simplicial neighbourhood in the first derived of the manifold. This method is suggested by a scheme used by Lefschetz (6)[†]; he forms a simplex con-

[†] Numbers refer to the bibliography at the end of the paper.

taining all the vertices of the complex and considers the complex as immersed in the sphere which is the boundary of this simplex. This approach has proved extremely powerful in a number of ways, but we have had to modify it in order to get a normal neighbourhood for our complex in the embedding space.

The second model for the dual complex consists of cells which are open subcomplexes of the first derived of the original complex, and are formed precisely as the dual cells have always been formed for manifolds. This method presents certain difficulties of detail, but it has the advantage that it is intrinsic to the complex.

For the purpose of analysing the duality relations, we define in this paper "cycles of dual type". They are cycles of the first derived of the complex which are homologous to cycles of dual cells. We also make use of Čech's notion of p -regularity as a condition for duality, although we have to extend this to cover the more general class of complex with which we are dealing.

The results of the paper are closely related to those which Čech (1) has obtained by algebraic methods; but it will be noticed that the geometric approach leads in almost all cases to considerably greater generality.

The terminology will be almost entirely that of Lefschetz (2); we shall, however, abandon his term "link complex" in favour of Alexander's "complement", and we shall follow Newman (3) in writing this as a quotient. In discussing multiplications of dual cycles (which have been investigated recently by Alexander, Kolmogoroff, Čech, and Whitney), we shall use Čech's terminology, and for the sake of brevity they will be described as Čech-multiplications.

The complex and its dual.

We deal with a finite complex K_n which has the property that every simplex of it is a face of one of its n -simplexes; from this K a closed subcomplex L is removed, which has a normal neighbourhood in K . For the discussion of the dual complex and its relation to K , it is necessary to assign a definite orientation to each σ_n of K ; when this has been done, the σ_n 's form an n -chain whose boundary can be calculated in the usual way; we take all the σ_{n-1} 's present in the boundary chain and all their faces, and call this subcomplex J . J is consequently a closed subcomplex of $K \bmod L$, but not necessarily closed in K .

For the dual complex we refer to two different models, each of which has its peculiar usefulness. To get the first, we suppose K to be embedded

as a subcomplex of some orientable m -dimensional manifold M_m ; we can also require that it has a normal neighbourhood in M , that is, that every simplex of M all of whose vertices lie on K is itself a simplex of K . Now, in the standard way, to each simplex of M we can assign a dual cell which is an open subcomplex of the first derived of M , which we call M' ; each dual cell is a normal cell, since M is a manifold. We call the complex of these dual cells M^* , and the subcomplex consisting of the cells dual to simplexes of $K \bmod L$ we call K_m^* . This complex, which is not in general closed, has the required incidences of a dual complex to $K \bmod L$, and, further, the boundary of a cell of K^* is a relative cycle of K^* consisting of cells dual to those of the dual boundary of the corresponding simplex of K . In other words, the operation of taking the dual boundary of a simplex of $K \bmod L$ is reflected in K^* by the operation of taking the ordinary boundary, considering the cells as chains of M' and ignoring simplexes not lying inside K^* .

The second model for the dual complex of $K \bmod L$ will be written as K_n^+ ; this will be reached by precisely the method normally used to get the dual cells of a manifold. It will be described in terms of complements; the complement of a simplex σ_p in a complex C_n is written as C_n/σ_p and it consists of all the simplexes τ_q of the closure of C , such that $\sigma_p \tau_q$ is a simplex of C . This is a complex, and its incidences follow at once from those of C ; that is, the incidence relations between τ_q and τ_r are defined to be those between $\sigma_p \tau_q$ and $\sigma_p \tau_r$ in C . If a definite orientation is assigned to the n -simplexes of C , this induces an orientation of the $(n-p-1)$ -simplexes of C/σ_p ; that is, we orient τ_{n-p-1} in such a way that $\sigma_p \tau_{n-p-1}$ gives a correctly oriented simplex of C . In this way we can regard C/σ_p as being an $(n-p-1)$ -chain or as a complex.

We may wish to define an open subcomplex of $K' \bmod L'$, the first derived of $K \bmod L$, which is the dual to some simplex σ_p of $K \bmod L$. In the subdivision, σ_p has assigned to it a definite vertex, say A_p , and a number of p -simplexes of the first derived lie inside σ_p ; call one of them σ_p' . Now we take K'/σ_p' as a complex and join the vertex A_p to it; now $A_p(K'/\sigma_p')$ is a closed subcomplex of K' , and we call

$$A_p(K'/\sigma_p') - K'/\sigma_p'$$

the cell dual to σ_p . It is an elementary matter to see that this open subcomplex is independent of the particular σ_p' inside σ_p which was chosen. Those who are familiar with van Kampen's thesis (4) will recognise K^+ as being the star-complex dual to K' , which is itself a star-complex. In our treatment we are less concerned with K^+ as an abstract complex than as a carrier for a subset of the chains of K' .

The approach to duality relations.

While every attempt is to be made to present a geometric attack on duality theorems, it would be futile to avoid algebraic arguments completely. We assume, then, the algebraic proof that the p -th homology group of an arbitrary complex is dual to the p -th dual group; this result comes directly from consideration of incidence matrices (5). Now the p -th dual group of K is isomorphic with the $(m-p)$ -th ordinary group of K^* , or with the $(n-p)$ -th ordinary group of K^+ . The method of proving duality relations, when K^* is used, is to show that, subject to certain conditions, the $(m-p)$ -th group of K^* is isomorphic to the $(n-p)$ -th group of K , or rather of some subcomplex of K ; and when K^+ is considered, we compare R_{n-p} of this complex with R_{n-p}^+ of K^+ , and the two $(n-p-1)$ -th torsion groups. Since the proofs are all proofs of isomorphism, no serious problems about coefficient groups arise; we can show that the homology groups of K^* with coefficient group g are isomorphic to the dual groups of K with the same coefficient group. In this case the tool for proving isomorphism is that of intersection with K ; for the coefficients on K , we use the integers, and in point of fact only $+1$ and -1 and 0 ; consequently the coefficient group for the intersection chains is again g . Since full generality can be combined with convenience, we regard g as being the group of integers, and we can also use this group for the ordinary groups of K .

Lefschetz (2) gives two alternative ways of defining intersections of chains, of which for the first part we use only the more geometric. Namely, in an orientable manifold C_n we have two chains D_p and D_q ; we say that they have normal intersection if no simplex of dimension greater than $p+q-n$ is a face of simplexes of each chain (or "is a face of each chain").

If we have two chains, having normal intersection, each of which intersects the boundary of the other normally, we can define that intersection in the following way: In C/σ_{p+q-n} , D_p/σ_{p+q-n} , and D_q/σ_{p+q-n} are two chains of dimension $n-q-1$ and $n-p-1$ respectively; and C/σ_{p+q-n} is a homology sphere of dimension $2n-p-q-1$, since C is a manifold. In such a sphere we can define looping coefficients for two non-intersecting cycles of complementary dimensions. Since each chain intersects the boundary of the other normally, where neither D_p/σ_{p+q-n} nor D_q/σ_{p+q-n} is vacuous, each must be a cycle; and, from the normal intersection of D_p and D_q , these cycles do not intersect. For each such σ_{p+q-n} of C we form the looping coefficient of $(D_p/\sigma_{p+q-n}, D_q/\sigma_{p+q-n})$ in C/σ_{p+q-n} ; and we define a chain by attaching to each σ_{p+q-n} of C this number if it is

defined and otherwise zero. This chain is the intersection chain $D_p \cdot D_q$.

If D_p and $F(D_p)$ each have normal intersection with each of D_q and $F(D_q)$, we can show that

$$F(D_p \cdot D_q) = D_p \cdot F(D_q) + (-1)^{n-q} F(D_p) \cdot D_q.$$

It is necessary to assume the normal intersection of $F(D_p)$ with $F(D_q)$ in order to define the intersections on the right-hand side.

We assume these results and the necessary results on looping coefficients which make them possible.

A further pre-requisite to the following work is Alexander's duality theorem. This states that, if K is a complex immersed in a sphere H_m , the p -th homology group of K is dual to the $(m-p-1)$ -th group of $H_m - \bar{K}$, the space of H_m residual to K , except that the 0-th group of either contains one more free generator than the corresponding group of the other. These results follow from consideration of looping coefficients or from examining the groups of K^* , in the terminology that we have adopted, where now H_m is our M_m .

It is perhaps regrettable that, in a paper claiming to prove an inclusive duality theorem, one classical theorem has to be assumed; and that a paper giving intersection theory for a more general class of complex should appeal to a great deal of fundamental work on classical intersections. In defence it may be said that Alexander's duality theorem and the intersection theory outlined above are important only in the proofs which appeal to K^* , and that a different proof of an identical result, using only K^+ , is independent of these assumptions. Both methods, however, will be given because the geometric ideas involved are highly dissimilar. And, as regards intersections, the result is really to show to what extent it is possible to generalize to arbitrary complexes the definitions and results of intersections discovered for manifolds.

The embedding method.

The first model to be considered is K^* . This is derived from the embedding of K in some orientable manifold M_m . One method of doing this has been given, and extensively used, by Lefschetz (6); and indeed it was this approach that gave rise to these investigations. He takes all the vertices of the closure of K , and considers the simplex σ_{m+1} formed by these; the boundary of this simplex is a sphere, H_m , which contains the complex. When convenient, Lefschetz adds an arbitrary number of auxiliary vertices. This embeds K in an orientable manifold.

Since we desire K to have a normal neighbourhood, we take all the simplexes of H not lying in the closure of K , and perform a stellar sub-

division on them. Now we have K normally embedded in a sphere. Lefschetz makes particular use of the simple combinatorial structure of the original H_m ; but for our purpose normal embedding is essential; and when this is required, we lose this structural simplicity. Consequently we present this as an existence proof, to show that the embedding is always possible; but we do not make special arguments involving the spherical character of the embedding manifolds, and therefore we do not in the first place require more than that the embedding complex has the local characteristics defining a manifold, and the orientability of this manifold.

We are now ready to examine the relations between the groups of K^* and those of some indicated subcomplex of K' . The link to be used is that of intersection, and this fact leads us immediately to see what subcomplex we must take.

First, how can we describe the set of simplexes common to K^* and to K' ? With each simplex of M' , the first derived of M , we can associate a sequence of simplexes of M , each a face of its successor in the sequence. Such a simplex belongs to K^* if, and only if, the first simplex of the sequence belongs to $K \bmod L$; and it belongs to $K' \bmod L'$ if, and only if, the last simplex of the sequence belongs to $K \bmod L$. Since we have assumed L to be a closed subset of K , the former condition implies the latter. The set of simplexes picked out in this way will be all the simplexes of K' except those having some vertex on L' : for, if a simplex has any vertex on L' , one element of the sequence is not a simplex of $K \bmod L$, and so the first is not. Now the complex so described is one of a well-known type and is designated by $K' - L'$.

Thus, so far as intersections of cycles of K^* with $K \bmod L$ are concerned, we may think of the complex on which they lie as being $K' - L'$. But we also want the intersection chains to be cycles of the carrying complex. Suppose, therefore, that Γ_q^* is a cycle of K^* having normal intersection with $K \bmod L$ and with J ; taking the n -simplexes of K' to form an n -chain and calling this C' , we know that

$$F(C' \cdot \Gamma_q^*) = C' \cdot F(\Gamma_q^*) + (-1)^{m-q} F(C') \cdot \Gamma_q^*.$$

Since Γ_q^* has no boundary in K^* , from the way in which K^* was constructed we can see that $F(\Gamma_q^*)$ can have no intersection with C' . Consequently the boundary of the intersection chain lies entirely on $F(C')$, or lies entirely on J' . This leads us naturally to take $K' - L' \bmod J'$ to be the complex whose homologies we should examine for duality with those of $K \bmod L$.

We designate $K' - L' \bmod J'$ by $\tilde{K}$. Thus, with any cycle of K^* having normal intersection with $K' \bmod L'$ and with J' , we associate

the intersection chain defined as we have described above, and that chain is a cycle of $\tilde{K}$.

If, now, we take a dual cycle (that is, a cycle of cells of K^*), we know that this does have normal intersection with K and with J ; or, to speak more accurately, its first derived has normal intersection with C' . This is the usual result about dual and ordinary chains of a manifold, and does not require proof here.

It is important in the interest of precision to distinguish between K as a complex and as a chain of its n -simplexes; when it is considered in the latter light, it is written as C , and similarly for the first derived.

Cycles of dual type.

We now need to introduce a slightly modified type of intersection. Suppose that D_p and D_q intersect normally, and that $F(D_p)$ and D_q also have normal intersection, but that we do not know about the intersections with $F(D_q)$. In this case there may be a σ_{p+q-n} for which D_p/σ_{p+q-n} and D_q/σ_{p+q-n} are both non-vacuous, but in which D_q/σ_{p+q-n} is not a cycle; if this is so, we cannot define a looping coefficient. We therefore adopt the convention that the coefficient of such a σ_{p+q-n} in the intersection chain is zero. It is always a simplex on $F(D_q)$, and we may regard this process as yielding intersections mod $F(D_q)$. By the argument which Lefschetz uses for the more usual case, we arrive at the result that

$$F(D_p \cdot D_q) = (-1)^{n-q} F(D_p) \cdot D_q \text{ mod } F(D_q).$$

Let us now consider cycles of $K^{*'} which have normal intersection with $K' \text{ mod } J'$; since we are considering only results mod J' , we do not need to demand normality with J' ; and, since the boundary of the cycle of K^{*}' is away from K' , no question of normality arises there. The resulting intersection is therefore a cycle of $\tilde{K}$. Any such cycle of $\tilde{K}$ we call a cycle of dual type.$

THEOREM 1. *Every bounding cycle of $\tilde{K}$ is of dual type.*

Let $\Gamma_q = \tilde{F}(C_{q+1})$, that is, the part of the boundary lying on $\tilde{K}$; and suppose that $C_{q+1} = \sum a_i \sigma_{q+1}^i$. Since σ_{q+1}^i is not on J' , and is incident on some σ_n of C' , we know that C'/σ_{q+1}^i is an $(n-q-2)$ -cycle, lying in M'/σ_{q+1}^i . But M'/σ_{q+1}^i is a sphere of dimension $m-q-2$, and in it we can find an $(m-n-1)$ -cycle Δ_{m-n-1}^i which loops C'/σ_{q+1}^i with coefficient 1. For each σ_{q+1}^i of C_{q+1} we find such a Δ_{m-n-1}^i and form

$$C_{m-n+q+1} = \sum a_i \sigma_{q+1}^i \Delta_{m-n-1}^i.$$

This chain has normal intersection with $K' \bmod J'$, and so has its boundary; these facts follow from the fact that Δ_{m-n-1}^i is a cycle containing no vertices of K' . We can, therefore, use our modular intersection formula,

$$F(C_{m-n+q+1} \cdot C') = (-1)^{m-n} F(C_{m-n+q+1}) \cdot C' \bmod J'.$$

Now $C_{m-n+q+1} \cdot C' = \sum a_i \sigma_{q+1}^i = C_{q+1}$, so that the left-hand side is Γ_q ; and we have shown that Γ_q is the intersection with C' of a cycle of K^{**} which intersects $K' \bmod J'$ normally. This means that Γ_q is a cycle of dual type, and the theorem is proved.

We may suppose that with each σ_p^i of $\tilde{K}$ we have constructed the associated Δ_{m-n-1}^i as described above; it is not uniquely defined, but let us choose one arbitrarily for each simplex. We call the chain $\sigma_p^i \Delta_{m-n-1}^i$ the associated $(m-n+p)$ -chain of σ_p^i . It is in this construction that we make essential use of the condition that each simplex of K is a face of an n -simplex.

THEOREM 2. *The sum of two cycles of $\tilde{K}$ of dual type is of dual type.*

This is elementary: to each summand we have the cycle of K^{**} , and the sum of these two cycles will intersect $\tilde{K}$ in the required sum.

Theorems 1 and 2 give the result that, if Γ_q is a cycle of $\tilde{K}$ of dual type, every cycle homologous to Γ_q is of dual type; consequently we can speak of a homology class of $\tilde{K}$ as being of dual type.

THEOREM 3. *If Γ_{m-n+p}^* is a bounding cycle of K^{**} having normal intersection with $\tilde{K}$, Γ_p , then Γ_p is a bounding cycle.*

Let $\Gamma_{m-n+p}^* = F^*(C_{m-n+p+1}^*)$; we cannot immediately say that $C_{m-n+p+1}^*$ has normal intersection with C' , but we can always find a homologous chain having that property. For, if there is a σ_r not a face of Γ_{m-n+p}^* , $r > p+1$, on K' which is a face of $C_{m-n+p+1}^*$, and if this is not true for any σ_s , $s > r$, we can push $C_{m-n+p+1}^*$ away from this σ_r . For $C_{m-n+p+1}^*/\sigma_r$ is a cycle of dimension $m-n+p-r$ lying in M'/σ_r (for σ_r does not lie on the boundary), and containing no vertices of K'/σ_r (for no σ_s , $s > r$, was a face of $C_{m-n+p+1}^*$). But $m-n+p-r < m-n-1$, since $r > p+1$; and, from Alexander's duality theorem, because the dimension of K'/σ_r is $n-r-1$, every cycle of $(M'/\sigma_r - K'/\sigma_r)$ which is of dimension less than $m-n-1$ bounds in that set.

So $C_{m-n+p+1}^*/\sigma_r = F(D_{m-n+p-r+1})$, where D contains no vertex of K' . Then $\{C_{m-n+p+1}^* + (-1)^r F(\sigma_r D_{m-n+p-r+1})\}$ is a chain bounded by Γ_{m-n+p}^* .

which does not have σ_r as a face, and in fact reduces by unity the number of r -simplexes of $\tilde{K}$ lying on the chain. We repeat this process for all the σ_r 's and then the σ_{r-1} 's and so on, until we eventually arrive at a chain $B_{m-n+p+1}^*$ bounded by Γ_{m-n+p}^* which itself intersects $\tilde{K}$ normally.

Now, because Γ_{m-n+p}^* and $B_{m-n+p+1}^*$ both have normal intersection with K' , we can use the boundary formula

$$F(B_{m-n+p+1}^* \cdot C') = (-1)^{m-n} F(B_{m-n+p+1}^*) \cdot C' \pmod{J'}.$$

The left-hand side is the boundary of a $(p+1)$ -chain of $\tilde{K}$, and we have consequently shown that $\Gamma_{m-n+p}^* \cdot C'$ is a bounding cycle of $\tilde{K}$.

THEOREM 4. *Every homology class of K^* has at least one representant having normal intersection with $\tilde{K}$.*

This follows from the facts that the actual "dual" cycles of K^* do have normal intersection and that every homology class of K^* contains the first derived of a "dual" cycle of K^* .

As a corollary of Theorems 3 and 4 we can deduce that, in terms of this intersection, we can associate with each homology class of K^* a unique homology class of $\tilde{K}$; ("each" because of Theorem 4, and "unique" from Theorem 3).

Since the homology classes of K^* are isomorphic with those of K^* and consequently with the dual groups of $K \pmod{L}$, we have shown that there is a homomorphism of the whole p -th dual group of $K \pmod{L}$ upon the whole of the $(n-p)$ -th dual type homology group of $\tilde{K}$; the dual type homology group is, of course, a subgroup of the ordinary homology group. The method of proving duality is to show that, under certain conditions, the homomorphism is an isomorphism, and that, under others, this subgroup is actually the whole group.

This whole approach is closely analogous to Čech's (1); he takes a closed complex with a covering cycle Γ_n , and he maps the dual group of dimension p on to the ordinary group of dimension $n-p$, by forming the Čech-multiple $\Gamma_n \cdot B^p$ giving an ordinary C_{n-p} . Here we free ourselves from the restriction of having a closed complex, and from that of a covering cycle.

p-regularity and its geometric content.

Čech (1) has defined p -regularity for a closed complex having a covering cycle; he calls such a complex, K , p -regular if K/σ_q has R_{n-p-1} and

θ_{n-p-2} identical with the corresponding numbers of an $(n-q-1)$ -sphere. There is a convenient way of extending this definition to a complex having no covering cycle, and we use the same name as above; we say that $K \bmod L$ is p -regular if the above condition is fulfilled for all simplexes of $K \bmod L$ not lying on J , but, for σ_q on J , K/σ_q must have R_{n-p-1} and θ_{n-p-2} equal to those of a point. For this to happen, J must essentially be a genuine wall of the complex, and not simply a formal interior boundary such as that of a 2-dimensional projective plane.

Now this property is not invariant under sub-division (7) and so we cannot conveniently work with cycles of $K' - L' \bmod J'$, but must use those of $K - L \bmod J$. The homology groups of $K' - L' \bmod J'$ are isomorphic to those of $K - L \bmod J$. For, since L has a normal neighbourhood in K , every simplex of $K \bmod L$ has a vertex not lying on L ; take any ordering of these vertices. We transform each vertex of $K' \bmod L'$ into the "heaviest" vertex of the simplex of $K \bmod L$ containing it. This is a standard simplicial transformation of $K' - L'$ into $K - L$ which also sends J' into J ; cycles of $K' - L' \bmod J'$ go into cycles of $K - L \bmod J$ whose first deriveds are homologous over $\tilde{K}$ to the original ones. Consequently it sends homology classes of $\tilde{K}$ into homology classes of $K - L \bmod J$ in a 1-to-1 fashion. In fact the groups are isomorphic.

We can also define a transformation of $K^{*'} into a new complex by applying this given transformation to the vertices of $\tilde{K}$ and leaving fixed the vertices outside; we may call the resulting complex $\tilde{K}_m^{*}$. The transformation, in the same way, defines an isomorphism between the groups of $K^{*'}$ and those of $\tilde{K}^{*}$.$

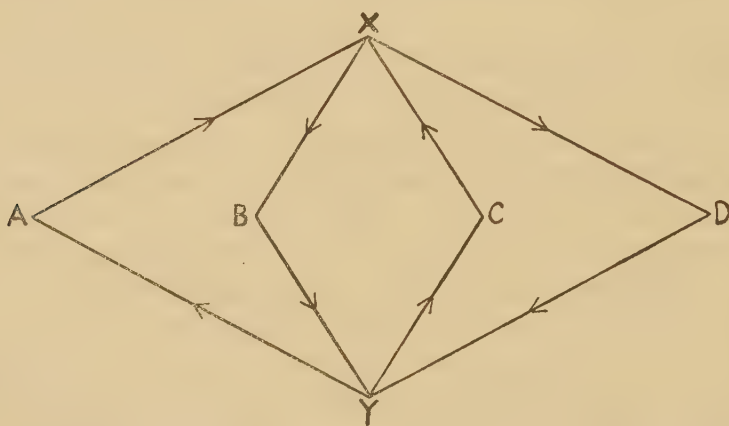
As before, we can consider intersections of cycles of $\tilde{K}^{*}$ with C provided that the intersection is normal, and Theorems 1 to 4 are true for these intersections. Theorems 1 to 3 are proved precisely as before, or we could appeal for them all to the argument which proves Theorem 4 for this case, namely that each homology class of $\tilde{K}^{*}$ has a representant having normal intersection with C . The argument is simply that the transform of a cycle Γ_p^{*} having normal intersection with C' is a cycle having normal intersection with C ; for under the transformation each simplex common to Γ_p^{*} and to C' goes into a simplex of at most the same dimension, also common to the transforms; and the set of simplexes common to the transforms is just the transform of the set of simplexes common to Γ_p^{*} and C' : this is true because the transformation is the identity off K' .

Then again we have by means of normal intersections a homomorphism of the groups of $\tilde{K}^{*}$ on to those of $K - L \bmod J$.

We now turn to our main theorem on p -regularity.

THEOREM 5. *If $K \bmod L$ is p -regular, any Γ_{m-n+p}^* of $\bar{K}^*$ having normal intersection with $K \bmod J$ which is the zero chain on $K-L \bmod J$ bounds a $C_{m-n+p+1}^*$ in $\bar{K}^*$ having also normal intersection with $K \bmod J$, which is zero on $K-L \bmod J$.*

This theorem may seem of more interest if a simple example is given where K is not p -regular and the theorem does not hold. Consider the accompanying 1-dimensional complex, in which L and J are lacking: take it as immersed in a two-sphere. Then $\bar{K}^*$ is essentially a simplicial neighbourhood of it in this two-sphere; take a vertex of $\bar{K}^*$ lying inside



the quadrilateral $XBYC$, say P ; and one lying in the exterior as drawn, say Q . Suppose that PX and QX are both simplexes of $\bar{K}^*$. Now $PX + QX$ is a relative cycle of $\bar{K}^*$ having normal intersection with K , which is algebraically zero. But this cycle does not bound in $\bar{K}^*$; crudely speaking, we cannot free the cycle from K : if we could free it, of course, it would be homologous to zero, because any cycle free from K is zero on $\bar{K}^*$.

It is easy to see that here K_1 is not 0-regular: if it was 0-regular, in every complement we should have $R_{n-p-1} (= R_0)$ to be that of a sphere of the corresponding dimension. But, in the complement of X , $R_0 = 4$ instead of 2.

Now for the proof of the theorem.

Since Γ_{m-n+p}^* intersects K normally, they have no common faces of dimension greater than p . Since the intersection is algebraically zero on $K-L \bmod J$, any σ_p of $K-L \bmod J$ which is a face of Γ_{m-n+p}^* has $\Gamma_{m-n+p}^*/\sigma_p$ and C/σ_p non-linking cycles of $\bar{K}^*/\sigma_p$. But, since K is p -regular, K/σ_p has $R_{n-p-1} = 1$, $\theta_{n-p-2} = 0$; or $(\bar{K}^*/\sigma_p - K/\sigma_p)$ has $R_{m-n-1} = 1$,

$\theta_{m-n-1} = 0$ by Alexander's duality theorem; in fact in this there is only one generating cycle of the non-bounding $(m-n-1)$ -cycles. But $\Gamma_{m-n+p}^*/\sigma_p$ is not a multiple of this cycle because we have seen that it does not link C/σ_p ; then $\Gamma_{m-n+p}^*/\sigma_p$ bounds some D_{m-n} in the complement which contains no vertices of K/σ_p . If we now take

$$\Gamma_{m-n+p}^* + (-1)^p F(\sigma_p D_{m-n})$$

we get a homologous cycle of $\bar{K}^*$ of which σ_p is no longer a face and which thus contains one less p -simplex in common with K than Γ_{m-n+p}^* . Furthermore, the chain $\sigma_p D_{m-n}$ has normal intersection with K which is zero. In this way we find a homologous chain of $\bar{K}^*$ sharing no σ_p 's with $K - L \bmod J$.

For the σ 's that it may now share with J , and for the σ_q 's of dimension less than p which it may share with K , the argument is similar. For each such simplex, the complement in K has $R_{n-p-1} = \theta_{n-p-2} = 0$. And so the residual space to this complement in the complement in $\bar{K}^*$ has $R_{m-n+p-q-1} = \theta_{m-n+p-q-1} = 0$: consequently the complement of the simplex in the Γ_{m-n+p}^* must be bounding in this residual space. By precisely the method already used we push the Γ_{m-n+p}^* off all the simplexes of K , beginning with those of highest dimension and continuing down, until we arrive at a homologous cycle entirely free from $K - L$, which is therefore zero in $\bar{K}^*$. It should be pointed out that, since we have not necessarily got normal intersection with J , there may be common faces of dimension greater than p on J : these are, however, of no peculiar difficulty. Now at each step we added to the old cycle the boundary of a chain having normal intersection with K , which is zero. Consequently we have shown that Γ_{m-n+p}^* bounds a $C_{m-n+p+1}^*$ which has normal zero intersection with K .

It should be remarked that care must always be taken about dimensions when Alexander's duality theorem is used, because slight exceptions arise if either of the two complementary dimensions vanishes; in the above work we have been dealing with dimensions $n-p-1$ and $m-n+p-q-1$. The former vanishes only when $p = n-1$; then, in K/σ_{n-1} , $R_0 = 2$, and the duality theorem tells us that, in $(\bar{K}^*/\sigma_{n-1} - K/\sigma_{n-1})$, $R_{m-n-1} = 1$ (and not 2), which is the result required for the argument.

The dimension $m-n+p-q-1$ needs a little more attention. For $m > n+1$, $m-n+p-q-1$ can vanish only for a σ_q in J ; but the condition of p -regularity here implies that $R_0(\bar{K}^*/\sigma_q - K/\sigma_q) = 1$, which is again the required result. If $q = m-n+p$, the dimension in question becomes negative, but the difficulty in this case is no more than apparent.

If $m = n+1$, $m-n+p-q-1$ may vanish for a σ_q lying in $K \bmod J$, for which $R_0(\bar{K}_{n+1}^*/\sigma_q - K_n/\sigma_q) = 2$. But, since Γ_{m-n+p}^* has zero inter-

section with K , $\Gamma_{m-n+p}^*/\sigma_q$ (which is a 0-chain) lies entirely on one of the two parts of this residual space, and so again is a bounding cycle. If $m = n$, the conditions of the theorem imply that Γ_{m-n+p}^* has no simplexes in common with $K \bmod J$, which is a case that has already been treated.

We can now state two important corollaries of this theorem.

COROLLARY 1. *If $K \bmod L$ is p -regular, the homomorphism of the $(m-n+p)$ -th group of $\bar{K}^*$ on to the p -th group of $K-L \bmod J$ is an isomorphism.*

That is, if Γ_{m-n+p}^* intersects $K-L$ normally in a bounding cycle of $K-L \bmod J$, it is bounding in $\bar{K}^*$. Since this bounding cycle is of dual type (Theorem 1), take the Δ_{m-n+p}^* intersecting $K-L$ in it; then $\Gamma_{m-n+p}^* - \Delta_{m-n+p}^*$ intersects $K-L$ normally and has zero intersection on $K-L \bmod J$; therefore it bounds in $\bar{K}^*$; but, by the very construction of Δ_{m-n+p}^* (see the proof of Theorem 1), Δ_{m-n+p}^* bounds in $\bar{K}^*$. Therefore Γ_{m-n+p}^* bounds in $\bar{K}^*$. This proves the first corollary; two different homology classes of $\bar{K}^*$ cannot give the same homology class of $K-L \bmod J$; for then their difference would give the zero homology class, and we should have a non-bounding class intersecting in bounding cycles.

COROLLARY 2. *If $K \bmod L$ is p -regular, the $(p+1)$ -th dual type homology classes, which form a subgroup of the $(p+1)$ -th ordinary homology group of $K-L \bmod J$, are actually the whole group.*

In other words, every $(p+1)$ -cycle of $K-L \bmod J$ is a dual type cycle. To prove this, we take any such $(p+1)$ -cycle; to each simplex we form its associated $(m-n+p+1)$ -chain of $\bar{K}^*$ (see Theorem 1). Take the whole $(m-n+p+1)$ -chain of $\bar{K}^*$ associated with the $(p+1)$ -cycle and form its boundary. It is geometrically obvious that the resulting $(m-n+p)$ -cycle of $\bar{K}^*$ has normal intersection with K which is zero on $K-L \bmod J$. That the intersection is zero perhaps requires a proof.

The $(m-n+p+1)$ -chain, $C_{m-n+p+1}^*$, has itself normal intersection with K and so has its boundary, so that we may apply the boundary formula

$$F(C_{m-n+p+1}^* \cdot C) = (-1)^{m-n} F(C_{m-n+p+1}^*) \cdot C \bmod J.$$

But the left-hand side is zero, $\bmod J$, because the intersection is assumed to be a cycle; and the right-hand side is the intersection of the boundary of $C_{m-n+p+1}^*$ with C , neglecting the sign. Consequently this intersection is the zero chain $\bmod J$.

Now $F(C_{m-n+p+1}^*)$ is a Γ_{m-n+p}^* of $\bar{K}^*$ having zero intersection $\bmod J$: since we assume $K \bmod L$ to be p -regular, Theorem 5 tells us that Γ_{m-n+p}^*

bounds a $D_{m-n+p+1}^*$ which has zero intersection with C . Then

$$(C_{m-n+p+1}^* - D_{m-n+p+1}^*)$$

is a cycle of $\bar{K}^*$ whose intersection with C is the same as that of $C_{m-n+p+1}^*$, namely our original $(p+1)$ -cycle. Therefore every $(p+1)$ -cycle of $K-L \bmod J$ is of dual type.

We can now state our fundamental duality theorem:

THEOREM 6. *If $K \bmod L$ is p - and $(p-1)$ -regular the classical duality relation holds between the $(n-p)$ -th group of $K \bmod L$ and the p -th group of $K-L \bmod J$.*

For we have the duality between the $(n-p)$ -th group of $K \bmod L$ and the $(m-n+p)$ -th group of $\bar{K}^*$, and, by Corollaries 1 and 2, we get an isomorphism of this group with the p -th group of $K-L \bmod J$.

This may be expressed by saying that

$$R_{n-p}(K \bmod L) = R_p(K-L \bmod J),$$

$$\theta_{n-p-1}(K \bmod L) = \theta_p(K-L \bmod J),$$

where in each case integers are used for coefficients.

The main steps in this argument are reminiscent of the steps that Čech (1) uses in the proof of his partial duality theorem; for instance, we have used his idea of setting up a homomorphism of a dual group in a subgroup of the complementary group. But the methods of proof bear little resemblance, Čech being concerned with a purely algebraic approach to the problem.

The method of K^+ .

We have already described the complex K^+ ; it consists of cells which are open subcomplexes of the first derived $K'-L'$, and, in fact, the set of all these cells completely covers $K'-L'$. The inconveniences connected with this method are: first, that the boundary of such a cell considered as a chain of $K'-L'$ is not the same as the boundary when it is considered as a dual cell of the dual complex; and secondly, that we cannot subdivide the complex K^+ ; each cell does not, as a chain of K' , have the homology characters of a genuine cell of that dimension.

The first objection is, however, not serious. Let σ_p^i be a simplex of K ; we call its dual cell E_{n-p}^{i+} and its underlying chain can be described as $A(K'/\sigma_p')$, where A is the point of subdivision of σ_p and σ_p' is any p -simplex of K' lying inside σ_p .

Now

$$\begin{aligned} F(E_{n-p}^+) &= F[A(K'/\sigma_p')] = K'/\sigma_p' - A F(K'/\sigma_p') \\ &= K'/\sigma_p' + (-1)^p A[F(K')/\sigma_p'], \end{aligned}$$

but K'/σ_p' is precisely the boundary if we consider E_{n-p}^+ as a cell of the complex K^+ ; we may write this $F^+(E_{n-p}^+)$.

Then

$$F(E_{n-p}^+) = F^+(E_{n-p}^+) + (-1)^p A[F(K')/\sigma_p'].$$

The residual term is zero unless σ_p' is on J' , and in that case it is a chain of J' . Consequently if we take $K' - L' \bmod J'$, ignoring simplexes of J' , we find that

$$F(E^+) = F^+(E^+).$$

In this way we are again led to consider the complex $K' - L' \bmod J'$, which we called $\tilde{K}$ above. And we have seen that with dual cycles of $K \bmod L$ we have associated ordinary cycles of $\tilde{K}$.

Now consider $\tilde{K}$; certain cycles on it are genuine dual cycles as just described; it is not hard to see that these cycles are also the intersections of cycles of K^* with C . They are then cycles of dual type in the sense already used: and, from the theorems proved about dual type cycles, they consist of just those cycles which are homologous, on $\tilde{K}$, to some actual dual cycle Γ^+ .

We could take this as our definition of dual type homology classes of $\tilde{K}$, and we should immediately see that we have a homomorphism of the dual homology classes of dimension p (dual to those of dimension $n-p$) on the whole group of dual type classes of dimension p of $K - L \bmod J$.

But complications arise which make this proof less direct than these considerations would lead us to expect. The effect of p -regularity on the dual cells is easy to see; if σ_q is a simplex of $K \bmod L$ not of J , then its dual cell, as an open subcomplex of K' , has R_{n-p} and θ_{n-p-1} (relative homologies) cell-like. This means essentially that any $(n-p)$ -cycle of K' can be pushed off dual cells of higher dimension and made to rest entirely on dual $(n-p)$ -cells, so that each $(n-p)$ -cycle will be homologous to a dual $(n-p)$ -cycle. But if this argument were to be valid we should arrive at the conclusion that $R_{n-p}(K - L \bmod J) = R_p(K \bmod L)$, which is not the previous result and need not actually be true. The reason why this argument fails is that the cells dual to simplexes of J do not display this convenient homology characteristic when taken $\bmod J$.

The natural attack on this double difficulty is to take $K - L \bmod J$ as our fundamental complex and consider cells dual to simplexes of it lying

in K' . Exactly the same scheme as before is appropriate. Let us introduce N' as the boundary of the simplicial neighbourhood of L' in K' .

The set of cells dual to the simplexes of $K-L \bmod J$ now covers the complex $K'-J' \bmod N'$. Each simplex of $K-L \bmod J$ has a complement with R_{n-p-1} and θ_{n-p-2} sphere-like; so that, in this dual complex, each cell, as a subcomplex of K' , has R_{n-p} and θ_{n-p-1} cell-like. This is true without exception for these dual cells.

We can now state

THEOREM 7. *If $K \bmod L$ is p -regular, and C_{n-p} is a chain of $K'-J' \bmod N'$ bounded by an actual dual cycle Γ_{n-p-1}^+ , then there is an actual dual chain C_{n-p}^+ which is homologous to some multiple of C_{n-p} .*

For we can examine whether C_{n-p} enters any E_n^+ ; if it does, since E_n^+ has $R_{n-p} = 0$, locally C_{n-p} is weakly homologous to zero; hence, by taking some multiple of C_{n-p} , we can find a homologous chain not entering this E_n^+ . We do this successively through all the dimensions down to $n-p$. For, since the boundary is a genuine dual cycle, for any dual cell that it enters it is locally a (relative) cycle. This argument shows that, for some t , $tC_{n-p} \sim C'_{n-p}$, where C'_{n-p} lies only on the E_{n-p}^+ 's. But each E_{n-p}^+ has $R_{n-p} = 1$; hence, if C'_{n-p} enters any E_{n-p}^+ , it covers the whole of it; and consequently C'_{n-p} may be written as a chain of dual cells.

COROLLARY 1. *If Γ_{n-p-1}^+ is weakly homologous to zero over $K'-J' \bmod N'$, it is dually weakly homologous to zero.*

COROLLARY 2. *Every Γ_{n-p} of $K'-J' \bmod N'$ is weakly homologous to a dual cycle.*

For it has no boundary, and so its boundary is a Γ_{n-p-1}^+ .

THEOREM 8. *If $K \bmod L$ is p -regular, and $t\Gamma_{n-p-1} \sim 0$, where Γ_{n-p-1} is a cycle of $K'-J' \bmod N'$, then Γ_{n-p-1} is homologous to a Γ_{n-p-1}^+ .*

By a similar argument, Γ_{n-p-1} is weakly homologous to zero locally in any dual cell it enters; but locally $\theta_{n-p-1} = 0$, or any cycle weakly homologous to zero bounds simply. So Γ_{n-p-1} can be taken off any dual cell of dimension greater than or equal to $n-p$. Let us call the new cycle Δ_{n-p-1} which lies only on E_{n-p-1}^+ 's. We cannot immediately conclude that it is a Γ_{n-p-1}^+ because we do not know that each E_{n-p-1}^+ has $R_{n-p-1} = 1$. But we know that $t\Delta_{n-p-1}$ bounds some C_{n-p} . By the argument of the above theorem (which needed really not that the boundary of the chain was a dual cycle, but that it lay entirely on E_{n-p-1}^+ 's) $t'\Delta_{n-p-1}$ bounds a genuine dual chain; consequently $t'\Delta_{n-p-1}$ is a genuine dual cycle, and it follows at once that Δ_{n-p-1} is also a dual cycle.

Now suppose $K \bmod L$ to be p - and $(p-1)$ -regular. Consider the classes of cycles of $K' - J' \bmod N'$, classified in terms of weak homologies; that is, Γ_k and Δ_k belong to the same class if we can find any s and t , neither of them zero, such that $s\Gamma_k + t\Delta_k$ bounds. By the second corollary every such class contains at least one representant which is a dual cycle; and by the first corollary, since we have also $(p-1)$ -regularity, two such representants of the same class belong to the same weak dual homology class. The converse is clearly true, so that we have an isomorphism between the $(n-p)$ -th weak homology group of $K' - J' \bmod N'$ and the p -th weak dual group of $K - L \bmod J$. Thus

$$R_{n-p}(K' - J' \bmod N') = R_p(K - L \bmod J).$$

We have now to show that $\theta_{n-p-1}(K' - J' \bmod N') = \theta_p(K - L \bmod J)$. Consider the $(n-p-1)$ -th torsion group of $K' - J' \bmod N'$. If Γ_{n-p-1} is a member of it, by Theorem 8 it is homologous to a Γ_{n-p-1}^+ . So each class in this group contains one representant which is a dual cycle. If now $t\Gamma_{n-p-1} \sim 0$, then $t\Gamma_{n-p-1}^+$ bounds on $K' - J' \bmod N'$. But seeing that we have p - and $(p-1)$ -regularity we can strengthen Theorem 7; for in each E^+ now R_{n-p} and θ_{n-p} are cell-like; so that any chain bounded by an $n-p-1$ dual cycle is strongly homologous to a dual chain. Consequently $t\Gamma_{n-p-1}^+$ is the boundary of a dual chain. And equally, if Γ_{n-k-1}^+ and Δ_{n-p-1}^+ are two representants of the same class of $K' - J' \bmod N'$, they are homologous over $K' - J' \bmod N'$ and so are dually homologous.

The converses are trivial, so that we have: corresponding to each torsion class of $K' - J' \bmod N'$ of dimension $n-p-1$, having coefficient t , there is a unique torsion class of the dual complex, also having coefficient t . This gives us our second result that

$$\theta_{n-p-1}(K' - J' \bmod N') = \theta_p(K - L \bmod J).$$

Tucker (8) discusses complexes in which certain sets of cells are amalgamated to form cells of a new complex, and examines the circumstances in which the two complexes have the same homology characters. He actually discusses only the case where each amalgamation has the homology characters of a cell of the appropriate dimension; and one effect of this is that he is able to define the incidence number between two such amalgamated sets in terms of the incidence in the original complex between their fundamental cycles. If we apply his methods, we see that each amalgamation has R_{n-p+1} , R_{n-p} , θ_{n-p} , and θ_{n-p-1} cell-like; consequently we can take over these methods for the incidence matrix between dimensions $n-p+1$ and $n-p$. When we pick out the submatrix which gives the incidences of the new complex, we ignore certain torsion

coefficients, but we do get the result that the $(n-p)$ -th Betti number is the same for each complex, although the $(n-p)$ -th torsion groups may differ.

When we come to the $(n-p)$ -th and $(n-p-1)$ -th incidence matrix, we have no determinate submatrix to pick out, because the $n-p-1$ amalgamations may have several $(n-p-1)$ -cycles; but from the fact that θ_{n-p} and θ_{n-p-1} are null, we may use Tucker's reduction of the matrix, and we can see that, whichever way we pick out the submatrix for the new complex, the parts which are ignored cannot contribute any torsion coefficients, and we find that θ_{n-p-1} is the same for the two complexes. We could not so easily arrive at this result if some $n-p-1$ amalgamation had no $(n-p-1)$ -cycle; but, from the way in which our complex is constructed, we see that this cannot be the case.

We see then that here too a purely algebraic argument can be given an intuitive significance by examining closely the geometric content. The methods used above yield Tucker's result in full; he forms his cells into cell-blocks and makes of them an abstract complex whose incidence matrices have certain invariants identical with those of the original complex. By our argument, we show that each homology class on the original complex has a representant composed of cell-blocks and that two homologous cell-block cycles bound a chain of cell-blocks.

In order to complete this proof we have to show that the $(n-p)$ -th Betti number and $(n-p-1)$ -th torsion coefficients of $K'-J' \bmod N'$ and of $K' \bmod L'$ are the same. We do this by showing that each has characters the same as those of $K'-J' \bmod L'$.

We have a complete isomorphism between the groups of $K'-J' \bmod N'$ and of $K'-J' \bmod L'$; to any cycle of $K'-J' \bmod L'$ we associate that cycle which consists of those of its simplexes lying in $K'-J' \bmod N'$, with the same coefficients. This sends homology classes into unique homology classes. And two non-homologous cycles of the first cannot become homologous under this transformation; for, by subtracting the boundary of the region bounded, we should have a non-bounding cycle of $K'-J' \bmod L'$ going into zero on $K'-J' \bmod N'$; it would thus have to lie entirely in the neighbourhood of L' . But we can make a simplicial transformation of this neighbourhood of L' into L , by ordering the vertices of L and sending every vertex of the neighbourhood into the heaviest vertex in the simplex containing it. Since L has a normal neighbourhood in K , every simplex, all of whose vertices lie on L , lies itself on L ; consequently this is a genuine transformation on to L , which has the usual deformation property. Now any cycle of the neighbourhood $\bmod L$ will go into a chain of L and, since the transformation is a deformation, it must previously have bounded $\bmod L$. This completes the first half of the proof.

We must now show that

$$R_{n-p}(K' - J' \bmod L') = R_{n-p}(K' \bmod L'),$$

$$\theta_{n-p-1}(K' - J' \bmod L') = \theta_{n-p-1}(K' \bmod L').$$

This is done by proving that each non-bounding $(n-p)$ class of $K' \bmod L'$ has a representant lying away from J' on $K' - J' \bmod L'$, and similarly for the torsion $(n-p-1)$ classes.

We may piece together the neighbourhood of J' into dual cells in the familiar way; and the conditions that p -regularity imposes on the simplexes of J tell us that in each of these cells $R_{n-p} = 0$, $\theta_{n-p-1} = 0$, and from the $(p-1)$ -regularity $R_{n-p+1} = 0$, $\theta_{n-p} = 0$.

By the old argument, any $(n-p)$ -chain of the neighbourhood of J' with boundary away from J can be pushed on to dual cells of dimension $n-p$; and, since even these have $R_{n-p} = 0$, the final cycle must lie completely off the neighbourhood. Similarly for the torsion cycles of dimension $n-p-1$. Repetition of the argument is unnecessary; the result is that each class in question has a representant on $K' - J' \bmod L'$, and two such representants are themselves homologous on $K' - J' \bmod L'$. Consequently the required groups are isomorphic.

As the final result we have again

$$R_{n-p}(K \bmod L) = R_p(K - L \bmod J),$$

$$\theta_{n-p-1}(K \bmod L) = \theta_p(K - L \bmod J).$$

Derivation of classical duality theorems.

First and most obviously, Čech's partial duality theorem (1) is included; he has no J and no L , and his conditions for p -regularity are slightly stronger than those used here. Consequently his result, that for a p - and $(p-1)$ -regular complex having a covering cycle, $R_{n-p} = R_p$, $\theta_{n-p-1} = \theta_p$, is deducible as a special case of our theorem.

Secondly we shall show that the general duality theorem given by Lefschetz (2) follows from this one; since all the other duality theorems are special cases of his, we need go no further than that. He takes K to be an orientable manifold, with a closed subcomplex, Q , and a closed subcomplex of this, Q_1 ; $Q - Q_1$ he calls Q_2 . Q is taken to have a normal neighbourhood in K .

He next proves the duality which holds between $(K - Q_1) \bmod Q_2$ and $(K - Q_2) \bmod Q_1$. Consider now $K' - Q_2' \bmod Q_1'$; Q_1' corresponds to L in our theorem, and the boundary of the simplicial neighbourhood of Q_2' is J' .

With this understanding, the complex is p -regular for all p , and we have now full duality between $K' - Q_2' \bmod Q_1'$ and $K' - Q_1' \bmod J'$. We have only to show that the groups of $K' - Q_1' \bmod J'$ are isomorphic with those of $K' - Q_1' \bmod Q_2'$. A similar result has been proved above for the case where Q_2' was a closed subcomplex; the crux of the proof was to show that no cycle $\bmod Q_2'$ lying in the simplicial neighbourhood of Q_2' could be non-bounding $\bmod Q_2'$. Since Q_2' is actually a closed subcomplex of the complex $K' - Q_1'$ (or rather, those of its simplexes lying in the complex form a closed set), and since Q_2 has a normal neighbourhood in K , precisely the same argument applies. Since the homology characters are known to be the same whether we take the neighbourhoods in K or in K' , we can immediately derive the full duality between $K - Q_1 \bmod Q_2$ and $K - Q_2 \bmod Q_1$.

It should be pointed out that the methods of this paper and those used by Lefschetz to prove this duality theorem do not differ fundamentally. For, if we think of the embedding method, the manifold in which we embedded $K - Q_2 \bmod Q_1$ could be simply K ; and of course the K^+ method is merely an extension of the usual method of proof for manifolds to more general complexes.

For non-orientable manifolds we get duality only if we take as coefficients the integers $\bmod 2$. In our theorem we should then have L and J missing, because the simplexes of J all occur in the boundary with even coefficients. Therefore this result too is a direct consequence of the general theorem.

Intersections on a general complex.

Previously we have made use of intersections defined in terms of looping coefficients of complements; but henceforth we confine ourselves to intersections between ordinary chains of $K \bmod L$ and chains of K^+ . Between two such chains it is easy to define the intersection in the way it has been defined by Lefschetz for manifolds.

We say that $(\sigma_p^i, E_{n-q}^{j+})$ is zero unless σ_q^j is a face of σ_p^i ; and if σ_q^j is a face of σ_p^i , we take all the $(p-q)$ -simplexes of the first derived of σ_p^i which have the point of subdivision of σ_q^j as the vertex corresponding to the simplex of lowest dimension. Since we assume σ_p^i and σ_q^j to be given to us oriented, as chains, we can assign an orientation to any of these $(p-q)$ -simplexes, τ_{p-q} , in the following way: first write down the vertices of σ_q^j to give the right orientation, then the vertex which when added gives the $(q+1)$ -simplex represented in τ_{p-q} , and so on, until we have all the vertices of σ_p^i written down in a definite order; if this yields the right orientation of σ_p^i , we give τ_{p-q} the coefficient $+1$; if not, we give it the

coefficient -1 . This chain we call the intersection chain $(\sigma_p^i \cdot E_{n-q}^{j+})$, where σ_p^i and σ_q^j are taken with a definite orientation.

With this definition it is a matter of elementary computation to show that

$$F(\sigma_p^i \cdot E_{n-q}^{j+}) = \sigma_p^i \cdot F^+(E_{n-q}^{j+}) + (-1)^q F(\sigma_p^i) \cdot E_{n-q}^{j+}.$$

This formula tells us that, if we form the intersection of a cycle of $K \bmod L$ with a dual cycle of K^+ (which is a cycle of $K' - L' \bmod J'$), the resulting chain is a cycle of $K' - L'$.

Also the intersection of a bounding cycle of $K \bmod L$ with a dual cycle is a bounding cycle on $K' - L'$; and that of a cycle of $K \bmod L$ with a dually bounding dual cycle is again bounding on $K' - L'$.

Consequently by means of this intersection we can associate with any pair of a homology class of $K \bmod L$ and a dual homology class of K^+ a unique homology class of $K' - L'$.

If now we make an ordering of the vertices of K and L , but make all the vertices of L lighter than all those of K , we can define a simplicial transformation of $K' - L'$ into $K - L$, by sending each vertex of $K' - L'$ into the heaviest vertex of its containing simplex. Since L has a normal neighbourhood in K , no vertex goes into L , and we have the required result. This transformation preserves all homologies. Now we have associated with any pair, consisting of a cycle of $K \bmod L$, and a dual cycle, a unique cycle of $K - L$, based on this ordering. It is well known (6) that, in case L is missing, this cycle is exactly the Čech-multiple of the two cycles. This result simply shows that we can repeat the process on $K \bmod L$, and by a careful enough ordering of the vertices the multiple can be obtained as a cycle on $K - L$; and that the corresponding homology results hold on $K - L$.

It may be worth while to point out that these intersections can be developed not from K^+ but from K^* : the realization of the dual cycle can take place on K^* and we have access to the classical manifold results on intersection to give the intersection between a cycle of K^* and one of $K \bmod L$. It is clear that the resulting intersection chain is identical with that previously arrived at with a possible change of sign.

So far we have defined only intersections between chains of one complex and chains of another, giving as a result chains on a third. In its most general form that type of work is barren of content, and most interest attaches to intersections when all three of the complexes coincide; in order to achieve this we must first assume that J and L are lacking, so that we are dealing with a closed complex which has a covering cycle. Our previous work now gives us a definition of intersection between a cycle of

K and one of K^+ giving a cycle of K' . But we do not yet have the required result for homology classes of K : for, suppose that we take B_p^+ and C_p^+ , both cycles of K' , homologous on K' ; yet, because we do not know that they are dually homologous, we cannot say that their intersections with a cycle of K will be homologous on K' .

As a simple example of this difficulty, take two 2-spheres S and T , and suppose that they have one common point A . In the dual complex there are, of course, vertices, and each vertex is homologous to any other on K' . But let us try to define the intersection of the cycle S and the homology class of vertices; we may take as representant a vertex which is a dual cycle; if we take a vertex on S , the intersection is that point (or its negative), and if we take one on T the intersection is zero. In fact the intersection cannot be defined to within homology, because actual homology over K' of dual cycles does not imply dual homology. A vertex on S is not dually homologous to one on T .

But we know of a condition that we can impose, which disposes of this problem. From the first corollary to Theorem 5, we can say that, if K is p -regular, any dual cycle of dimension p which bounds on K' bounds dually. Consequently in a p -regular complex we can define the intersection between any dual type homology class of dimension p and any other homology class whatsoever, which is a uniquely defined homology class of K' . For we take any dual representant of the class and form its intersection with any representant of the other class: the homology class of the resulting cycle is now independent of which representants were chosen.

This result includes the result of Čech (1) that, in a p - and q -regular complex, intersection can be defined between dual type p -classes and dual type q -classes; the increase in generality seems to be due to the geometric approach. Čech takes a dual type A_p and expresses it as $\Gamma_n \cdot A^{n-p}$; and the B_q as $\Gamma_n \cdot B^{n-q}$; the intersection is defined as

$$\Gamma_n \cdot A^{n-p} \cdot B^{n-q} = \Gamma_n \cdot C^{2n-p-q} = C_{p+q-n}.$$

The uniqueness of the homology classes of A^{n-p} and B^{n-q} is guaranteed by the p - and q -regularity of K .

Suppose now that Čech allowed a new multiplication between Γ_n and any ordinary cycle, whose result was the cycle itself; this would be geometrically convincing and would not disturb his algebra of cycles. Then in a p -regular complex a dual type p -cycle would be represented as $\Gamma_n \cdot A^{n-p}$; the product with any B_q would be $\Gamma_n \cdot C_{p+q-n} = C_{p+q-n}$.

Certainly, whether or not this multiplication is introduced, under the Čech scheme the homology class of C_{p+q-n} is uniquely defined; and equally

certainly this product would coincide with the intersection product in a manifold.

As a particular case of this type of intersection, it might be well to point out that, if K is p - and $(p-1)$ -regular, every p -cycle is of dual type, and so we have intersection defined between any p -homology class and any other class.

It has been stated that this intersection between cycles and dual cycles could have been considered from the point of view of K^* . In the p -regular complex, to any dual type p -homology class of K we assign uniquely an $(m-n+p)$ -homology class of K^* and thus get a unique intersection class on K' of dimension $p+q-n$. This intersection, we have seen, is equivalent to one type of Čech-multiplication.

The second type (in Čech's paper the first type) refers to two dual cycles and gives as a result a dual cycle defined to within dual homology. We here outline a proof† that this multiplication is identical with intersection in K^{**} . The multiplication is based on an ordering of the vertices, Ω : we will assume the vertices of L to be the lightest under Ω . Simplexes of K are taken to be written with their vertices ordered as in Ω . The multiple of σa with $a\sigma'$ is $\sigma a\sigma'$: the multiple of two simplexes not of this type is zero.

We now introduce $\psi(\sigma)$ defined for any simplex of K ; it is $\bar{K}^*/\sigma - K/\sigma$, considered as a chain. We designate by t the simplicial transformation of $K' \bmod L'$ into $K \bmod L$ based on Ω and we extend it to be the identity off K ; so that $t(K^{**}) = \bar{K}^*$.

We now derive certain formulae, identical with those given by Lefschetz (6), the difference being that t is now a different and less easily handled transformation than the one that he uses.

First, $t[(\sigma_1 a)^*] = \sum_{\sigma_1 a\tau \text{ in } K}^{\tau} a\tau\psi(\sigma_1 a\tau)$, as is easily verified: the definition of the complement as a chain ensures that the correct sign is given to each simplex.

We now examine the intersection of this with $(b\sigma_2)^*$: for any τ this is zero, unless $a\tau$ has $b\sigma_2$ as a face. Then b must be equal to a to get an intersection at all, and τ need be taken to run over only the set of simplexes having σ_2 as a face. Further, when we have seen what the intersection is, we take the transform of this intersection and we get degenerate simplexes

† H. Freudenthal in his paper "Alexanderscher und Gordonscher Ring und ihrer Isomorphie", *Annals of Math.* (2), 38 (1937), 647-655, proves an equivalent result by an analogous method. The author's result was obtained independently.

unless the vertices of σ_2 are the first vertices in τ . We can therefore say

$$t[(a\sigma_2)^* \cdot t(\sigma_1 a)^*] = t[(a\sigma_2)^* \cdot \sum_{\sigma_1 a \sigma_2 \tau \text{ in } K} a\sigma_2 \tau \psi(\sigma_1 a \sigma_2 \tau)].$$

The intersection on the right-hand side can be easily shown to be normal. A simplex of the derived of the second term belongs to it if, and only if, it contains the point of subdivision of $a\sigma_2$, and of no other of its faces, and is of full dimension otherwise. Call such a simplex ρ . The complement of ρ in the derived of the second term is the first derived of $F(a\sigma_2)$; a rather complicated examination of the situation reveals that the whole complement of ρ in $K^{*'}$ is the join of this complement to its complement in $(a\sigma_2)^*$. But two spheres have looping coefficient 1 in their join, and consequently the coefficient of ρ in the intersection is 1.

Now, applying the transformation, we see that, if b is the last vertex of σ_2 , the right-hand side is equal to

$$\sum_{\sigma_1 a \sigma_2 \tau \text{ in } K} b \tau \psi(\sigma_1 a \sigma_2 \tau) = t[(\sigma_1 a \sigma_2)^*].$$

Suppose, now, that we have two dual cycles; we consider pairs of their dual cells in turn, and this formula shows us that, if we take the transform of one of the cycles and form the intersection, we get the transform of their Čech-multiple. But the transformation preserves homology classes, and we can state that the Čech-multiple of two dual cycles gives us a dual cycle defining their intersection homology class in $K^{*'}$.

If K is a manifold, the K^* model can be got by immersion in itself: consequently this shows that the Čech-multiplication on $K \bmod L$, where K is a manifold, is isomorphic with intersections for complementary dimensions on $K-L$.

Let us then think of K immersed in K^* ; we have three types of intersections. First, intersections between two cycles of K^* ; since K^* is a manifold, no difficulties are experienced here; each cycle is a dual cycle of the complex K , and the homology class of the intersection cycle can be found by the simple mechanical method of the Čech-multiplication for two dual cycles: this is true for an open K provided that we make a suitable ordering of the vertices.

Secondly, we have intersection between chains of K^* and chains of K ; this intersection is of the simplest type, and classical intersection theory gives as a result a chain of K . When cycles are used, we get a cycle of $K-L$, and again its homology class can be discovered by the other method of Čech-multiplication.

Lastly, we may wish to consider intersections between actual cycles of K ; this is defined if at least one of the cycles is of dual type, and if K is regular in the dimension of that cycle. In that case the intersection class is uniquely defined and can be arrived at by applying classical intersection theory.

In fact the well-known geometrical tools can give us the algebraic multiplications, and make it clear that these multiplications can also be used on open complexes. And, further than that, we show that we can also use these tools on complexes of greater generality than orientable manifolds.

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INVARIANT AND COVARIANT SYSTEMS ON AN ALGEBRAIC VARIETY

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Received and read 19 January, 1939.]

Introduction.

The present paper is concerned with elaborating and extending the theory of canonical systems of equivalence on an algebraic variety, which I have introduced in previous papers†. Among many possible applications of this theory, one which suggests itself is the general problem of intersections of algebraic manifolds, the determination of the residual variety of intersection of a number of varieties all of which pass through a given variety. This problem is not dealt with in the present paper, but the formal development of the general theory given below is designed to pave the way for a treatment of the question, to which I hope to return later.

Much of the formal manipulation of systems of equivalence seems to involve a good deal of complicated algebra, in which the geometrical content of the subject tends to be obscured. With a view to minimising the complications of this nature, I have devoted the first section of the paper to establishing a number of identities of which frequent use will be made. In the next section I introduce the notion of the *ring of equivalence* on an algebraic variety; this is a commutative ring whose elements are systems of equivalence, and in which the algebraic operation of multiplication corresponds to the geometrical operation of intersection. I have elaborated this notion more than is required for the rest of the paper because it seems to have interesting potentialities. In §3 the definition of the canonical system $\{X_h\}$ of dimension h on a V_d is recalled, and another

† Todd, 8, 9, 10. See also Eger, 2.

set of invariant systems $\{Y_h\}$ is introduced. These systems are expressible in terms of the canonical systems and of their intersections, and conversely, the systems $\{X_h\}$ can be expressed in terms of the systems $\{Y_h\}$ and their intersections; the explicit forms of these relations show a curious kind of reciprocity. In §4 I initiate the theory of birational *covariant* systems of a pair of algebraic varieties V_r, V_d , of which V_d lies on V_r . A very large number of such covariant systems can be defined in terms of the invariant systems on V_r and V_d and of their mutual intersections; but the present account is confined in the main to two sets of systems, which I denote by $\{X_k[V_d, V_r]\}$ and $\{Y_k[V_d, V_r]\}$, which possess an important property of section-invariance, namely that the section of $\{X_k[V_d, V_r]\}$ (or $\{Y_k[V_d, V_r]\}$) by a general variety S_{r-1} of $r-1$ dimensions on V_r which meets V_d in a variety S_{d-1} of $d-1$ dimensions is the system $\{X_{k-1}[S_{d-1}, S_{r-1}]\}$ (or $\{Y_{k-1}[S_{d-1}, S_{r-1}]\}$). These varieties seem to play an important part in intersection problems, probably on account of this property of section-invariance. In the last section of the paper I make application of the results developed in the earlier sections to obtain an expression, in invariant form, for the improper double locus which a V_d lying on a V_r in general possesses when $2d \geq r$. The result obtained includes (in a different form) results previously obtained by Yoxall† and Eger‡.

1. Preliminary identities.

1.1. Let R be an arbitrary commutative ring with a unit element e . Let $x_1, x_2, \dots$ be an arbitrary sequence of elements of R , and define $x_0 = e$. If the number t of elements in the sequence is finite we may suppose the sequence to extend to infinity, with the convention that $x_r = 0$ if $r > t$. We consider the formal power series

$$\phi(\lambda) \equiv \sum_{r=0}^{\infty} x_r \lambda^r \quad (1.01)$$

in an indeterminate λ . We denote by $\psi(\lambda)$ the power series which is formally inverse to $\phi(\lambda)$, and write

$$\psi(\lambda) \equiv \sum_{r=0}^{\infty} (-)^r y_r \lambda^r, \quad (1.02)$$

where

$$\phi(\lambda) \cdot \psi(\lambda) \equiv e. \quad (1.03)$$

† Yoxall, 11.

‡ Eger, 1.

Substituting from (1.01) and (1.02) in (1.03) we see that $y_0 = e$, and that

$$\sum_{i=0}^k (-)^i x_{k-i} y_i = 0 \quad (k = 1, 2, \dots). \quad (1.04)$$

The relations (1.04) give successively $y_1, y_2, y_3, \dots$ in terms of the x_r 's, and it is easily seen that y_k is a polynomial in $x_1, x_2, \dots, x_k$ which has integer coefficients and which is isobaric of weight k . The symmetry of (1.04) shows that x_k is the same function of $y_1, y_2, \dots, y_k$ as y_k is of $x_1, x_2, \dots, x_k$. The explicit expression for y_k is in fact given by the determinant

$$y_k = \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_{k-1} & x_k \\ e & x_1 & x_2 & \dots & x_{k-2} & x_{k-1} \\ 0 & e & x_1 & \dots & x_{k-3} & x_{k-2} \\ \dots & & \dots & & \dots & \dots \\ 0 & 0 & 0 & & e & x_1 \end{vmatrix}. \quad (1.05)$$

This is easily seen by induction. For, putting $k = 1$ in (1.04) we obtain $y_1 = x_1$, so (1.05) holds when $k = 1$. Assuming that it holds for y_h with $h < k$ we expand the determinant in (1.05) by the top row. The cofactor of x_i is easily verified to be $(-)^{i-1} y_{k-i}$, so that the value of the determinant is

$$\sum_{i=1}^k (-)^{i-1} x_i y_{k-i} = y_k,$$

by (1.04). In particular, we find that

$$y_1 = x_1, \quad y_2 = x_1^2 - x_2, \quad y_3 = x_1^3 - 2x_1x_2 + x_3.$$

1.2. Now let n be a fixed positive integer, and consider the polynomials ξ_r, η_r of degree r in the indeterminate λ , defined by

$$\xi_r \equiv \sum_{i=0}^r (-)^i \binom{n-r+i}{i} x_{r-i} \lambda^i, \quad (1.06)$$

$$\eta_r \equiv \sum_{i=0}^r (-)^i \binom{n+r-1}{i} y_{r-i} \lambda^i, \quad (1.07)$$

where the sequences $x_1, x_2, \dots, y_1, y_2, \dots$ are related as in (1.04). We make the usual convention that the binomial coefficient $\binom{a}{b}$ is to be replaced by zero if $a < b$. Thus, if $r > n$, ξ_r vanishes.

We prove that

$$\sum_{j=0}^k (-)^j \xi_{k-j} \eta_j = 0 \quad (k = 1, 2, \dots). \quad (1.08)$$

We introduce a second indeterminate ρ . To prove (1.08) it is sufficient to show that

$$\left(\sum_{r=0}^{\infty} \xi_r \rho^r \right) \cdot \left(\sum_{s=0}^{\infty} (-)^s \eta_s \rho^s \right) = e, \quad (1.09)$$

for (1.08) follows from (1.09) by equating the coefficient of ρ^k on the left of (1.09) to zero.

Now, using (1.06), we have

$$\begin{aligned} \sum_{r=0}^{\infty} \xi_r \rho^r &= \sum_{r=0}^{\infty} \sum_{i=0}^r (-)^i \binom{n-r+i}{i} x_{r-i} \lambda^i \rho^r = \sum_{r=0}^{\infty} \sum_{h=0}^r (-)^{r-h} \binom{n-h}{r-h} x_h \rho^h (\lambda \rho)^{r-h} \\ &= \sum_{h=0}^{\infty} \sum_{r=h}^{\infty} (-)^{r-h} \binom{n-h}{r-h} x_h \rho^h (\lambda \rho)^{r-h} = \sum_{h=0}^{\infty} \sum_{q=0}^{\infty} (-)^q \binom{n-h}{q} x_h \rho^h (\lambda \rho)^q \\ &\quad - \sum_{h=0}^{\infty} x_h \rho^h (1 - \lambda \rho)^{n-h}, \end{aligned}$$

since $\binom{n-h}{q} = 0$ when $q > n-h$. Hence, if we put

$$\sigma = \frac{\rho}{1 - \lambda \rho}, \quad (1.10)$$

we see by (1.01) that

$$(1 - \lambda \rho)^{-n} \sum_{r=0}^{\infty} \xi_r \rho^r = \phi(\sigma). \quad (1.11)$$

Again, from (1.07),

$$\begin{aligned} \sum_{s=0}^{\infty} (-)^s \eta_s \rho^s &= \sum_{s=0}^{\infty} \sum_{i=0}^s (-)^{s+i} \binom{n+s-1}{i} y_{s-i} \lambda^i \rho^s \\ &= \sum_{s=0}^{\infty} \sum_{h=0}^s (-)^h \binom{n+s-1}{s-h} y_h \rho^h (\lambda \rho)^{s-h} \\ &= \sum_{h=0}^{\infty} \sum_{s=h}^{\infty} (-)^h \binom{n+s-1}{s-h} y_h \rho^h (\lambda \rho)^{s-h} \\ &= \sum_{h=0}^{\infty} \sum_{p=0}^{\infty} (-)^h \binom{n+h-1+p}{p} y_h \rho^h (\lambda \rho)^p \\ &= \sum_{h=0}^{\infty} (-)^h y_h \rho^h (1 - \lambda \rho)^{-(n+h)}, \end{aligned}$$

and so, from (1.02) and (1.10),

$$(1-\lambda\rho)^n \sum_{s=0}^{\infty} (-)^s \eta_s \rho^s = \psi(\rho). \quad (1.12)$$

(1.09) follows from (1.11), (1.12) and (1.03), and from (1.09) follows (1.08).

1.3. Let $x_1', x_2', \dots$ be a second sequence of elements of the ring R , define $x_0' = e$ and put

$$\phi'(\lambda) \equiv \sum_{r=0}^{\infty} x_r' \lambda^r. \quad (1.13)$$

As in 1.1 we consider the inverse power series

$$\psi'(\lambda) = \sum_{r=0}^{\infty} (-)^r y_r' \lambda^r, \quad (1.14)$$

where

$$\phi'(\lambda) \cdot \psi'(\lambda) = e, \quad (1.15)$$

and define a sequence of elements $y_1', y_2', \dots$. Put now

$$x_r^* \equiv \sum_{i=0}^r (-)^i x_i y_{r-i}' \quad (r = 0, 1, 2, \dots), \quad (1.16)$$

$$y_r^* \equiv \sum_{i=0}^r (-)^i y_i x_{r-i}' \quad (r = 0, 1, 2, \dots). \quad (1.17)$$

We show that

$$\sum_{j=0}^k (-)^j x_{k-j}^* y_j^* = 0 \quad (k = 1, 2, \dots). \quad (1.18)$$

For

$$\phi(-\lambda) \cdot \psi'(-\lambda) = \left(\sum_{r=0}^{\infty} (-)^r x_r \lambda^r \right) \left(\sum_{s=0}^{\infty} y_s' \lambda^s \right) = \sum_{t=0}^{\infty} x_t^* \lambda^t,$$

and

$$\psi(-\lambda) \cdot \phi'(-\lambda) = \left(\sum_{r=0}^{\infty} y_r \lambda^r \right) \left(\sum_{s=0}^{\infty} (-)^s x_s' \lambda^s \right) = \sum_{u=0}^{\infty} (-)^u y_u^* \lambda^u,$$

and so, from (1.03) and (1.15),

$$\left(\sum_{t=0}^{\infty} x_t^* \lambda^t \right) \left(\sum_{u=0}^{\infty} (-)^u y_u^* \lambda^u \right) = e,$$

whence (1.18) follows by equating to zero the coefficient of λ^k .

2. The ring of equivalence on a V_d .

Let V_d be an irreducible algebraic variety of d dimensions. An effective algebraic variety V_k of k dimensions lying on V_d is called *pure* if it can be expressed as the sum of a finite number of irreducible varieties each of which is of dimension k . The aggregate of pure effective V_k 's on V_d forms an abelian semi-group under the operation of addition, and by the introduction of virtual elements this can be extended into an abelian group $\mathfrak{G}_k$. Each element of $\mathfrak{G}_k$ can be expressed as the sum or difference of a finite number of pure irreducible V_k 's lying on V_d , and we may refer to such an element as a *pure virtual variety of k dimensions*.

The notion of equivalence between varieties of k dimensions on V_d , elaborated in numerous papers by Severi†, leads to a separation of the elements of $\mathfrak{G}_k$ into cosets, each coset consisting of the aggregate of varieties equivalent to any one of its members‡. The factor group of the coset containing the zero element of $\mathfrak{G}_k$ is an additive abelian group $\mathfrak{H}_k$ whose elements are the *complete virtual systems of equivalence* of V_k 's on V_d . Such a group exists for each value of k from 0 to d inclusive; and the group $\mathfrak{H}_d$ is a free abelian group with a single generator, being simply the additive group of all integer multiples of V_d .

Let V_k and V_l' be two pure varieties on V_d , and suppose that $k+l \geq d$. The *virtual intersection* $(V_k \cdot V_l')$ of V_k and V_l' is a pure variety of $k+l-d$ dimensions. It has a definite symbolic meaning even if the actual intersection of V_k and V_l' has components of dimension greater than $k+l-d$ (as will happen, for instance, if V_l' contains V_k), and in such a case, although there is no actual pure intersection variety of $k+l-d$ dimensions, the symbol $(V_k \cdot V_l')$ defines a pure variety belonging to a definite system of equivalence, and so corresponds to a definite member of $\mathfrak{H}_{k+l-d}$. Further, it is a simple consequence of the definition of equivalence that, as V_k and V_l' vary respectively in systems of equivalence, their virtual intersection $(V_k \cdot V_l')$ also varies in a system of equivalence; this follows from the transitivity of the equivalence relation, and from the fact that equivalence on a subvariety of V_d implies equivalence on V_d itself§.

Thus to any ordered pair of elements $(x^{(k)}, y^{(l)})$ such that $x^{(k)}$ belongs to $\mathfrak{H}_k$ and $y^{(l)}$ to $\mathfrak{H}_l$ corresponds a definite member $(x^{(k)} \cdot y^{(l)})$ of $\mathfrak{H}_{k+l-d}$. If $k+l < d$ we define formally $(x^{(k)} \cdot y^{(l)})$ as being zero. Clearly

$$(x^{(k)} \cdot y^{(l)}) = (y^{(l)} \cdot x^{(k)}). \quad (2.01)$$

† See, e.g., Severi, 5, where earlier references are given.

‡ This notion is elaborated in Todd, 7.

§ For a proof of the latter statement see Severi, 5, 402.

Also, if $y^{(l)}$ is another element of $\mathfrak{H}_l$,

$$(x^{(k)} \cdot y^{(l)} + y'^{(l)}) = (x^{(k)} \cdot y^{(l)}) + (x^{(k)} \cdot y'^{(l)}), \quad (2.02)$$

the addition on the left being group-addition in $\mathfrak{H}_l$, and that on the right being group-addition in $\mathfrak{H}_{k+l-d}$. Further, if $z^{(m)}$ is an element of $\mathfrak{H}_m$,

$$(x^{(k)} \cdot (y^{(l)} \cdot z^{(m)})) = ((x^{(k)} \cdot y^{(l)}) \cdot z^{(m)}), \quad (2.03)$$

each side being zero if $k+l+m < 2d$, and an element of $\mathfrak{H}_{k+l+m-2d}$ otherwise.

We now consider the aggregate $\mathbf{R}$ of all ordered sets of elements $x = [x^{(0)}, x^{(1)}, \dots, x^{(d)}]$, where $x^{(k)}$ is an element of $\mathfrak{H}_k$ for $k = 0, 1, \dots, d$. We define two operations in $\mathbf{R}$, which we represent by the signs of addition and multiplication, as follows:

$$[x^{(0)}, x^{(1)}, \dots, x^{(d)}] + [y^{(0)}, y^{(1)}, \dots, y^{(d)}] = [x^{(0)} + y^{(0)}, x^{(1)} + y^{(1)}, \dots, x^{(d)} + y^{(d)}], \quad (2.04)$$

$$[x^{(0)}, x^{(1)}, \dots, x^{(d)}] \cdot [y^{(0)}, y^{(1)}, \dots, y^{(d)}] = [t^{(0)}, t^{(1)}, \dots, t^{(d)}], \quad (2.05)$$

where
$$t^{(k)} = \sum_{p+q=d+k} (x^{(p)} \cdot y^{(q)}), \quad (2.06)$$

the summations in (2.06) and on the right of (2.04) being group-additions in the corresponding $\mathfrak{H}_k$.

The relations implied by the fact that each $\mathfrak{G}_k$ is an additive abelian group clearly imply that if x, y, z are any three elements of $\mathbf{R}$, then

$$x + y = y + x,$$

$$x + (y + z) = (x + y) + z,$$

and that

$$x + 0 = x,$$

where 0 denotes the set $[0, 0, \dots, 0]$ of zero-elements of the groups $\mathfrak{H}_k$. Further, given any two elements x and y there is an element z of $\mathbf{R}$ such that $z = x + y$.

The relations (2.01), (2.02), (2.03) imply that

$$x \cdot y = y \cdot x,$$

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

Consequently the aggregate $\mathbf{R}$ under the operations defined in (2.04), (2.05), (2.06) is a *commutative ring*. We call this the *ring of equivalence* on V_d . It possesses a unit element, namely $[0, 0, \dots, 0, e]$, where e is the generating operation of $\mathfrak{H}_d$.

An element $x = [x^{(0)}, x^{(1)}, \dots, x^{(d)}]$ of $\mathbf{R}$ which is not the zero-element is said to have *dimension* k if $x^{(k)} \neq 0$ but $x^{(h)} = 0$ for all $h > k$; it is called a *pure* element of dimension k if $x^{(k)} \neq 0$ but $x^{(h)} = 0$ for all $h \neq k$. The pure elements of $\mathbf{R}$ of dimension k correspond to the elements of $\mathfrak{H}_k$, and hence to the complete systems of equivalence of V_k 's on V_d . The pure elements of dimension d all have the form $[0, 0, \dots, 0, ne]$ where e is the generating operation of $\mathfrak{H}_d$ and n is an integer. It is clear from (2.06) that these elements can be treated as scalars in $\mathbf{R}$, since

$$[x^{(0)}, x^{(1)}, \dots, x^{(d)}] \cdot [0, 0, \dots, 0, ne] = [nx^{(0)}, nx^{(1)}, \dots, nx^{(d)}].$$

It will be convenient to refer to these elements simply as scalars.

The following properties of the ring $\mathbf{R}$ may easily be verified:

(a) Each element of $\mathbf{R}$ of dimension k can be expressed as the sum of pure elements of $\mathbf{R}$ of dimension less than or equal to k . For

$$[x^{(0)}, x^{(1)}, \dots, x^{(d)}] = [x^{(0)}, 0, \dots, 0] + [0, x^{(1)}, \dots, 0] + \dots + [0, 0, \dots, x^{(d)}].$$

(b) The sum of two pure elements of dimension k is zero or a pure element of dimension k .

(c) The product of two pure elements of $\mathbf{R}$ of respective dimensions k, l is zero or a pure element of dimension $k+l-d$ if this number is not negative, and is zero otherwise.

(d) The product of any element of $\mathbf{R}$ with a non-vanishing scalar is an element of the same dimension (or, exceptionally, zero since the groups $\mathfrak{H}_k$ may possess elements of finite period corresponding to divisors of zero on V_d).

(e) Every element of $\mathbf{R}$ of dimension less than d is nilpotent, and every product with more than d factors of dimension less than d vanishes. This follows from (c).

We are concerned mainly with the pure elements of $\mathbf{R}$ in what follows: we use the symbol e to denote the unit element of the ring.

3. *The invariant systems of equivalence on a V_d .*

3.1. We now consider a non-singular V_d lying in a linear space $[r]$ of r dimensions, where $r > 2d$. We denote, as usual, by $(a_0, a_1, \dots, a_d)$ the condition imposed on a $[d]$ lying in the $[r]$ to meet a given space $[a_i]$ in an $[i]$ for $i = 0, 1, \dots, d$; the space $[a_i]$ containing $[a_j]$ if $i > j$, so that $0 \leq a_0 < a_1 \dots < a_d \leq r$. In particular, we define

$$\tau_k = (r-d-1, r-d, \dots, r-d+k-2, r-d+k, r-d+k+1, \dots, r), \quad (3.01)$$

$$\sigma_k = (r-d-k, r-d+1, r-d+2, \dots, r). \quad (3.02)$$

Thus τ_k expresses the condition that a $[d]$ should lie in a prime through a given $[r-d+k-2]$, while σ_k expresses the condition that a $[d]$ should meet a given $[r-d-k]$; each of these conditions is of power k . It is known† that

$$\tau_k = \begin{vmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_{k-1} & \sigma_k \\ 1 & \sigma_1 & \dots & \sigma_{k-2} & \sigma_{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \sigma_1 \end{vmatrix}. \quad (3.03)$$

The tangent $[d]$'s to V_d which lie in a prime through a given $[r-d+k-2]$, Π say, touch V_d at the points of a variety M_{d-k} of $d-k$ dimensions. It is easy to prove that, as Π varies, M_{d-k} varies in a system of equivalence on V_d . For, if Π varies in a pencil in a fixed $[r-d+k-1]$, then the corresponding M_{d-k} varies in a linear system on the M_{d-k+1} which is the locus of points of contact of tangent spaces to V_d lying in primes through the base $[r-d+k-3]$ of the pencil. These varieties M_{d-k} are thus equivalent on V_d . Now any pair of $[r-d+k-2]$'s of the $[r]$ can be connected by a chain of spaces of the same dimension any consecutive pair of which lie in a $[r-d+k-1]$. Since the relation of equivalence on V_d is transitive it follows that any pair of varieties M_{d-k} are equivalent. The order μ_k of M_{d-k} is the k -th class of V_d .

In a similar way the tangent $[d]$'s to V_d which meet a given $[r-d-k]$ touch V_d at points of a variety N_{d-k} of $d-k$ dimensions, varying in a system of equivalence as the $[r-d-k]$ varies. The order ν_k of N_{d-k} is the k -th type of V_d . We may notice that the N_{d-1} and M_{d-1} corresponding to a given $[r-d-1]$ are the same. For convenience we define $M_d \equiv N_d \equiv V_d$.

The relation (3.03) shows that, on V_d , we have the equivalence

$$M_{d-k} \equiv \begin{vmatrix} N_{d-1} & N_{d-2} & \dots & N_{d-k} \\ 1 & N_{d-1} & \dots & N_{d-k+1} \\ 0 & 0 & \dots & N_{d-1} \end{vmatrix}, \quad (3.04)$$

the products being evaluated as intersections on V_d . Now let m_k, n_k denote the elements of the ring of equivalence $\mathbf{R}$ on V_d which correspond to the complete systems of equivalence defined by M_{d-k}, N_{d-k} respectively. Then (3.04) leads to the relation

$$m_k = \begin{vmatrix} n_1 & n_2 & \dots & n_k \\ e & n_1 & \dots & n_{k-1} \\ 0 & 0 & \dots & n_1 \end{vmatrix}, \quad (3.05)$$

expressing m_k in terms of $n_1, n_2, \dots, n_k$. From the results of 1.1 it follows that the same relation holds if the m 's and the n 's are interchanged.

3.2. Let $|S|$ denote the linear system of prime sections of V_d . We consider the two systems of equivalence

$$X_{d-k} \equiv \sum_{i=0}^k (-)^i \binom{d-k+1+i}{i} (M_{d-k+i} \cdot S^i), \quad (3.06)$$

$$Y_{d-k} \equiv \sum_{i=0}^k (-)^i \binom{d+k}{i} (N_{d-k+i} \cdot S^i). \quad (3.07)$$

The system X_{d-k} is the *canonical system*[†] of dimension $d-k$ on V_d . If s is the element of $\mathbf{R}$ which corresponds to $|S|$, then the systems X_{d-k} and Y_{d-k} correspond to elements x_k, y_k of $\mathbf{R}$, where

$$x_k = \sum_{i=0}^k (-)^i \binom{d-k+1+i}{i} m_{k-i} s^i, \quad (3.08)$$

$$y_k = \sum_{i=0}^k (-)^i \binom{d+k}{i} n_{k-i} s^i. \quad (3.09)$$

[†] Todd, 9, 1955, equation (4), with a slight change in notation.

Since the m 's and n 's are connected by (3.05) it follows from the results of 1.2 (with the integer n equal to $d+1$) that

$$\sum_{j=0}^k (-)^j x_{k-j} y_j = 0 \quad (k = 1, 2, \dots),$$

so that, on V_d ,
$$\sum_{j=0}^k (-)^j (X_{d-k+j} \cdot Y_{d-j}) \equiv 0 \quad (k = 1, 2, \dots). \quad (3.10)$$

The equivalence (3.10) shows that the systems Y_{d-k} can be expressed in terms of the systems X_{d-k} and their mutual intersections, and are therefore invariant systems of equivalence† on V_d . In particular,

$$Y_{d-1} \equiv X_{d-1}, \quad Y_{d-2} \equiv (X_{d-1}^2) - X_{d-2}, \quad Y_{d-3} \equiv (X_{d-1}^3) - 2(X_{d-1} \cdot X_{d-2}) + X_{d-3}.$$

The symmetry of (3.10) shows that the X 's are expressible in terms of the Y 's by precisely similar equivalences.

We may remark that (3.06) and (3.07) can be solved for M_{d-k} and N_{d-k} respectively, leading to the relations

$$M_{d-k} \equiv \sum_{i=0}^k \binom{d-k+1+i}{i} (X_{d-k+i} \cdot S^i), \quad (3.11)$$

$$N_{d-k} \equiv \sum_{i=0}^k \binom{d+k}{i} (Y_{d-k+i} \cdot S^i), \quad (3.12)$$

which, as may easily be verified, satisfy (3.06) and (3.07).

These systems Y_{d-k} have been introduced principally in order to facilitate certain calculations; being connected with the systems X_{d-k} by (3.10) they are not essentially new invariant systems. In the next paragraph we obtain certain useful properties of these systems.

3.3. The ordinary canonical systems X_{d-k} have an important property expressed by the equivalence†

$$X_{d-k}[S] \equiv (S \cdot \{X_{d-k+1}[S] + X_{d-k+1}[V]\}) \quad \text{on } S, \quad (3.13)$$

where S is an irreducible V_{d-1} lying on V_d belonging to a sufficiently ample linear system, and $k > 0$. There is a corresponding relation for the systems

† I have been told by M. Eger that he has considered a set of invariant systems on a V_d (from a different point of view) which he calls "variétés canoniques extérieures", and which appear to be identical with these systems. I hope that Eger's work will shortly be published.

† Todd, 8, 137, or 10, also Eger, 2.

Y_{d-k} , which actually has a slightly simpler form. We prove, in fact, that

$$Y_{d-k}[S] \equiv (S \cdot Y_{d-k+1}[V]) + (S^2 \cdot Y_{d-k+2}[V]) \quad \text{on } S, \quad (3.14)$$

for $k > 0$.

We prove (3.14) by induction. It is true trivially when $k = 1$, with the formal interpretation $Y_{d-1}[S] \equiv S$, $Y_d[V] \equiv V$, $Y_{d+1}[V] \equiv 0$, and it is true when $k = 2$, for then $Y_{d-2}[S] \equiv X_{d-2}[S]$ and (3.14) reduces to (3.13) with $k = 2$. We suppose, then, that (3.14) holds for the systems $Y_{d-h}[S]$ with $h < k$. There will be no confusion if we write Y_{d-k} , X_{d-k} for $Y_{d-k}[V]$, $X_{d-k}[V]$ respectively. From (3.10) we see that

$$Y_{d-k+1} \equiv \sum_{i=0}^{k-2} (-)^i (Y_{d-k+2+i} \cdot X_{d-1-i}),$$

and so, the equivalences being on S ,

$$\begin{aligned} (S \cdot Y_{d-k+1}) &\equiv \sum_{i=0}^{k-2} (-)^i \left((S \cdot Y_{d-k+2+i}) \cdot (S \cdot X_{d-1-i}) \right)_S \\ &\equiv \sum_{i=0}^{k-2} (-)^i \left((S \cdot Y_{d-k+2+i}) \cdot \{X_{d-2-i}[S] - (S \cdot X_{d-1-i}[S])\} \right)_S, \\ &\hspace{15em} \text{by (3.13),} \\ &\equiv -(S^2 \cdot Y_{d-k+2}) + \sum_{i=0}^{k-2} (-)^i \left(X_{d-2-i}[S] \cdot \{ (S \cdot Y_{d-k+2+i}) \right. \\ &\hspace{15em} \left. + (S^2 \cdot Y_{d-k+3+i}) \} \right)_S \\ &\equiv -(S^2 \cdot Y_{d-k+2}) + \sum_{i=0}^{k-2} (-)^i \left(X_{d-2-i}[S] \cdot Y_{d-k+1+i}[S] \right)_S, \\ &\hspace{15em} \text{by the inductive hypothesis,} \\ &\equiv -(S^2 \cdot Y_{d-k+2}) + Y_{d-k}[S], \text{ by the analogue of (3.10) for } S. \end{aligned}$$

Transposing, we obtain (3.14) and therefore complete the proof by induction.

3.4. It is perhaps worth while remarking that, whereas the locus M_{d-k} of points of contact of tangent $[d]$'s which lie in a prime through an $[r-d+k-2]$ is in general non-singular if V_d is non-singular, the locus N_{d-k} of points of contact of tangent $[d]$'s which meet a given $[r-d-k]$ may very well have singularities for every position of the $[r-d-k]$; this will happen if k is so small that tangent spaces of V_d exist meeting an arbitrary $[r-d-k]$ in a line or space of higher dimension. This may be seen as follows. Let Π denote a given $[r-d-k]$ and let Σ' , Σ'' be two

$[r-d-k-1]$'s lying in Π , and meeting in an $[r-d-k-2]$, Ω . The varieties N'_{d-k-1} , N''_{d-k-1} of contact of tangent $[d]$'s of V_d which meet Σ' and Σ'' respectively both lie on the variety N_{d-k} of contact of tangent $[d]$'s which meet Π , and both contain the variety N_{d-k-2} of contact of tangent $[d]$'s which meet Ω . If there are tangent $[d]$'s which meet Π in a line their freedom is $d-2(k+1)$ and their locus of contact is a variety H_{d-2k-2} lying on N_{d-k} . Since a line in Π meets Σ' and Σ'' but does not in general meet Ω , H_{d-2k-2} lies on N'_{d-k-1} and N''_{d-k-1} but not, in general, on N_{d-k-2} ; thus the varieties N'_{d-k-1} , N''_{d-k-1} lying on N_{d-k} meet in N_{d-k-2} and in a variety H_{d-2k-2} of dimension less than $d-k-2$, and it is easy to see that N'_{d-k-1} and N''_{d-k-1} have no other common points. This situation clearly can only arise if H_{d-2k-2} is a singular locus on N_{d-k} .

We have supposed that $r > 2d$. If $r < 2d$ a variety V_d in $[r]$ in general possesses a locus Γ_{2d-r} of improper double points, a general point of which we regard as the seat of a pair of distinct points of V_d which happen to coincide in position in the $[r]$, and at such a point there are two distinct tangent $[d]$'s to V_d , which may coincide in position for particular points of Γ_{2d-r} ; such points are *pinch-points* on V_d . The aggregate of tangent spaces to such a V_d is completely determinate, and we may still define the varieties M_{d-k} , introduced above, by the same geometrical conditions. But the geometrical definition of N_{d-k} fails when $k > r-d$. We may then define the system of equivalence $\{N_{d-k}\}$ indirectly by using the relations (3.04), and we can still develop the theory of the systems Y_{d-k} . The theory of the present section can thus be extended to any variety V_d in $[r]$ which possesses only "ordinary" singularities, that is, improper double varieties, and in particular to varieties V_d in $[d+1]$ with ordinary singularities in the usual sense, or to multiple $[d]$'s with branch-loci of general character.

4. Covariant systems of a pair of varieties.

4.1. We propose in this section to initiate the theory of covariant systems of a pair of varieties, one of which contains the other. In a sense this theory may be regarded as a generalisation of the theory of the projective invariants of a V_d lying in a linear space $[r]$, which is really the theory of simultaneous birational invariants of the V_d and the $[r]$.

We consider, then, a non-singular V_r and a V_d lying on it which has only ordinary singularities appropriate to its dimension (and which is thus non-singular when $2d < r$). The simultaneous covariant systems which at once arise for consideration are the invariant systems of V_d and the

intersections of these with the invariant systems on V_r . When the dimension of one of these covariant systems is zero, the number of points in a set of the series in question is an invariant of V_d and V_r which we may call a *projective character* of V_d , relative to the V_r ; and these characters are additive characters of V_d . The number of independent invariant systems of dimension $d-k$ on V_d is $p(k)$, the number of partitions of k , these systems being all of the form $(X_{d-1}^{a_1} X_{d-2}^{a_2} \dots X_{d-l}^{a_l})$ with $a_1 + 2a_2 + \dots + la_l = k$; and the number of invariant systems of dimension $r-d+k$ on V_r is $p(d-k)$. Thus the number of projective characters which arise in this way is

$$\sum_{k=0}^d p(k) \cdot p(d-k) \quad (4.01)$$

(which is independent of r). It should be observed that, if V_r is a linear space $[r]$, not all these characters are independent, since there are various equivalences between the intersections of the canonical systems on an $[r]$ which do not hold for a general V_r .

For instance, when $d=2$, (4.01) shows that five additive characters arise for a surface on a V_r , whereas it is known that four suffice in $[r]$; and this is due to the fact that, in $[r]$,

$$r(X_{r-1}^2) - 2(r+1)X_{r-2} = 0.$$

It has been remarked† that, for a surface lying on a quadric in $[5]$, five additive characters are necessary to determine the surface.

4.2. We have just seen that a very large number of covariant systems of a V_d lying on V_r can be defined in terms of the mutual intersections of the invariant systems. Of these systems, one set seems to play a part of great importance. We define

$$Y_k[V_d, V_r] \equiv \sum_{i=0}^{d-k} (-1)^i (Y_{k+i}[V_d] \cdot X_{r-i}[V_r]) \quad (k=0, 1, \dots, d), \quad (4.02)$$

the equivalences being on V_d ; $Y_d[V_d, V_r]$ is just V_d itself. The notation is meant to suggest a kind of analogy with $Y_k[V_d]$.

Let V_{r-1} be a variety lying on V_r and containing V_d . We prove that

$$Y_k[V_d, V_r] \equiv Y_k[V_d, V_{r-1}] + (V_{r-1} \cdot Y_{k+1}[V_d, V_{r-1}]). \quad (4.03)$$

This follows readily from (3.13) and (4.02). For, by (3.13),

$$X_h[V_{r-1}] \equiv (V_{r-1} \cdot \{X_{h+1}[V_r] + X_{h+1}[V_{r-1}]\})$$

† See, e.g., Roth, 3, 72.

and so, from (4.02),

$$\begin{aligned}
 Y_k[V_d, V_{r-1}] &\equiv \sum_{i=0}^{d-k} (-)^i (Y_{k+i}[V_d] \cdot X_{r-1-i}[V_{r-1}])_{(V_{r-1})} \\
 &= \sum_{i=0}^{d-k} (-)^i (Y_{k+i}[V_d] \cdot X_{r-i}[V_r]) \\
 &\quad + \left(\sum_{i=0}^{d-k} (-)^i (Y_{k+i}[V_d] \cdot X_{r-i}[V_{r-1}])_{(V_{k-1})} \cdot V_{r-1} \right) \\
 &\equiv Y_k[V_d, V_r] - (V_{r-1} \cdot Y_{k+1}[V_d, V_{r-1}]),
 \end{aligned}$$

since the term for which $i=0$ in the second summand vanishes. This proves (4.03).

A more important property of the system $Y_k[V_d, V_r]$ is the following. Let S_{r-1} be a variety of $r-1$ dimensions on V_r which meets V_d in a variety S_{d-1} of $d-1$ dimensions. Then

$$Y_k[S_{d-1}, S_{r-1}] \equiv (S_{r-1} \cdot Y_{k+1}[V_d, V_r]) \quad (k=0, 1, \dots, d-1), \quad (4.04)$$

the equivalence being on S_{d-1} .

For

$$\begin{aligned}
 &(S_{r-1} \cdot Y_{k+1}[V_d, V_r]) \\
 &\equiv \left(S_{r-1} \cdot \sum_{i=0}^{d-k-1} (-)^i (Y_{k+1+i}[V_d] \cdot X_{r-i}[V_r]) \right), \quad \text{by (4.02),} \\
 &\equiv \sum_{i=0}^{d-k-1} (-)^i \left((S_{r-1} \cdot Y_{k+1+i}[V_d]) \cdot (S_{r-1} \cdot X_{r-i}[V_r]) \right)_{(S_{r-1})} \\
 &\equiv (S_{r-1} \cdot Y_{k+1}[V_d]) + \sum_{i=1}^{d-k-1} (-)^i \left((S_{r-1} \cdot Y_{k+1+i}[V_d]) \cdot \{X_{r-i-1}[S_{r-1}] \right. \\
 &\quad \left. - (S_{r-1} \cdot X_{r-i}[S_{r-1}]) \right)_{(S_{r-1})}, \quad \text{by (3.13),} \\
 &= (S_{r-1} \cdot Y_{k+1}[V_d]) + (S_{r-1}^2 \cdot Y_{k+2}[V_d]) \\
 &\quad + \sum_{i=1}^{d-k-2} (-)^i \left(X_{r-i-1}[S_{r-1}] \cdot \{ (S_{r-1} \cdot Y_{k+1+i}[V_d]) + (S_{r-1}^2 \cdot Y_{k+2+i}[V_d]) \} \right)_{(S_{r-1})} \\
 &\quad + (-)^{d-k-1} (S_{d-1} \cdot X_{r-d+k}[S_{r-1}])_{(S_{r-1})} \\
 &\equiv Y_k[S_{d-1}] + \sum_{i=1}^{d-k-2} (-)^i (X_{r-i-1}[S_{r-1}] \cdot Y_{k+i}[S_{d-1}])_{(S_{r-1})} \\
 &\quad + (-)^{d-k-1} (S_{d-1} \cdot X_{r-d+k}[S_{r-1}])_{(S_{r-1})}, \quad \text{by (3.14),} \\
 &\equiv Y_k[S_{d-1}, S_{r-1}],
 \end{aligned}$$

which proves (4.04).

This property we may term the property of "section-invariance," which, if not a strictly logical name, at least expresses the nature of the property. It is this property which makes the study of the systems $Y_k[V_d, V_r]$ of importance in dealing with intersection problems on a V_r .

An immediate corollary of (4.04) is that, if a V_{r-h} , S_{r-h} say, on V_r , meets V_d in a variety S_{d-h} of $d-h$ dimensions, then, on S_{d-h} ,

$$Y_k[S_{d-h}, S_{r-h}] \equiv (S_{r-h} \cdot Y_{h+k}[V_d, V_r]). \quad (4.05)$$

A similar proof shows that, if we define

$$X_k[V_d, V_r] \equiv \sum_{i=0}^{d-k} (-)^i (X_{k+i}[V_d] \cdot Y_{r-i}[V_r]), \quad (4.06)$$

then $X_k[V_d, V_r]$ possesses the property analogous to (4.05), that

$$X_k[S_{d-h}, S_{r-h}] \equiv (S_{r-h} \cdot X_{h+k}[V_d, V_r]). \quad (4.07)$$

An independent proof arises by observing that, on V_d ,

$$\sum_{i=0}^{d-k} (-)^i (X_{k+i}[V_d, V_r] \cdot Y_{d-i}[V_d, V_r])_{(V_d)} \equiv 0, \quad (4.08)$$

which follows from the considerations of 1.3, letting x_k, y_k, x'_k, y'_k denote the elements of the ring of equivalence on V_d which correspond respectively to $X_{d-k}[V_d], Y_{d-k}[V_d], (V_d \cdot X_{r-k}[V_r]), (V_d \cdot Y_{r-k}[V_r])$. Then (4.08) is a consequence of (1.18), and (4.07) follows from (4.08) and (4.05).

5. The inflexional tangents of a V_d in $[d+1]$.

5.1. As a first application of these results, we investigate in this section the locus of points of contact of tangents, drawn to a V_d with ordinary singularities in $[d+1]$ from a general point O , which have $k+1$ -point contact with V_d . The locus in question is of dimension $d-k$, and may be denoted by F_{d-k} . In particular, F_{d-1} is the locus of contact of ordinary tangents drawn through O , and is thus the Jacobian M_{d-1} (or N_{d-1}) of the linear system $|S|$ of prime sections of V_d which pass through O . The varieties F_{d-k} , for $k > 1$, all lie on F_{d-1} . We denote by F'_{d-k} a variety on V_d , equivalent to F_{d-k} , but not lying on F_{d-1} , for example the locus of contacts of $k+1$ -point tangents from a different point.

We require some preliminary results. In the first place, the Jacobian variety M_{d-k} of the sections of V_d by a linear ∞^{d-k+1} system of primes whose base $[k-1]$ passes through O , lies on F_{d-1} , the locus of contact of

tangent primes through O . The points common to two varieties M_{d-k} , M_{d-h} lying on F_{d-1} are points at which the tangent prime passes through both a $[k-1]$ and an $[h-1]$ through O , such a prime thus contains the joining $[k+h-2]$ of these spaces and hence its point of contact lies on the corresponding $M_{d-k-h+1}$. Thus

$$(M_{d-k} \cdot M_{d-h})_{(F_{d-1})} \equiv M_{d-k-h+1}. \quad (5.01)$$

We may use (3.04) to define the systems of equivalence N_{d-k} on V_d , the geometrical definition of these varieties failing for $k > 1$.

In particular, the locus P_{d-2} of pinch-points of V_d belongs to the system $\{N_{d-2}\}$, and lies on F_{d-1} . From (3.04),

$$N_{d-k} \equiv \sum_{i=1}^k (-)^{i+1} (N_{d-k+i} \cdot M_{d-i}).$$

If we suppose the varieties M_{d-i} to lie on F_{d-1} this defines a variety N_{d-k} on F_{d-1} . We may show that

$$(N_{d-k} \cdot P_{d-2})_{(F_{d-1})} \equiv N_{d-k-1} \quad (\text{on } V_d). \quad (5.02)$$

For, since $N_{d-2} \equiv (M_{d-1}^2) - M_{d-2}$, $P_{d-2} \equiv (M'_{d-1} \cdot F_{d-1}) - M_{d-2}$, where M_{d-2} lies on F_{d-1} and M'_{d-1} is a general variety of the system $\{M_{d-1}\}$ on V ; while, with the same significance attaching to the dashes,

$$N_{d-k} \equiv \sum_{i=1}^k (-)^{i+1} (N'_{d-k+i} \cdot M_{d-i}).$$

Hence

$$\begin{aligned} (N_{d-k} \cdot P_{d-2})_{(F_{d-1})} &\equiv \left(\sum_{i=1}^k (-)^{i+1} (N'_{d-k+i} \cdot M_{d-i}) \cdot \{(M'_{d-1} \cdot F_{d-1}) - M_{d-2}\} \right)_{(F_{d-1})} \\ &\equiv \sum_{i=1}^k (-)^{i+1} (N'_{d-k+i} \cdot M_{d-i} \cdot M'_{d-1}) \\ &\quad - \sum_{i=1}^k (-)^{i+1} (N'_{d-k+i} \cdot M_{d-i-1}) \quad \text{by (5.01),} \\ &\equiv (N_{d-k} \cdot M'_{d-1}) + \sum_{j=2}^{k+1} (-)^{j+1} (N'_{d-k-1+j} \cdot M_{d-j}) \quad \text{by (3.04),} \\ &\equiv (N_{d-k} \cdot M'_{d-1}) + \{N_{d-k-1} - (N'_{d-k} \cdot M_{d-1})\} \\ &\equiv N_{d-k-1} + (N_{d-k} \cdot M'_{d-1}) - (N'_{d-k} \cdot M_{d-1}) \\ &\equiv N_{d-k-1} \quad \text{on } V_d, \text{ which proves (5.02).} \end{aligned}$$

Observe that the above reasoning does not enable us to infer without further discussion that the equivalence (5.02) holds on F_{d-1} .

5.2. We now introduce a set of varieties U_{d-k} ($k = 1, 2, \dots$) lying on F_{d-1} , defined by

$$U_{d-k} \equiv \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (N_{d-k+i} \cdot S^i), \quad (5.03)$$

where the intersections are taken on V_d . We prove first that, on V_d ,

$$U_{d-k} \equiv \sum_{l=0}^k \binom{d+1}{l} (Y_{d-k+l} \cdot S^l). \quad (5.04)$$

This follows by substituting in (5.03) the expression for N_{d-k+i} given in (3.12). In fact, we see that

$$\begin{aligned} U_{d-k} &\equiv \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \sum_{j=0}^{k-i} \binom{d+k-i}{j} (Y_{d-k+i+j} \cdot S^{i+j}) \\ &\equiv \sum_{i=0}^{k-1} \sum_{l=i}^k (-1)^i \binom{k-1}{i} \binom{d+k-i}{l-i} (Y_{d-k+l} \cdot S^l) \\ &\equiv \sum_{l=0}^k \sum_{i=0}^{\min(l, k-1)} (-1)^i \binom{k-1}{i} \binom{d+k-i}{l-i} (Y_{d-k+l} \cdot S^l). \end{aligned}$$

Now, comparing the coefficients of x^l in the binomial expansions on the two sides of the identity

$$(1-x)^{k-1} (1-x)^{-(d+k-l+1)} = (1-x)^{-(d-l+2)},$$

we see that, if $l \leq k-1$,

$$\sum_{i=0}^l (-1)^i \binom{k-1}{i} \binom{d+k-i}{l-i} = \binom{d+1}{l},$$

and, for $l = k$,

$$\sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \binom{d+k-i}{k-i} = \binom{d+1}{k}.$$

Hence

$$U_{d-k} \equiv \sum_{l=0}^k \binom{d+1}{l} (Y_{d-k+l} \cdot S^l),$$

which proves (5.04).

It is worth while noticing that, if V_d is projected from O into a multiply covered $[d]$, and we regard the projected variety as "lying on" the $[d]$, then, in the notation of § 4,

$$U_{d-k} \equiv Y_{d-k}[V, V'], \quad (5.05)$$

where V' is the $[d]$ covered by the projection of V .

5.3. We are now in a position to enunciate the theorem to be proved, namely that the locus F_{d-k} of points of contact of $(k+1)$ -point tangents drawn to V_d from O is given by the equivalence

$$F_{d-k} \equiv \sum_{i=1}^k \frac{(k-1)!}{(k-i)!} (U_{d-i} \cdot F'_{d-k+i}), \quad (5.06)$$

where U_{d-i} is the variety on F_{d-1} defined by (5.03), and F'_{d-k+i} is a variety on V_d (but not on F_{d-1}) equivalent to F_{d-k+i} . The equivalence (5.06) gives successively F_{d-1} , F_{d-2} , ... and so on.

When $k=1$, (5.06) asserts that $F_{d-1} \equiv U_{d-1}$; but, by (5.03),

$$U_{d-1} \equiv N_{d-1} \equiv F_{d-1},$$

so that (5.06) holds when $k=1$.

We prove the general result by induction. Let B_{d-1} denote the double V_{d-1} lying on V_d , a general point of B_{d-1} being considered as the seat of two "places" on V_d , which coincide at points of the pinch variety P_{d-2} . The first polar of O with respect to V_d passes through B_{d-1} and meets V_d residually in F_{d-1} . Hence, if n is the order of V_d ,

$$F_{d-1} + B_{d-1} \equiv (n-1)S. \quad (5.07)$$

The variety F_{d-1} contains the pinch-variety P_{d-2} , and meets the double B_{d-1} further in the locus R_{d-2} of points of B_{d-1} at which one of the tangent primes to V_d passes through O . At a point of R_{d-2} the polar quadric consists of the two tangent primes to V_d and this passes through O . Hence R_{d-2} lies on the second polar of O . The locus F_{d-2} of points of contact of 3-point tangents from O is thus the intersection, residual to R_{d-1} , of F_{d-1} with the second polar of O . Thus

$$R_{d-2} + F_{d-2} \equiv (n-2)(S \cdot F_{d-1}). \quad (5.08)$$

The complete intersection of F_{d-1} and B_{d-1} consists of R_{d-2} and P_{d-2} , so that

$$R_{d-2} + P_{d-2} \equiv (F_{d-1} \cdot B_{d-1}). \quad (5.09)$$

From (5.07), (5.08), and (5.09) it follows that

$$\begin{aligned}
 F_{d-2} &= (n-2)(S \cdot F_{d-1}) - (F_{d-1} \cdot \{(n-1)S - F_{d-1}\}) + P_{d-2} \\
 &\equiv (F_{d-1}^2) - (S \cdot F_{d-1}) + N_{d-2} \quad (\text{since } P_{d-2} \equiv N_{d-2}) \\
 &\equiv (U_{d-1} \cdot F'_{d-1}) + N_{d-2} - (S \cdot N_{d-1}) \quad (\text{since } F_{d-1} \equiv N_{d-1} \equiv U_{d-1}) \\
 &\equiv (U_{d-1} \cdot F'_{d-1}) + U_{d-2} \quad \text{by (5.03),}
 \end{aligned}$$

which proves (5.06) when $k = 2$.

The intersection of F_{d-2} and B_{d-1} consists of the locus R_{d-3} of points of B_{d-1} at which an inflexional tangent to V_d passes through O , and the locus P_{d-3} of points of P_{d-2} at which the tangent prime to V_d passes through O . The line joining O to a point of R_{d-3} has four coincident intersections with V_d , so that O lies on the polar cubic of each point of R_{d-3} , and R_{d-3} lies on the third polar of O . The residual intersection of F_{d-2} and the third polar of O is the locus F'_{d-3} of points of contact of four-point tangents drawn to V_d from O . Thus

$$R_{d-3} + F'_{d-3} \equiv (n-3)(S \cdot F_{d-2}).$$

The locus P_{d-3} is the intersection of P_{d-2} and R_{d-2} . The polar quadric of a point P of P_{d-3} consists of the tangent prime to V_d at P , counted twice, and this prime passes through O . Thus the first polar of O has a double point at P . The tangent $[d-1]$ to F_{d-1} at P contains the tangent $[d-2]$'s to P_{d-2} and R_{d-2} , since these lie on F_{d-1} , and hence coincides with the tangent $[d-1]$ to B_{d-1} at P . Thus the tangent $[d-2]$ to F_{d-2} at P lies in the tangent $[d-1]$ to B_{d-1} , and hence F_{d-2} touches B_{d-1} at every point of P_{d-3} . Thus

$$R_{d-3} + 2P_{d-3} \equiv (F_{d-2} \cdot B_{d-1}).$$

Proceeding in this way, we see that F_{d-k+1} meets B_{d-1} in the variety R_{d-k} of points of B_{d-1} at which a k -point tangent to one sheet of V_d passes through O , and in a variety $P_{d-k} \equiv (F_{d-k+1} \cdot P_{d-2})_{F_{d-1}}$ on P_{d-2} at which a $(k-1)$ -point tangent to V_d passes through O ; the points of R_{d-k} lie on the k -th polar of O , and at any point of P_{d-k} , we can show that F_{d-k+1} and B_{d-1} have contact of such a nature that P_{d-k} counts $k-1$ times in their intersection. We therefore have the equivalences

$$R_{d-k} + F_{d-k} \equiv (n-k)(S \cdot F_{d-k+1}), \quad (5.10)$$

$$R_{d-k} + (k-1)(F_{d-k+1} \cdot P_{d-2})_{F_{d-1}} \equiv (F_{d-k+1} \cdot B_{d-1}). \quad (5.11)$$

Thus, from (5.10), (5.11), and (5.07),

$$F_{d-k} \equiv (F_{d-k+1} \cdot F_{d-1}) - (k-1)(S \cdot F_{d-k+1}) + (k-1)(F_{d-k+1} \cdot P_{d-2})_{(F_{d-1})}.$$

By the inductive hypothesis, however, since $P_{d-2} \equiv N_{d-2}$,

$$\begin{aligned} (F_{d-k+1} \cdot P_{d-2})_{F_{d-1}} &\equiv \sum_{i=1}^{k-1} \frac{(k-2)!}{(k-1-i)!} (U_{d-i} \cdot F'_{d-k+i+2} \cdot N_{d-2})_{(F_{d-1})} \\ &\equiv \sum_{i=1}^{k-1} \frac{(k-2)!}{(k-1-i)!} ((U_{d-i} \cdot N_{d-2})_{F_{d-1}} \cdot F'_{d-k+1+i}), \end{aligned}$$

and

$$\begin{aligned} (U_{d-i} \cdot N_{d-2})_{F_{d-1}} &\equiv \left(\sum_{j=0}^{i-1} (-)^j \binom{i-1}{j} N_{d-i+j} \cdot S^j \cdot N_{d-2} \right)_{(F_{d-1})} \\ &\equiv \sum_{j=0}^{i-1} (-)^j \binom{i-1}{j} (N_{d-i+j-1} \cdot S^j), \text{ by (5.02),} \\ &\equiv U_{d-i-1} + (S \cdot U_{d-i}), \text{ as follows easily from (5.03).} \end{aligned}$$

Hence

$$\begin{aligned} (F_{d-k+1} \cdot P_{d-2})_{F_{d-1}} &\equiv \sum_{i=1}^{k-1} \frac{(k-2)!}{(k-i-1)!} [(U_{d-i-1} \cdot F'_{d-k+1+i}) + (S \cdot U_{d-i} \cdot F'_{d-k+1+i})] \\ &\equiv \sum_{i=1}^{k-1} \frac{(k-2)!}{(k-i-1)!} (U_{d-i-1} \cdot F'_{d-k+1+i}) + (S \cdot F'_{d-k+1}), \end{aligned}$$

by the inductive hypothesis,

and so

$$\begin{aligned} F_{d-k} &\equiv (F_{d-k+1} \cdot F_{d-1}) - (k-1)(S \cdot F_{d-k+1}) + (k-1) \\ &\quad \times \left[\sum_{i=1}^{k-1} \frac{(k-2)!}{(k-i-1)!} (U_{d-i-1} \cdot F'_{d-k+1+i}) + (S \cdot F'_{d-k+1}) \right] \\ &\equiv (U_{d-1} \cdot F'_{d-k+1}) + \sum_{i=1}^{k-1} \frac{(k-1)!}{(k-i-1)!} (U_{d-i-1} \cdot F'_{d-k+1+i}) \\ &\equiv \sum_{j=1}^k \frac{(k-1)!}{(k-j)!} (U_{d-j} \cdot F'_{d-k+j}), \end{aligned}$$

since, on V_d ,

$$(F_{d-k+1} \cdot F_{d-1}) \equiv (U_{d-1} \cdot F'_{d-k+1}).$$

This completes the proof of (5.06).

6. *The double locus of a V_d in $[r]$.*

6.1. Let V_d be a locus of dimension d and order n in $[r]$. If $2d > r$, V_d in general has a locus Γ_h of improper double points, of dimension $h = 2d - r$. We express Γ_h in terms of the invariant systems of equivalence on V_d and their linear sections, and prove, in fact, that

$$\Gamma_h \equiv nC_h - \sum_{i=0}^{d-h} \binom{r+1}{i} (Y_{h+i} \cdot C_{d-i}), \quad (6.01)$$

where Y_{h+i} is the invariant system of $h-i$ dimensions on V_d , and C_{d-i} is a linear section of V_d , of dimension $d-i$. When $h = 0$, i.e. $2d = r$, this is equivalent to a result of Eger†.

The proof proceeds for a fixed d , by induction on r . When $r = d+1$, $h = d-1$, and V_d has a double Γ_{d-1} . The canonical system $X_{d-1} \equiv Y_{d-1}$ is cut out on V_d by primals of order $n-d-2$ passing through the double locus, and so

$$\begin{aligned} \Gamma_{d-1} &\equiv (n-d-2) C_{d-1} - Y_{d-1} \\ &\equiv nC_{d-1} - [Y_{d-1} + (d+2) C_{d-1}], \end{aligned}$$

which is (6.01) with $r = d+1$. The theorem is thus true when $r = d+1$.

6.2. To prove (6.01) generally we suppose it to hold for V_d in $[r-1]$. We project V_d into a variety V_d' of order n in $[r-1]$, from a general point O of the $[r]$. The double locus Γ'_{h+1} on V_d' has dimension $h+1$, and is the projection of the variety Δ_{h+1} on V_d cut out by the cone of chords passing through O . Using (6.01) to determine Γ'_{h+1} , and translating the result in terms of equivalence on V_d , we see that

$$\Delta_{h+1} \equiv nC_{h+1} - \sum_{i=0}^{d-h-1} \binom{r}{i} (Y_{h+1+i} \cdot C_{d-i}). \quad (6.02)$$

The variety Δ_{h+1} has Γ_h as a double locus, and it possesses an involution I of pairs of points cut out by lines through O . The locus of coincidences in this involution is the variety N_h of points of contact of tangents to V_d which pass through O .

Let $|\Sigma|$ be a general pencil of primes of the $[r]$, and consider the correspondence of pairs of points (P, Q) of Δ_{h+1} such that Q lies in the prime of

† Eger, 1.

$|\Sigma|$ which contains the companion P' of P in the involution I . There are ∞^{2h+1} pairs of corresponding points; to a point P corresponds the variety $V_h[P]$ cut out on Δ_{h+1} by the prime of Σ through the mate of P in I ; thus $V_h[P] \equiv (\Sigma \cdot \Delta_{h+1})$. The points P which correspond to a given point Q are the mates in I of the points of the section of Δ_{h+1} by the prime of $|\Sigma|$ which contains Q ; denote their locus by $V_h'[Q]$. The same variety $V_h'[Q]$ is clearly associated with any point of the section of Δ_{h+1} by the prime of $|\Sigma|$ through Q , and for different primes of $|\Sigma|$ we get an algebraic system of varieties $V_h'[Q]$ on Δ_{h+1} forming a rational system and therefore contained in a linear system. The section of Δ_{h+1} by the prime of $|\Sigma|$ which passes through O is self-corresponding in I , and so, for a point Q on this section, the corresponding $V_h'[Q]$ is the section itself. It follows, since all the $V_h'[Q]$ are contained in a linear system, that

$$V_h'[Q] \equiv (\Sigma \cdot \Delta_{h+1}).$$

Now if on any variety V_{h+1} we have a correspondence of ∞^{2h+1} pairs of points (P, Q) such that the varieties $V_h[P]$, $V_h'[Q]$ corresponding in the direct and inverse correspondences to P , Q respectively each vary in a linear system on V_{h+1} , then the locus of coincidences of the correspondence is equivalent to $V_h[P] + V_h'[Q]$. This is easily seen by induction. For suppose the result proved for varieties of h dimensions, let D_h be the locus of coincidences of the correspondence on V_{h+1} and let S_h be a general prime section of V_{h+1} . The correspondence between pairs of points on V_{h+1} induces a correspondence of ∞^{2h-1} pairs of points (P_1, Q_1) on S_h , in which the points Q_1 corresponding to P_1 are the points of the section of $V_h[P_1]$ by S_h , and the points P_1 corresponding to Q_1 are the points of the section of $V_h'[Q_1]$ by S_h . The coincidences in this correspondence are evidently the points of the section of D_h by S_h . Hence, by the inductive hypothesis,

$$\begin{aligned} (S_h \cdot D_h) &\equiv (S_h \cdot V_h[P_1]) + (S_h \cdot V_h'[Q_1]) \\ &\equiv (S_h \cdot V_h[P]) + (S_h \cdot V_h'[Q]), \end{aligned}$$

and from this the relation

$$D_h \equiv V_h[P] + V_h'[Q]$$

follows from the familiar criterion for linear equivalence†. The result is

† See Zariski, 12, 89. The proof, given for surfaces, extends at once to varieties.

thus true on a V_{h+1} , if it is true on a V_h , but on a curve it is simply the familiar expression for the set of coincidences in a correspondence of valency zero†. The proof is thus complete.

In the particular correspondence on Δ_{h+1} which we are considering, the set of coincidences consists (i) of the variety N_k of coincidences of the involution I , (ii) of the points of the double locus Γ_h , and (iii) of the points of Δ_{h+1} lying in the prime of $|\Sigma|$ which passes through O . Hence

$$N_h + \Gamma_h + (\Sigma \cdot \Delta_{h+1}) \equiv (\Sigma \cdot \Delta_{h+1}) + (\Sigma \cdot \Delta_{h+1}),$$

this can be written as an equivalence on V_d in the form

$$N_h + \Gamma_h \equiv (C_{d-1} \cdot \Delta_{h+1}), \quad (6.03)$$

since the section of V_d by a prime of $|\Sigma|$ belongs to the linear system $|C_{d-1}|$.

Using (6.03), (6.02), and (3.12), and remembering that $h = 2d - r$, we therefore see that

$$\begin{aligned} \Gamma_h &\equiv (C_{d-1} \cdot \Delta_{h+1}) - N_h \\ &\equiv \left(C_{d-1} \cdot \left\{ nC_{h+1} - \sum_{i=0}^{d-h-1} \binom{r}{i} (Y_{h+1+i} \cdot C_{d-i}) \right\} \right) - \sum_{i=0}^{d-h} \binom{r}{i} (Y_{h+i} \cdot C_{d-i}) \\ &\equiv nC_h - \sum_{i=0}^{d-h-1} \binom{r}{i} (Y_{h+1+i} \cdot C_{d-1-i}) - \sum_{i=0}^{d-h} \binom{r}{i} (Y_{h+i} \cdot C_{d-i}) \\ &\equiv nC_h - \sum_{i=0}^{d-h} \binom{r+1}{i} (Y_{h+i} \cdot C_{d-i}), \end{aligned}$$

which proves (6.01).

An alternative form of (6.01) is of interest. The canonical system X_{r-k} in $[r]$ is composed of varieties (virtual if k is odd) of order $(-)^k \binom{r+1}{k}$, and so

$$(V_d \cdot X_{r-k}[r]) \equiv (-)^k \binom{r+1}{k} C_{d-k}.$$

† See, e.g., Severi, 4, 224.

Hence (6.01) becomes

$$\begin{aligned}\Gamma_h &\equiv nC_h - \sum_{k=0}^{d-h} (-)^k (Y_{h+k}[V_d] \cdot X_{r-k}[r]) \\ &\equiv nC_h - Y_h[V_d, [r]],\end{aligned}$$

i.e., since $(V_d^2) \equiv nC_h$,

$$\Gamma_h \equiv (V_d^2) - Y_h[V_d, [r]]. \quad (6.04)$$

In the next section we generalize this result by replacing the $[r]$ by any non-singular V_r through V_d .

7. The double locus of a V_d on a V_r .

7.1. The object of the present section is to prove that, for a V_d lying on a non-singular V_r , the double variety Γ_h of dimension $h = 2d - r$ is given by the equivalence

$$\Gamma_h \equiv (V_d^2) - Y_h[V_d, V_r]. \quad (7.01)$$

We have already seen that this holds when V_r is a linear space†.

We obtain this result by proving inductively two chains of theorems, which we enunciate as follows:

THEOREM (A_k). *Let S be a V_d free from proper singularities, lying on a non-singular variety V of r dimensions, and let $\mathfrak{A}$ be a variety of $r - k$ dimensions, free from proper singularities, which lies on V and meets S in a variety Δ of dimension $d - 1$. When $d \geq k$, the varieties $\mathfrak{A}$ and S intersect further in a variety G of dimension $d - k$, and G is given by the equivalence (on $\mathfrak{A}$)*

$$G \equiv (\mathfrak{A} \cdot S) + \sum_{i=1}^k (-)^i \left((\Delta^i)_S \cdot Y_{r-2k+i}[\mathfrak{A}, V] \right)_{\mathfrak{A}}. \quad (7.02)$$

(There are other forms into which this expression can be thrown which involve $\mathfrak{A}$ and S in a more symmetrical manner. The present form is best suited to our immediate purpose.)

† This result was proved otherwise by Yoxall, 11, in the case $d = 2$, $r = 4$.

THEOREM (B_k). *The improper double variety Γ on a variety S of $r-k$ dimensions lying on V is given by*

$$\Gamma \equiv (S^2) - Y_{r-2k}[S, V]. \quad (7.03)$$

This is equivalent to (7.01), apart from a change of notation.

We prove these theorems inductively, obtaining (A_k) from (B_{k-1}) and then deducing (B_k) from (A_k).

Theorem (A_1) is trivial. Theorem (B_1) asserts that the double V_{r-2} , Γ , of a V_{r-1} , S , lying on V is given by

$$\begin{aligned} \Gamma &\equiv (S^2) - Y_{r-2}[S, V] \\ &\equiv (S^2) - Y_{r-2}[S] + (S \cdot X_{r-1}[V]). \end{aligned}$$

To prove this, let $[R]$ be the ambient space of V , and project V from a general $[R-r-1]$, Π , into a multiple $[r]$. The points of contact of tangents to V which meet Π project into points of the branch locus F_{r-1} of the multiple $[r]$; since this multiple space is birationally equivalent, without exceptional elements, to V , there will be no confusion if we denote equally by F_{r-1} the locus of contacts of these tangents with V . The cone which projects S_{r-1} from Π meets V again in a variety S'_{r-1} , and the projections of S_{r-1} and S'_{r-1} are *companion varieties* on the multiple $[r]$, covering the same set of points in the $[r]$ but lying on different sheets. If n is the order of S_{r-1} , and if Σ is a general prime of the $[R]$, the varieties S_{r-1} and S'_{r-1} together form the complete intersection of V with a cone of order n , and so

$$S_{r-1} + S'_{r-1} \equiv n(\Sigma \cdot V). \quad (7.04)$$

The variety S'_{r-1} clearly passes through the intersections of S_{r-1} and F_{r-1} , and meets S_{r-1} in another variety Γ'_{r-2} , the section by S_{r-1} of the cone of its chords which meet Π . Thus

$$(S_{r-1} \cdot S'_{r-1}) \equiv \Gamma'_{r-2} + (S_{r-1} \cdot F_{r-1}). \quad (7.05)$$

The projection of S_{r-1} on to the $[r]$, if Π is chosen generally, is a *simple* variety on $[r]$, covering a locus T_{r-1} of the $[r]$ birationally equivalent to S_{r-1} ; T_{r-1} being simply the projection of S_{r-1} from Π . The double locus on T_{r-1} evidently consists of the projections of Γ_{r-2} and Γ'_{r-2} . Hence, from (6.01) applied to T_{r-1} , we deduce the equivalence, first on T_{r-1} and thence on S_{r-1} ,

$$\Gamma_{r-2} + \Gamma'_{r-2} \equiv n(\Sigma \cdot S_{r-1}) - Y_{r-2}[S] - (r+1)(\Sigma \cdot S_{r-1}). \quad (7.06)$$

Hence, from (7.04), (7.05), (7.06),

$$\begin{aligned}
 \Gamma_{r-2} &\equiv (n-r-1)(\Sigma \cdot S_{r-1}) - Y_{r-2}[S_1] - \Gamma'_{r-2} \\
 &\equiv (n-r-1)(\Sigma \cdot S_{r-1}) - Y_{r-2}[S] - (S_{r-1} \cdot S'_{r-1}) + (S_{r-1} \cdot F_{r-1}) \\
 &\equiv (S_{r-1} \cdot \{F_{r-1} + (n-r-1)\Sigma\}) - Y_{r-2}[S] - (S_{r-1} \cdot \{n\Sigma - S_{r-1}\}) \\
 &\equiv (S_{r-1}^2) - Y_{r-2}[S] + (S \cdot \{F_{r-1} - (r+1)\Sigma\}) \\
 &\equiv (S_{r-1}^2) - Y_{r-2}[S] + (S \cdot X_{r-1}[V]),
 \end{aligned}$$

and this proves Theorem (B_1):

7.2. Let us now suppose that we have proved (A_{k-1}) and (B_{k-1}). We prove (A_k). We draw, on V , a variety B of $r-1$ dimensions passing through $\mathfrak{A}$. This contains Δ , and meets S again in a variety Δ' of dimension $d-1$. Let ϕ be the variety, of $d-2$ dimensions, common to Δ and Δ' . Thus, on S ,

$$(B \cdot S) \equiv \Delta + \Delta', \quad (7.07)$$

$$(\Delta \cdot \Delta')_S \equiv \phi. \quad (7.08)$$

The locus G is the residual intersection on B of the varieties $\mathfrak{A}$ and Δ' , residual to ϕ , and the dimension of $\mathfrak{A}$ is $r-k = (r-1) - (k-1)$. We can therefore apply (A_{k-1}), and obtain the equivalence (on $\mathfrak{A}$)

$$G \equiv (\mathfrak{A} \cdot \Delta')_B + \sum_{i=1}^{k-1} (-1)^i \left((\phi^i)_{\Delta'} \cdot Y_{r+1-2k+i}[\mathfrak{A}, B] \right)_{\mathfrak{A}}. \quad (7.09)$$

A slight difficulty arises here, since the variety B , constrained to contain $\mathfrak{A}$, has the double variety of dimension $r-2k$ of $\mathfrak{A}$ as a locus of double points, and also has a variety of double points (of the same dimension $r-2k$) at simple points of $\mathfrak{A}$. It is easily seen that the presence of these singularities does not affect the *geometrical* discussion, but the interpretation of (7.09) needs a little consideration. We do not know, at present, what effect such singularities may have on the canonical systems on B . However, in (7.09), it is only these systems of dimension greater than or equal to $r+1-2k$ which appear, and the only possible effect of the multiple points of B on these is that they should be fixed base-points, of certain multiplicity, for the systems $Y_{r+1-2k+i}[\mathfrak{A}, B]$. These base-loci being of dimension $r-2k$ at most cannot affect the

equivalence (7.09) except, perhaps, by imposing certain base-conditions on G , and these we are not at present concerned with, since we seek only an equivalence for G .

We proceed to transform (7.09): In the first place, from (7.07) and (7.08) we see that

$$(\phi^i)_{\Delta'} \equiv (\Delta' \cdot \Delta^i)_S \equiv \left(\{ (B \cdot S) - \Delta \} \cdot \Delta^i \right)_S, \\ i.e. \quad (\phi^i)_{\Delta'} \equiv \left(B \cdot (\Delta^i)_S \right) - (\Delta^{i+1})_S. \quad (7.10)$$

Next, by (4.03),

$$Y_j[\mathfrak{U}, B] \equiv Y_j[\mathfrak{U}, V] - (B \cdot Y_{j+1}[\mathfrak{U}, B]) \quad \text{on } \mathfrak{U}, \quad (7.11)$$

and so (7.09) becomes†

$$G \equiv \left(\mathfrak{U} \cdot \{ (B \cdot S) - \Delta \} \right)_B + \sum_{i=1}^{k-1} (-)^i \left(\{ (B \cdot \Delta_S^i) - \Delta_S^{i+1} \} \cdot Y_{r+1-2k+i}[\mathfrak{U}, B] \right)_{\mathfrak{U}} \\ \equiv (\mathfrak{U} \cdot S) - (\mathfrak{U} \cdot \Delta)_B - (B \cdot \Delta \cdot Y_{r+2-2k}[\mathfrak{U}, B])_{\mathfrak{U}} \\ + \sum_{i=2}^k (-)^i \left(\Delta_S^i \cdot \{ (B \cdot Y_{r+1-2k+i}[\mathfrak{U}, B]) + Y_{r-2k+i}[\mathfrak{U}, B] \} \right)_{\mathfrak{U}} \\ \equiv (\mathfrak{U} \cdot S) - (\mathfrak{U}_{B^2} \cdot \Delta)_{\mathfrak{U}} - (\Delta \cdot \{ Y_{r+1-2k}[\mathfrak{U}, V] - Y_{r+1-2k}[\mathfrak{U}, B] \})_{\mathfrak{U}} \\ + \sum_{i=2}^k (-)^i (\Delta_S^i \cdot Y_{r-2k+i}[\mathfrak{U}, V])_{\mathfrak{U}}, \text{ by (7.11),} \\ \equiv (\mathfrak{U} \cdot S) + \sum_{i=1}^k (-)^i (\Delta_S^i \cdot Y_{r-2k+i}[\mathfrak{U}, V])_{\mathfrak{U}} - \left(\Delta \cdot \{ \mathfrak{U}_{B^2} - Y_{r+1-2k}[\mathfrak{U}, B] \} \right)_{\mathfrak{U}}.$$

But $\mathfrak{U}$ has a double variety of dimension $r-2k$ at most, and no double variety of dimension $r-2k+1$. Hence, by Theorem (B_{k-1}) applied to the variety $\mathfrak{U}$ lying on B ,

$$(\mathfrak{U}_{B^2}) - Y_{r+1-2k}[\mathfrak{U}, B] \equiv 0,$$

$$\text{and so} \quad G \equiv (\mathfrak{U} \cdot S) + \sum_{i=1}^k (-)^i (\Delta_S^i \cdot Y_{r-2k+i}[\mathfrak{U}, V])_{\mathfrak{U}} \quad \text{on } \mathfrak{U},$$

which proves (A_k) .

7.3. Assuming (A_k) , we now proceed to prove (B_k) . As in the proof of (B_1) , we project V into a multiple $[r]$ from a general vertex of projection

† For typographical reasons we write Δ_S^i for $(\Delta^i)_S$.

Π . The cone which projects S from Π meets V again in a variety S' of $r-k$ dimensions, which meets S in the variety Δ of $r-k-1$ dimensions in which S meets the locus F_{r-1} of contacts of tangents to V which meet Π , and in the variety Γ' of $r-2k$ dimensions cut out on S by the cone of its chords which meet Π . Thus, if S is of order n ,

$$S + S' \equiv n(V \cdot \Sigma^k), \quad (7.12)$$

$$(S \cdot F_{r-1}) \equiv \Delta, \quad (7.13)$$

and by Theorem (A_k) , since Γ' is the residual intersection of S and S' , which both contain Δ ,

$$\Gamma' \equiv (S \cdot S') + \sum_{i=1}^k (-)^i \left((\Delta^i)_{S'} \cdot Y_{r-2k+i}[S, V] \right)_{S'}. \quad (7.14)$$

The projection of S into $[r]$ is a simple locus in the $[r]$, if Π is chosen generally, and its double variety consists of the projections of Γ' and of Γ , the double locus on S . Hence, from (6.01),

$$\Gamma + \Gamma' \equiv n(S \cdot \Sigma^k) - \sum_{i=0}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i). \quad (7.15)$$

Thus, from (7.12), (7.14), (7.15),

$$\begin{aligned} \Gamma \equiv n(S \cdot \Sigma^k) - \sum_{i=0}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) &+ (S^2) - n(S \cdot \Sigma^k) \\ &- \sum_{i=1}^k (-)^i \left((\Delta^i)_{S'} \cdot Y_{r-2k+i}[S, V] \right)_{S'}, \end{aligned}$$

that is,

$$\begin{aligned} \Gamma \equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) \\ - \sum_{i=1}^k (-)^i \left((\Delta^i)_{S'} \cdot Y_{r-2k+i}[S, V] \right)_{S'}. \end{aligned} \quad (7.16)$$

To reduce this, we first prove that

$$(\Delta^i)_{S'} \equiv (-)^{i-1} (S \cdot U_{r-i}), \quad (7.17)$$

where U_{r-i} is the variety on V_r introduced in 5.2 (with r instead of d). Since $U_{r-1} \equiv F_{r-1}$ the case $i=1$ of (7.17) is included in (7.13).

The intersection of S' and F_{r-1} consists in part of Δ , and the residual intersection Δ' is also of $r-k-1$ dimensions. In the projected figure in

$[r]$, Δ lies on the branch locus F_{r-1} , and Δ' is the companion variety to Δ on the multiple $[r]$. The points common to Δ and Δ' are therefore common to F_{r-1} and its companion variety, and therefore lie on the variety F_{r-2} along which three sheets of the multiple $[r]$ branch. Thus, on V_r , the intersection of Δ and Δ' is the variety common to Δ and the locus F_{r-2} of contact of inflexionally tangent spaces of V_r which pass through Π . Accordingly,

$$(S' \cdot F_{r-1}) \equiv \Delta + \Delta',$$

$$(\Delta \cdot \Delta')_{S'} \equiv (S \cdot F_{r-2}),$$

(since the points common to S and F_{r-2} lie on the variety Δ common to S and F_{r-1}). Hence

$$\begin{aligned} (\Delta^2)_{S'} &\equiv \left(\Delta \cdot \{ (S' \cdot F_{r-1}) - \Delta' \} \right)_{S'} \equiv (\Delta \cdot F_{r-1}) - (S \cdot F_{r-2}) \\ &\equiv (S \cdot \{ F_{r-1}^2 - F_{r-2} \}) \equiv -(S \cdot U_{r-2}), \end{aligned}$$

by (7.13) and (5.06), and this proves (7.17) when $i = 2$. For the general case we use induction. Assuming the result to hold for the exponent $i-1$, we have

$$\begin{aligned} (\Delta^i)_{S'} &\equiv \left((\Delta^{i-1})_{S'} \cdot (\Delta^2)_{S'} \right)_\Delta \equiv \left((-)^{i-1} (S \cdot U_{r-i+1}) \cdot -(S \cdot U_{r-2}) \right)_\Delta \\ &\equiv (-)^i (S \cdot (U_{r-i+1} \cdot U_{r-2})_{(F_{r-1})}), \end{aligned}$$

and therefore we have only to show that

$$(U_{r-i+1} \cdot U_{r-2})_{F_{r-1}} \equiv U_{r-i}. \quad (7.18)$$

But, using (5.03) and (5.02), (remembering that $P_{r-2} \equiv N_{r-2}$, and that $N_{r-1} \equiv F_{r-1}$), we have

$$\begin{aligned} (U_{r-i+1} \cdot U_{r-2})_{(F_{r-1})} &\equiv \left(\sum_{j=0}^{i-2} (-)^j \binom{i-2}{j} (N_{r-i+1+j} \cdot \Sigma^j) \cdot \{ N_{r-2} - (N_{r-1} \cdot \Sigma) \} \right)_{(F_{r-1})} \\ &\equiv \sum_{j=0}^{i-2} (-)^j \binom{i-2}{j} (N_{r-i+j} \cdot \Sigma^j) - \sum_{l=1}^{i-1} (-)^{l-1} \binom{i-2}{l-1} (N_{r-i+l} \cdot \Sigma^l) \\ &\equiv \sum_{j=0}^{i-2} (-)^j \left\{ \binom{i-2}{j} + \binom{i-2}{j-1} \right\} (N_{r-i+j} \cdot \Sigma^j) + (-)^{i-1} (N_{r-1} \cdot \Sigma^{i-1}) \\ &\equiv \sum_{j=0}^{i-1} (-)^j \binom{i-1}{j} (N_{r-i+j} \cdot \Sigma^j) \equiv U_{r-i}, \end{aligned}$$

which proves (7.18) and hence (7.17).

Substituting from (7.17) in (7.16), and then using (4.02), we see that

$$\begin{aligned}
 \Gamma &\equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) + \sum_{i=1}^k (U_{r-i} \cdot Y_{r-2k+i}[S, V]) \\
 &\equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) \\
 &\quad + \sum_{i=1}^k \sum_{j=0}^{k-i} (-)^j (U_{r-i} \cdot Y_{r-2k+i+j}[S] \cdot X_{r-j}[V]) \\
 &\equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) \\
 &\quad + \sum_{i=1}^k \sum_{l=i}^k (-)^{l-i} (U_{r-i} \cdot Y_{r-2k+l}[S] \cdot X_{r-l+i}[V]) \\
 &\equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) \\
 &\quad + \sum_{l=1}^k (-)^l \left(Y_{r-2k+l}[S] \cdot \sum_{i=1}^l (-)^i (U_{r-i} \cdot X_{r-l+i}[V]) \right).
 \end{aligned}$$

But, from (5.04),

$$\begin{aligned}
 &\sum_{i=1}^l (-)^i (U_{r-i} \cdot X_{r-l+i}[V]) \\
 &\equiv \sum_{i=1}^l \sum_{j=0}^i (-)^i \binom{r+1}{j} (Y_{r-i+j}[V] \cdot X_{r-l+i}[V] \cdot \Sigma^j) \\
 &\equiv \sum_{i=1}^l (-)^i (Y_{r-i}[V] \cdot X_{r-l+i}[V]) + \sum_{i=1}^l \sum_{j=1}^i (-)^i \binom{r+1}{j} (Y_{r-i+j}[V] \cdot X_{r-l+i}[V] \cdot \Sigma^j) \\
 &\equiv -X_{r-l}[V] + \sum_{j=1}^l \left(\binom{r+1}{j} \Sigma^j \cdot \sum_{i=j}^l (-)^i (Y_{r-i+j}[V] \cdot X_{r-l+i}[V]) \right) \quad \text{by (3.10),} \\
 &\equiv -X_{r-l}[V] + \sum_{j=1}^l \left(\binom{r+1}{j} \Sigma^j \cdot \sum_{m=0}^{l-j} (-)^{m+j} (Y_{r-m}[V] \cdot X_{r-l+j+m}[V]) \right) \\
 &\equiv -X_{r-l}[V] + (-)^l \binom{r+1}{l} \Sigma^l,
 \end{aligned}$$

since by (3.10) the last inner sum is zero except when $l=j$.

Hence

$$\begin{aligned}
 \Gamma &\equiv (S^2) - Y_{r-2k}[S] - \sum_{i=1}^k \binom{r+1}{i} (Y_{r-2k+i}[S] \cdot \Sigma^i) \\
 &\quad + \sum_{l=1}^k (-)^l \left(Y_{r-2k+l}[S] \cdot \left\{ -X_{r-l}[V] + (-)^l \binom{r+1}{l} \Sigma^l \right\} \right) \\
 &\equiv (S^2) - Y_{r-2k}[S] - \sum_{l=1}^k (-)^l (Y_{r-2k+l}[S] \cdot X_{r-l}[V]) \\
 &\equiv (S^2) - Y_{r-2k}[S, V],
 \end{aligned}$$

which proves (B_k) and completes the induction.

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THE UNITS OF GROUP-RINGS

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[Received and read 16 February, 1939.]

1. *Introduction.*

Let G be any group, and K any ring. The formal sums

$$(1.1) \quad r_1 e_1 + r_2 e_2 + \dots + r_n e_n, \quad r_i \in K, \quad e_i \in G \quad (i = 1, 2, \dots, n),$$

when addition and multiplication are defined in the obvious way, form a ring, the group-ring of G over K , which will be denoted by $R(G, K)$. Henceforward, we suppose that K has the modulus 1, and we denote the identity in G by e_0 . Then $R(G, K)$ has the modulus $1 \cdot e_0$. Since no confusion can arise thereby, the element $1 \cdot e$ in $R(G, K)$ will be written as e , and whenever it is convenient, the elements e_0 in G and re_0 in $R(G, K)$ as 1 and r respectively. The symbol e , with or without subscripts, will always denote an element in G .

If the elements E_1, E_2 in $R(G, K)$ satisfy $E_1 E_2 = 1$, E_1 will be said to be a left unit, and E_2 a right unit in $R(G, K)$. If η is a left (or right) unit in K , then ηe is a left (or right) unit in $R(G, K)$. Such a unit will be described as trivial. The units in $R(G, K)$ form a group if and only if every right unit is also a left unit. This is so, for instance, if both G and K are Abelian. It is also true if G is a finite group, and K is any ring of complex numbers, for then the regular representation† of G can be extended to give an isomorphism of $R(G, K)$ in the ring of ordinary matrices.

The first object of this paper is to study units in $R(G, C)$, where C is the ring of rational integers. In § 2 we take G to be a finite Abelian group,

† A. Speiser, *Theorie der Gruppen von endlicher Ordnung* (2 ed., Berlin, 1927), 172.
J. H. M. Wedderburn, *Lectures on matrices* (New York, 1934), 149.

and consider the more general coefficient ring C' , where C' is the integer ring of a finite algebraic extension k' of the rational field k . We first investigate the algebra $R(G, k')$, and determine the structure of its integer ring, and of the unit group of that integer ring. We then show that a unit of finite order in $R(G, C')$ is trivial, and that the ranks of the unit groups of $R(G, C')$ and of the integer ring of $R(G, k')$ are equal. This suffices to prove, for the particular case of the rational ring C , that $R(G, C)$ has non-trivial units unless either the orders of all the elements of G divide four, or they all divide six, in which case $R(G, C)$ has only trivial units. In § 3 we consider groups all of whose elements have finite order, and show that, if G is such a group, and $R(G, C)$ has only trivial units, then either G is Abelian, or it is the direct product of a quaternion group and an Abelian group, all of whose elements other than the identity have order two. In § 4 we show that $R(G, C)$ has only trivial units if G belongs to a class of infinite discrete groups that includes free groups and free Abelian groups.

Section 5 is concerned with a condition on G which can be expressed as follows: Consider the following transformations on a matrix $\|a_{ij}\|$, $i, j = 1, \dots, n$, whose elements are in $R(G, C)$:

- (a) Multiplying a row on the left, or a column on the right by $\pm e$; that is, replacing the row $(a_{i1}, \dots, a_{in})$ by $(\pm ea_{i1}, \dots, \pm ea_{in})$, or the column $(a_{1i}, \dots, a_{ni})$ by $(\pm a_{1i}e, \dots, \pm a_{ni}e)$, $e \in G$;
- (b) Adding to one row (column) of $\|a_{ij}\|$ a left (right) multiple of another row (column) by an element in $R(G, C)$; that is, replacing the row $(a_{i1}, \dots, a_{in})$ by $(a_{i1} + ra_{j1}, \dots, a_{in} + ra_{jn})$, or the column $(a_{1i}, \dots, a_{ni})$ by $(a_{1i} + a_{1j}r, \dots, a_{ni} + a_{nj}r)$, $i \neq j$, $r \in K$;
- (c) Bordering $\|a_{ij}\|$ with a row and a column of zeros meeting in a 1; that is, putting $a_{in+1} = a_{n+1i} = 0$, $i = 1, \dots, n$, $a_{n+1n+1} = 1$; or the inverse of this.

Then the condition on G is that if $\|a_{ij}\|$ has a left inverse $\|a_{ij}^*\|$, so that

$$(1.3) \quad \sum_{r=1}^n a_{ir}^* a_{rj} = \delta_{ij} \quad (i, j = 1, \dots, n),$$

then $\|a_{ij}\|$ is transformable into the one-rowed matrix $\|1\|$ by a sequence of transformations (1.2).

The converse, that if a matrix $\|a_{ij}\|$ has a left inverse, so has any matrix derivable from it by transformations (1.2), is, of course, always true.

If G is Abelian and $R(G, C)$ has a non-trivial unit E , the condition is not satisfied. For the determinant $|a_{ij}|$ can then be defined as in elementary algebra, and is altered under transformations (1.2) only by a factor $\pm e$. The one-rowed matrix $\|E\|$ with determinant E cannot, therefore, be transformed into $\|1\|$ though it has the left inverse $\|E^{-1}\|$. On the other hand we show in § 5 that the condition is satisfied for certain groups other than that consisting of the identity alone.

These results are applicable to the topology of a polyhedron P if we take G to be its fundamental group. For K. Reidemeister† has shown that the invariants of certain "incidence matrices" under transformations (1.2) are invariants‡ of P . Also J. H. C. Whitehead§ could only prove a certain topological theorem for a polyhedron whose fundamental group satisfies the above matrix condition.

I should like to thank Mr. Whitehead for many helpful suggestions and much good advice that he has given me, both in the investigation of these problems and in the writing of this paper.

2. Finite Abelian groups.

In this section we determine the structure of the unit group of $R(G, C')$, where G is a finite Abelian group, and C' is the integer ring of a finite algebraic extension k' of the rational field k . To that end we consider $R(G, k')$, which is a commutative algebra over k' . It is known|| that $R(G, k')$ is semi-simple, and therefore is the direct sum of simple algebras over k' , each of which, being commutative, is isomorphic to an algebraic extension¶ of k' . Moreover there exists†† an extension $k'(\zeta)$ of k' , such that in the corresponding reduction of $R(G, k'(\zeta))$ each simple algebra over $k'(\zeta)$ is isomorphic to $k'(\zeta)$. That is, if g is the order of G , there exist elements $\eta_0, \eta_1, \dots, \eta_{g-1}$ in $R(G, k'(\zeta))$ such that

$$(2.1) \quad \begin{aligned} (a) \quad & \eta_i \eta_j = 0, \quad i \neq j, \quad i, j = 0, 1, \dots, g-1; \\ (b) \quad & \eta_i^2 = \eta_i, \quad i = 0, 1, \dots, g-1. \end{aligned}$$

† "Homotopiegruppen von Komplexen", *Abhandlung aus dem Mathematischen Seminar der Hamburgischen Universität*, 10 (1934), 211-215.

‡ Reidemeister proved they are combinatorial invariants, and Whitehead, *Quart. J. of Math.* (Oxford Series), 10 (1939), 81-83, has since proved they are topological invariants of P .

§ *Proc. London Math. Soc.* (2), 45 (1939), 243-327.

|| Wedderburn, *loc. cit.*, 168.

¶ Wedderburn, *loc. cit.*, 159 [Theorem 5 (iii)].

†† Wedderburn, *loc. cit.*, 168.

The first step is the explicit determination of these elements. We then carry out explicitly the reduction of $R(G, k')$ to a sum of fields. This enables us immediately to ascertain the structure of the unit group of the integer ring of $R(G, k')$. Finally we prove the two results concerning the unit group of $R(G, C')$, namely that its elements of finite order are all trivial, and that its rank is the same as that of the unit group of the integer ring of $R(G, k')$.

If h is the maximum order of any element in G , ζ can in fact be taken to be a primitive h -th root of unity. For since G is Abelian there are precisely $\dagger$ g irreducible mutually inequivalent representations of G , and each is of order 1. Denote them by $\Gamma^0, \Gamma^1, \dots, \Gamma^{g-1}$, and the elements of G by $e_0, e_1, \dots, e_{g-1}$. We can regard Γ^i as a homomorphism of G in a multiplicative group of complex numbers, and write

$$(2.2) \quad \Gamma^i(e_j) = \chi_j^i \quad (i, j = 0, 1, \dots, g-1).$$

Since the order of any element in G divides h , χ_j^i is an h -th root of unity and therefore a number in $k'(\zeta)$. We have $\ddagger$, if $\bar{\chi}_j^i$ denotes the reciprocal of χ_j^i ,

$$(2.3) \quad \begin{cases} (a) & \sum_{r=0}^{g-1} \chi_r^i \bar{\chi}_r^j = \delta_{ij} g \quad (i, j = 0, 1, \dots, g-1), \\ (b) & \sum_{r=0}^{g-1} \chi_r^i \bar{\chi}_j^r = \delta_{ij} g \quad (i, j = 0, 1, \dots, g-1). \end{cases}$$

Define elements $\eta_0, \eta_1, \dots, \eta_{g-1}$ in $R(G, k'(\zeta))$ by

$$(2.4) \quad \eta_i = \frac{1}{g} \sum_{r=0}^{g-1} \chi_r^i e_r \quad (i = 0, 1, \dots, g-1).$$

Given s, e_r can be written uniquely in the form $e_t e_s$, and then $\chi_r^i = \chi_t^i \chi_s^i$; so that we have

$$\eta_i = \frac{1}{g} \sum_{t=0}^{g-1} \chi_t^i \chi_s^i e_t e_s = \chi_s^i \eta_i e_s,$$

or

$$(2.5) \quad \eta_i e_s = \bar{\chi}_s^i \eta_i \quad (i, s = 0, 1, \dots, g-1).$$

Multiplying by χ_s^j and summing over $s = 0, 1, \dots, g-1$, we obtain (2.1) in virtue of (2.3a).

$\dagger$ Speiser, *loc. cit.*, compare Theorem 136, p. 159, with Theorem 154, p. 176.

$\ddagger$ Speiser, *loc. cit.*, 171, 173. The developments of this section as far as the proof of equations (2.1) are the Abelian case of the first part of paragraph 58 of Speiser's book.

The linear independence of $\eta_0, \eta_1, \dots, \eta_{g-1}$ follows from equations (2.1) and since their number is equal to the rank of $R(G, k'(\zeta))$ they form a basis for that ring. That is,

$$(2.6) \quad E = \sum_{r=0}^{g-1} a_r e_r = \sum_{r=0}^{g-1} b_r \eta_r,$$

where E is any element in $R(G, k'(\zeta))$; and, by equations (2.1), (2.5) and (2.4),

$$(2.7) \quad \begin{cases} (a) & b_i = \sum_{r=0}^{g-1} a_r \bar{\chi}_r^i \quad (i = 0, 1, \dots, g-1), \\ (b) & a_i = \frac{1}{g} \sum_{r=0}^{g-1} b_r \chi_r^i \quad (i = 0, 1, \dots, g-1). \end{cases}$$

In particular, since e_0 is the identity and therefore $\chi_0^i = 1, i = 0, 1, \dots, g-1$, or directly from equations (2.1),

$$(2.8) \quad e_0 = \eta_0 + \eta_1 + \dots + \eta_{g-1}.$$

If σ is any transformation in the Galois group of $k'(\zeta)$ relative to k' , the correspondence $e_j \rightarrow \sigma(\Gamma^i(e_j)) = \sigma(\chi_j^i)$ is a homomorphism, and therefore is one of the representations $\Gamma^0, \Gamma^1, \dots, \Gamma^{g-1}$. We denote it by $\Gamma^{\sigma(i)}$ and say that $\Gamma^{\sigma(i)}$ is conjugate to Γ^i . We denote by ξ_i a root of unity whose order is the least common multiple of the orders of $\chi_0^i, \chi_1^i, \dots, \chi_{g-1}^i$, so that $k'(\xi_i)$ is the least extension of k' in which the representation Γ^i exists. Clearly ξ_i is an h -th root of unity. But, since h is the maximum order of any element in G , we can write G as the direct product† of a cyclic group $\{e\}$ of order h , and another group G' . If Γ^i is the representation $e \rightarrow \zeta, e' \rightarrow 1, e' \in G'$, or any of its conjugates, then plainly ξ_i is a primitive h -th root of unity. We have thus established the last part of Theorem 1:

THEOREM 1. *Let $\Gamma^0, \Gamma^1, \dots, \Gamma^{p-1}$ be a complete set of irreducible, mutually inequivalent, and, relative to k' , non-conjugate representations of a finite Abelian group G . Then the ring $R(G, k')$ is the direct sum of the fields $R_0, R_1, \dots, R_{p-1}$, where R_α is isomorphic to $k'(\xi_\alpha)$, defined as above, $\alpha = 0, 1, \dots, p-1$. The numbers $\xi_0, \xi_1, \dots, \xi_{p-1}$ are h -th roots of unity, of which at least one is primitive.*

It remains to prove the first part of the Theorem.

† This is an immediate consequence of the theorem concerning the expression of a finite Abelian group as the direct product of cyclic groups of prime-power order. See e.g., Speiser, *loc. cit.*, Theorem 48, p. 62.

Let U_a be the subgroup of the Galois group consisting of those transformations σ for which $\sigma(a) = a$. Then σ is in U_a if and only if

$$\sigma(\chi_i^a) = \chi_i^a \quad (i = 0, 1, \dots, g-1);$$

that is, if, and only if, σ leaves invariant every number in $k'(\xi_a)$. Therefore, by the fundamental theorem of the Galois theory†, a number is left invariant by every transformation in U_a , if and only if it is in $k'(\xi_a)$.

By (2.7a) a necessary condition for E , given by (2.6), to be in $R(G, k')$ is that

$$(2.9) \quad \sigma(b_i) = b_{\sigma(i)} \quad (i = 0, 1, \dots, g-1),$$

holds for every transformation σ in the Galois group. By (2.7b) this condition is also sufficient. If equations (2.9) hold, b_a is in $k'(\xi_a)$ ($a = 0, 1, \dots, p-1$). Conversely, given b_a in $k'(\xi_a)$ ($a = 0, 1, \dots, p-1$), equations (2.9) can be solved uniquely for b_i ($i = 0, 1, \dots, g-1$). For, by hypothesis, to each number i of the set $0, 1, \dots, g-1$ we can assign a transformation τ_i in the Galois group such that $\tau_i(i) = a$ is one of the numbers $0, 1, \dots, p-1$. Then $b_i = \tau_i^{-1}(b_a)$, so that a solution, if it exists, is unique. On the other hand, if $\sigma(i) = j$, we have $\tau_j(j) = \beta$, where β is one of the numbers $0, 1, \dots, p-1$, and therefore $\tau_j \sigma \tau_i^{-1}(a) = \beta$. Hence we must have $a = \beta$, and $\tau_j \sigma \tau_i^{-1}$ is in U_a . Since b_a is in $k'(\xi_a)$, $\tau_j \sigma \tau_i^{-1}(b_a) = b_a$ i.e. $\sigma \tau_i^{-1}(b_a) = \tau_j^{-1}(b_a)$. Defining b_i as $\tau_i^{-1}(b_a)$, therefore, we have a solution of equations (2.9). There is thus a 1-1 correspondence between elements in $R(G, k')$ and sequences $(b_0, b_1, \dots, b_{p-1})$, b_a in $k'(\xi_a)$ ($a = 0, 1, \dots, p-1$). If we denote E , given by (2.6) and satisfying (2.9), by $(b_0, b_1, \dots, b_{p-1})$, we have, by (2.1),

$$(2.10) \quad \begin{cases} (a) & (b_0, b_1, \dots, b_{p-1}) + (c_0, c_1, \dots, c_{p-1}) = (b_0 + c_0, b_1 + c_1, \dots, b_{p-1} + c_{p-1}), \\ (b) & (b_0, b_1, \dots, b_{p-1}) (c_0, c_1, \dots, c_{p-1}) = (b_0 c_0, b_1 c_1, \dots, b_{p-1} c_{p-1}). \end{cases}$$

If, therefore, R_a consists of those sequences for which $b_\beta = 0$, $\beta \neq a$, the theorem is true.

The element E in $R(G, k')$ is said to be integral if it satisfies an equation of the form

$$E^r + a_{r-1} E^{r-1} + \dots + a_1 E = 0, \text{ with } a_1, \dots, a_{r-1} \text{ rational integers.}$$

The integral elements in $R(G, k')$ form a ring, the integer ring of $R(G, k')$, and the integral element E is a unit of this ring if there exists another integral element E' such that $EE' = 1$. By (2.10), the element

† B. L. van der Waerden, *Moderne Algebra*, 1 (2 ed., Berlin, 1937), 163.

E , given by (2.6), is integral in $R(G, k')$ if, and only if, b_a is integral in $k'(\xi_a)$ ($a = 0, 1, \dots, p-1$); and it is a unit in this integer ring if, and only if, b_a is a unit in $k'(\xi_a)$. Hence we have

THEOREM 2. *The unit group of the integer ring of $R(G, k')$ is the direct product of groups isomorphic to the unit groups of $k'(\xi_0), k'(\xi_1), \dots, k'(\xi_{p-1})$.*

If E , given by (2.6), is in $R(G, C')$, by (2.7a) it is in the integer ring of $R(G, k')$. The unit group of $R(G, C')$ is therefore a subgroup of the unit group of this integer ring.

THEOREM 3. *A unit of finite order in $R(G, C')$ is trivial.*

Let E , given by (2.6), be a unit of order m . Then

$$b_i^m = 1 \quad (i = 0, 1, \dots, g-1)$$

and therefore b_i and each of its conjugates has absolute value 1. The same is true of χ_j^i . Since E is a unit, we can choose i so that a_i is not zero. Now

$$|a_i| = \left| \frac{1}{g} \sum_{r=0}^{g-1} b_r \chi_r^i \right| \leq \frac{1}{g} \sum_{r=0}^{g-1} |b_r \chi_r^i| = 1,$$

and the same is true for each conjugate of a_i . But the product of these conjugates is the norm of a_i , a rational non-zero integer. Hence equality holds above, and

$$b_0 \chi_i^0 = b_1 \chi_i^1 = \dots = b_{g-1} \chi_i^{g-1} = a_i, \quad b_j = a_i \bar{\chi}_i^j \quad (j = 0, 1, \dots, g-1).$$

By (2.7b) and (2.3b), $a_j = 0$, if $j \neq i$, so that the unit E is trivial.

This disposes of the finite part of the unit group. We have next

THEOREM 4. *There exists an integer n such that the n -th power of any unit in the integer ring of $R(G, k')$ is in $R(G, C')$.*

Let n be the number of residue classes mod g in $k'(\zeta)$ that are prime to g , and let E , given by (2.6), be a unit in the integer ring of $R(G, k')$. Then

$$b_i \quad (i = 0, 1, \dots, g-1)$$

is a unit in $k'(\zeta)$ and is therefore prime to g . By Fermat's theorem for ideals†,

$$b_i^n \equiv 1 \pmod{g},$$

i.e.

$$b_i^n - 1 = c_i g \quad [c_i \text{ an integer in } k'(\zeta)].$$

† D. Hilbert, *Gesammelte Abhandlungen*, 1 (Berlin, 1932), 82.

By (2.8) therefore,

$$E^n = e_0 + \sum_{i=0}^{g-1} (b_i^n - 1) \eta_i = e_0 + \sum_{i=0}^{g-1} c_i \cdot g \eta_i.$$

By (2.4) the coefficients in $g\eta_i$ are algebraic integers. So, therefore, are those in E^n , which is therefore an element in $R(G, C')$.

THEOREM 5. *The unit groups of $R(G, C')$ and of the integer ring of $R(G, k')$ have the same rank.*

Since E^{-n} is the n -th power of E^{-1} , it also is in $R(G, C')$, so that E^n is a unit in $R(G, C')$. Moreover the n -th powers of an independent set of units are themselves independent, so that the rank of the unit group of $R(G, C')$ cannot be less than the rank of the unit group of the integer ring of $R(G, k')$. But the former group is a subgroup of the latter, and cannot therefore have a greater rank, whence the theorem follows.

Applying these results to $R(G, C)$ we obtain

THEOREM 6. *If G is a finite Abelian group, $R(G, C)$ has non-trivial units unless G is the direct product of*

(i) *l cyclic groups of order two, $l \geq 0$,*

and (ii) *either (a) m cyclic groups of order three, $m \geq 0$ or (b) n cyclic groups of order four, $n \geq 0$,*

in which cases $R(G, C)$ has only trivial units.

The "direct product of 0 groups ..." is, of course, to be interpreted conventionally as the group consisting of the identity alone.

By Theorem 3, a unit of finite order in $R(G, C)$ is trivial. Conversely, since C contains no units of infinite order, a trivial unit in $R(G, C)$ is of finite order. Hence $R(G, C)$ has only trivial units if, and only if, the rank of its unit group is zero. By Theorem 5, Theorem 2, and the last sentence of Theorem 1, the rank of the unit group of $R(G, C)$ is zero if, and only if, the rank of the unit group of $k(\zeta)$ is zero, that is to say†, if, and only if, $h = 2, 3, 4$, or 6 . Finally the groups mentioned in the enunciation are just those‡ for which $h = 2, 3, 4$, or 6 .

3. Groups all of whose elements have finite order.

THEOREM 7. *Let G^* be the direct product of a group G and a cyclic group of order two. If all the units in $R(G, C)$ are trivial, so are all the units in $R(G^*, C)$.*

† Hilbert, *loc. cit.*, 102 (Theorem 47).

‡ This is an immediate consequence of the theorem concerning the expression of a finite Abelian group as the direct product of cyclic groups of prime-power order. See, e.g., Speiser, *loc. cit.*, Theorem 48, p. 62.

Let the cyclic group of order two be generated by f . An element in $R(G^*, C)$ can be written uniquely in the form $\alpha + \beta f$, where α, β are in $R(G, C)$. If

$$(\alpha + \beta f)(\gamma + \delta f) \equiv \alpha\gamma + \beta\delta + (\alpha\delta + \beta\gamma)f = 1,$$

then $\alpha\gamma + \beta\delta = 1, \quad \alpha\delta + \beta\gamma = 0.$

Hence $(\alpha + \beta)(\gamma + \delta) = 1, \quad (\alpha - \beta)(\gamma - \delta) = 1.$

Because $R(G, C)$ has only trivial units, we must have $\alpha + \beta = \pm e_1$, $\alpha - \beta = \pm e_2$. Hence $\alpha = \frac{1}{2}(\pm e_1 \pm e_2)$. But α is in $R(G, C)$, so that we must have $e_1 = e_2$, so that $\alpha + \beta = \pm(\alpha - \beta)$. If $\alpha + \beta = \alpha - \beta = \pm e_1$, we have $\alpha = \pm e_1$, $\beta = 0$, and so $\alpha + \beta f = \pm e_1$. If $\alpha + \beta = -(\alpha - \beta) = \pm e_1$, we have similarly, $\alpha + \beta f = \pm e_1 f$. In all cases $\alpha + \beta f$ is a trivial unit, and the theorem is therefore proved.

THEOREM 8. *Let G be a quaternion group. Then all the units in $R(G, C)$ are trivial.*

Let the group G be generated by x, y, z , subject to

$$x^2 = y^2 = z^2 = xyz = P,$$

say, and let $A = a_0 + a_1x + a_2y + a_3z + b_0P + b_1xP + b_2yP + b_3zP$ be a unit in $R(G, C)$. The correspondence $x \rightarrow i, y \rightarrow j, z \rightarrow k, P \rightarrow -1$ is a homomorphism of $R(G, C)$ on the ring of quaternions with integer coefficients. In this ring the only units† are $\pm 1, \pm i, \pm j, \pm k$. Hence A is carried into one of these quantities. That is, for some index i ($i = 0, 1, 2, \text{ or } 3$).

$$(3.1) \quad a_i - b_i = \pm 1, \quad a_j - b_j = 0 \quad (j \neq i, j = 0, 1, 2, 3).$$

Let G' be a direct product of two cyclic groups of order two, and let its elements be $1, x', y', z'$. The correspondence $x \rightarrow x', y \rightarrow y', z \rightarrow z', P \rightarrow 1$ is a homomorphism of $R(G, C)$ on $R(G', C)$. In this ring, by Theorem 6, or by Theorem 7, the only units are $\pm 1, \pm x', \pm y', \pm z'$. Hence A is carried into one of these quantities. That is, for some index i' ($i' = 0, 1, 2, \text{ or } 3$),

$$(3.2) \quad a_{i'} + b_{i'} = \pm 1, \quad a_j + b_j = 0 \quad (j \neq i', j = 0, 1, 2, 3).$$

† The norm $a^2 + b^2 + c^2 + d^2$ of the unit $a + bi + cj + dk$ must be equal to 1.

Since a_i, b_i are integral, we have, on comparing (3.1) and (3.2), $i = i'$. Hence either

$$a_i = \pm 1, \quad b_i = 0, \quad a_j = b_j = 0 \quad (j \neq i),$$

or
$$a_i = 0, \quad b_i = \pm 1, \quad a_j = b_j = 0 \quad (j \neq i).$$

In either case the unit A is trivial.

THEOREM 9. *Let G be a group such that all the units in $R(G, C)$ are trivial. Then, if P, Q are in $R(G, C)$, $PQ = 0$ implies $QP = 0$.*

Suppose on the contrary that $PQ = 0, QP \neq 0$. Then $(QP)^2 = 0$, so that

$$(1 - 3QP)(1 + 3QP) = 1.$$

Since $QP \neq 0$, $1 - 3QP$ cannot be of the form $\pm e$, and is therefore a non-trivial unit.

THEOREM 10. *Let G be a group such that all the units in $R(G, C)$ are trivial. Then every cyclic sub-group of G of finite order is self-conjugate.*

Let e_1 generate a sub-group of order n , and let e_2 be any element in G . Let

$$P = e_2(1 - e_1), \quad Q = 1 + e_1 + e_1^2 + \dots + e_1^{n-1}.$$

Then $PQ = 0$, so that, by Theorem 9, $QP = 0$. That is

$$e_2 + e_1 e_2 + \dots + e_1^{n-1} e_2 = e_2 e_1 + e_1 e_2 e_1 + \dots + e_1^{n-1} e_2 e_1.$$

For some r , therefore,

$$e_2 e_1 = e_1^r e_2, \quad \text{i.e.} \quad e_2 e_1 e_2^{-1} = e_1^r.$$

That is to say, the subgroup $e_2 \{e_1\} e_2^{-1}$ coincides with $\{e_1\}$ and $\{e_1\}$ is self-conjugate.

THEOREM 11. *If all the elements of a group G have finite order, $R(G, C)$ has non-trivial units unless G is either*

- (i) *an Abelian group the orders of whose elements all divide four;*
- or (ii) *an Abelian group the orders of whose elements all divide six;*
- or (iii) *the direct product of a quaternion group and an Abelian group the orders of whose elements all divide two.*

In these cases $R(G, C)$ has only trivial units.

Let G be a group whose elements all have finite order such that $R(G, C)$ has only trivial units. By Theorem 6, the orders of the elements of G can only be 2, 3, 4, or 6. If G is Abelian it must therefore be of type

(i) or (ii). If G is not Abelian, by Theorem 10, every cyclic subgroup of G is self-conjugate, and G is therefore Hamiltonian. It is known[†] that a finite Hamiltonian group is the direct product of a group of type (iii) and an Abelian group all of whose elements have odd order; and this theorem can easily be extended to cover groups all of whose elements have finite order[‡]. If, however, the second factor in this product were not the identity alone, G would contain an element of order $4p$, where p is an odd prime, which is impossible. Hence G is of type (iii).

Conversely, let G be a group of type (i), (ii), or (iii), and let E be a unit in $R(G, C)$, where

$$E = \lambda_1 e_1 + \dots + \lambda_r e_r.$$

Then $e_1, \dots, e_r$ generate a finite subgroup G' of G . G' is itself of one of the three given types, and E is a unit in $R(G', C)$. But all the units in $R(G', C)$ are trivial, by Theorem 6 if G' is Abelian, and by Theorems 7 and 8 if G' is not Abelian. Hence E is a trivial unit, and $R(G, C)$ has only trivial units.

4. Infinite discrete groups.

A group H is said to be indexed if we are given a homomorphism γ of H in the additive group of rational integers, such that $\gamma(H)$ does not consist of zero alone. In general, a group can be indexed in more than one way. If e is an element in H , $\gamma(e)$ will be called the degree of e (relative to γ). If K is any ring, an element in $R(H, K)$ which can be put in the form

$$P = m_1 e_{i_1} + m_2 e_{i_2} + \dots + m_r e_{i_r},$$

[†] H. Hilton, *Finite groups* (Oxford, 1908), 178.

[‡] Hilton's proof (*loc. cit.*) requires the following alterations:

(i) The theorem at the top of page 177 is replaced by:

There are no Hamiltonian groups the orders of all of whose elements are powers of p , if p is a prime greater than 2. If $p = 2$, a Hamiltonian group all of whose elements have orders a power of p is the direct product of a quaternion group, and a group all of whose elements other than the identity have order 2.

The proof is unchanged.

(ii) To show that a Hamiltonian group is the direct product of groups all the elements in any one of which have orders which are powers of the same prime, we need two observations. First, the commutator of two elements in a Hamiltonian group is a power of each of them. Elements having relatively prime orders therefore commute. Secondly, if two elements in a Hamiltonian group have orders p^a and p^b , the subgroup that they generate is isomorphic to a factor group of the group given by $X^{p^a} = 1$, $Y^{p^b} = 1$, $XYX^{-1} = Y^r$, for some value of r , and the order of this group is p^{a+b} . The elements of a Hamiltonian group having a power of p as order therefore form a group.

where $e_{i_1}, e_{i_2}, \dots, e_{i_r}$ are all of degree α , will be called homogeneous of degree α . Any element P in $R(H, K)$ can be written in the form

$$(4.1) \quad P = P_{\alpha_1} + P_{\alpha_2} + \dots + P_{\alpha_p},$$

P_{α_i} homogeneous of degree α_i , $\alpha_1 < \alpha_2 < \dots < \alpha_p$. Plainly, if $P = 0$, then, in any such expansion, $P_{\alpha_i} = 0$, $i = 1, \dots, p$. If similarly

$$(4.2) \quad Q = Q_{\beta_1} + Q_{\beta_2} + \dots + Q_{\beta_q},$$

Q_{β_i} homogeneous of degree β_i , $\beta_1 < \beta_2 < \dots < \beta_q$, then we have

$$(4.3) \quad PQ = P_{\alpha_1} Q_{\beta_1} + \dots + P_{\alpha_p} Q_{\beta_q},$$

where the unwritten terms have degrees exceeding the degree $\alpha_1 + \beta_1$ of $P_{\alpha_1} Q_{\beta_1}$ but less than the degree $\alpha_p + \beta_q$ of $P_{\alpha_p} Q_{\beta_q}$.

A group G is said to be indicable throughout if every subgroup of G not consisting of the identity alone can be indexed. Any free group can obviously be indexed, and any subgroup of a free group is either a free group or the identity alone†, so that any free group is indicable throughout. Similarly any free Abelian group is indicable throughout. In an Appendix to this paper we show that, if two groups are indicable throughout, so are their free product and their direct product.

THEOREM 12. *If G is indicable throughout and K has no zero-divisors, $R(G, K)$ has no zero-divisors.*

In other words we must show that, if

$$(4.4) \quad P = m_1 e_{i_1} + m_2 e_{i_2} + \dots + m_r e_{i_r}, \quad Q = n_1 e_{j_1} + n_2 e_{j_2} + \dots + n_s e_{j_s}$$

($r \geq 1$, $s \geq 1$; $m_{i_\lambda} \neq 0$, $n_{j_\lambda} \neq 0$; $e_{i_\lambda} \neq e_{i_\mu}$, $e_{j_\lambda} \neq e_{j_\mu}$, if $\lambda \neq \mu$) are elements of $R(H, K)$, then $P \cdot Q \neq 0$. Plainly it is sufficient to prove this assertion in the case $e_{i_1} = e_{j_1} = 1$. The proof is by induction on $r + s$. If $r = s = 1$ the assertion is true because K has no zero-divisors.

Suppose then that it is true for $r + s < n$ ($n > 2$) and suppose that $r + s = n$. Then $e_{i_1}, \dots, e_{i_r}, e_{j_1}, \dots, e_{j_s}$ generate a subgroup H of G not consisting of the identity alone. H can therefore be indexed, and we can suppose P, Q , to be given by (4.1), (4.2), with none of $P_{\alpha_1}, P_{\alpha_p}, Q_{\beta_1}, Q_{\beta_q}$ zero. P and Q cannot both be homogeneous of degree zero, or we should have $\gamma(H) = 0$. But e_{i_1}, e_{j_1} are both of degree zero. Hence either $p > 1$ or $q > 1$. In either

† O. Schreier, "Die Untergruppen der freien Gruppen", *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, 5 (1927), 161-183.

case we have, $P_{\alpha_1} Q_{\beta_1} \neq 0$ by the hypothesis of the induction. By (4.3), therefore, $PQ \neq 0$.

THEOREM 13. *If G is indicable throughout and K has no zero-divisors, all the units of $R(G, K)$ are trivial.*

This is a corollary of the assertion: If P, Q are given by (4.4) with $r+s > 2$, then

$$PQ = p_1 e_{\kappa_1} + p_2 e_{\kappa_2} + \dots + p_t e_{\kappa_t}$$

($p_\lambda \neq 0, e_{\kappa_\lambda} \neq e_{\kappa_\mu}$ if $\lambda \neq \mu$), and $t \geq 2$. As before, it is sufficient to prove the assertion when $e_{i_1} = e_{j_1} = 1$. Let H have the same meaning as in the proof of Theorem 12. Since $r+s > 2$, H does not consist of the identity alone. It can therefore be indexed and we can suppose P, Q to be given by (4.1), (4.2) with none of $P_{\alpha_1}, P_{\alpha_p}, Q_{\beta_1}, Q_{\beta_q}$ zero. As in the proof of Theorem 12 we have either $p > 1$ or $q > 1$, and therefore $\alpha_1 + \beta_1 \neq \alpha_p + \beta_q$. But, by Theorem 12, $P_{\alpha_1} Q_{\beta_1} \neq 0, P_{\alpha_p} Q_{\beta_q} \neq 0$. These, as we have seen, have different degrees, so that by (4.3) our assertion is true.

5. Matrices.

Let G be a group satisfying the following condition:

Given $a_1, a_2, \dots, a_n$ in $R(G, C)$ such that the equation

$$(5.1) \quad \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 1$$

can be solved for $\lambda_1, \lambda_2, \dots, \lambda_n$ in $R(G, C)$, then we can transform $a_1, a_2, \dots, a_n$ into $1, 0, \dots, 0$ by a sequence of transformations of the following types:

- (5.2) (a) replacing a_i by $a_i + ra_j$ ($j \neq i$), r being an element in $R(G, C)$;
 (b) replacing a_i by $\pm ea_i$, e being an element in G .

Then G satisfies the matrix condition of § 1. For we may take $a_1, a_2, \dots, a_n$ to be the last column of any matrix with a left inverse whose elements are in $R(G, C)$. By (1.3), equation (5.1) can then be solved. We can therefore reduce the last column of the matrix to the form $0, 0, \dots, 0, 1$, by a manipulation of rows according to transformations (1.2). Hence the order of the matrix can be reduced, and by repeating the process we eventually arrive at the matrix $\|1\|$.

THEOREM 14. *The above condition is satisfied by the cyclic groups of orders two, three, and four, and by the direct product of two cyclic groups of order two.*

We shall prove the theorem for the cyclic group of order four. The other cases are similar.

Let G be generated by e , with $e^4 = 1$, and let $a_1, a_2, \dots, a_n$ be elements in $R(G, C)$ such that (5.1) can be solved. The correspondence $e \rightarrow i$ is a homomorphism of $R(G, C)$ on the ring of Gaussian integers. In this ring there is an algorithm†; any sequence $\rho_1, \rho_2, \dots, \rho_n$ can be reduced to the form $\sigma, 0, \dots, 0$ by a sequence of transformations of the following types:

- (5.3) (a) replacing $\rho_j + r\rho_k$ ($j \neq k$), r being a Gaussian integer;
 (b) replacing ρ_j by $-\rho_j$ or by $\pm i\rho_j$.

Let $\rho_1, \dots, \rho_n$ be the images of $a_1, \dots, a_n$ in the homomorphism. Any transformation (5.3) on $\rho_1, \dots, \rho_n$ can be induced by an appropriate transformation (5.2) on $a_1, \dots, a_n$. This sequence can therefore be transformed by means of transformations (5.2) into a sequence of the form:

$$a_1' = a_1' + \beta_1' e + \gamma_1' e^2 + \delta_1' e^3, \quad a_2' = (a_2' + \beta_2' e)(1 + e^2),$$

$$\dots, \quad a_n' = (a_n' + \beta_n' e)(1 + e^2).$$

Using in a similar way the homomorphism $e \rightarrow -1$ of $R(G, C)$ on the ring of integers, we transform this sequence in turn into one of the form:

$$a_1'' = a_1'' + \beta_1'' e + \gamma_1'' e^2 + \delta_1'' e^3, \quad a_2'' = (a_2'' + \beta_2'' e)(1 + e^2),$$

$$a_3'' = a_3''(1 + e)(1 + e^2), \quad \dots, \quad a_n'' = a_n''(1 + e)(1 + e^2);$$

and this can itself be transformed into a sequence of the form:

$$(5.4) \quad a_1''' = a_1''' + \beta_1''' e + \gamma_1''' e^2 + \delta_1''' e^3, \quad a_2''' = (a_2''' + \beta_2''' e)(1 + e^2),$$

$$a_3''' = a_3'''(1 + e)(1 + e^2), \quad a_4''' = \dots a_n''' = 0.$$

Now let us call a transformation (5.2a), or the resultant of a sequence of such transformations, admissible if, when it is applied to (5.4), the result is of the same form:

$$(5.5) \quad a_1^{(iv)} = a_1^{(iv)} + \beta_1^{(iv)} e + \gamma_1^{(iv)} e^2 + \delta_1^{(iv)} e^3, \quad a_2^{(iv)} = (a_2^{(iv)} + \beta_2^{(iv)} e)(1 + e^2),$$

$$a_3^{(iv)} = a_3^{(iv)}(1 + e)(1 + e^2), \quad a_4^{(iv)} = \dots a_n^{(iv)} = 0.$$

† van der Waerden, *loc. cit.*, 62.

Consider now the functions

$$\phi_1 = \alpha_1''' + \beta_1''' + \gamma_1''' + \delta_1''', \quad \phi_2 = \alpha_2''' + \beta_2''', \quad \phi_3 = \alpha_3'''.$$

An admissible transformation (5.2a) alters at most one of them. Hence an admissible sequence of transformations such that, when the transformations of the sequence are applied successively to (5.4), at each stage one of $|\phi_1|$, $|\phi_2|$, $|\phi_3|$ is reduced, must be of finite length. Suppose now that (5.5) represents the result of applying such a sequence of maximal length to (5.4). Then no admissible transformation applied to (5.5) reduces the absolute value of any of the corresponding functions

$$\phi_1 = \alpha_1^{(iv)} + \beta_1^{(iv)} + \gamma_1^{(iv)} + \delta_1^{(iv)}, \quad \phi_2 = \alpha_2^{(iv)} + \beta_2^{(iv)}, \quad \phi_3 = \alpha_3^{(iv)}.$$

But the transformation $\alpha_3^{(iv)} \rightarrow \alpha_3^{(iv)} + m(1+e^2)(1+e)\alpha_1^{(iv)}$ leaves $\alpha_1^{(iv)}$ and $\alpha_2^{(iv)}$ unaltered, and carries $\alpha_3^{(iv)}$ into $(\alpha_3^{(iv)} + 4m\phi_1)(1+e)(1+e^2)$. It is therefore an admissible transformation, and changes ϕ_3 into $\phi_3 + 4m\phi_1$. Hence, for all integral values of m , we have $|\phi_3 + 4m\phi_1| \geq |\phi_3|$. Therefore either $\phi_1 = 0$ or $2|\phi_3| \leq |\phi_1|$. Similarly, either $\phi_3 = 0$ or $|\phi_1| \leq 2|\phi_3|$. That is to say, either $\phi_1 = 0$, $\phi_3 = 0$ or $|\phi_1| = 2|\phi_3|$. But the equation in $\alpha_i^{(iv)}$ corresponding to (5.1) is soluble for λ_i in $R(G, C)$. Hence, putting $e = 1$, we have, for some integral values of μ_1, μ_2, μ_3 ,

$$\mu_1\phi_1 + 2\mu_2\phi_2 + 4\mu_3\phi_3 = 1.$$

Hence $\phi_1 \neq 0$, $|\phi_1| \neq 2|\phi_3|$. Therefore $\phi_3 = 0$. Similarly, $\phi_2 = 0$, so that

$$\alpha_1^{(iv)} = \alpha_1^{(iv)} + \beta_1^{(iv)}e + \gamma_1^{(iv)}e^2 + \delta_1^{(iv)}e^3,$$

$$\alpha_2^{(iv)} = \alpha_2^{(iv)}(1-e)(1+e^2), \quad \alpha_3^{(iv)} = \dots = \alpha_n^{(iv)} = 0.$$

Treating similarly the functions $\psi_1 = \alpha_1^{(iv)} - \beta_1^{(iv)} + \gamma_1^{(iv)} - \delta_1^{(iv)}$, $\psi_2 = \alpha_2^{(iv)}$, we can reduce this to the form $a^{(v)}$, 0, ..., 0. The theorem now follows, since all the units in $R(G, C)$ are trivial.

THEOREM 15. *The matrix condition of § 1 holds for the free cyclic group.*

Let $\|a_{ij}\|$ be a matrix whose elements are in $R(G, C)$, where G is the free cyclic group generated by x . Obviously we can transform $\|a_{ij}\|$ by means of transformations (1.2) so that it contains only positive powers of x . Moreover we can transform it so that it contains no power of x higher than the first. For let x^n , $n > 1$, be the highest power of x in $\|a_{ij}\|$, and

let $a_{ij} = p_{ij}x^n + q_{ij}$, where q_{ij} contains no power higher than the $(n-1)$ -th, and p_{ij} is an integer. Then we have

$$\begin{aligned} \|p_{ij}x^n + q_{ij}\| &\rightarrow \left\| \begin{array}{cc} p_{ij}x^n + q_{ij} & 0 \\ 0 & \delta_{ij} \end{array} \right\| \rightarrow \left\| \begin{array}{cc} p_{ij}x^n + q_{ij} & \delta_{ij}x \\ 0 & \delta_{ij} \end{array} \right\| \\ &\rightarrow \left\| \begin{array}{cc} q_{ij} & \delta_{ij}x \\ -p_{ij}x^{n-1} & \delta_{ij} \end{array} \right\|, \end{aligned}$$

and the assertion follows by induction on n .

Suppose, therefore, that $a_{ij} = b_{ij} + c_{ij}x$, where b_{ij} and c_{ij} are integers. Since the ordinary elementary transformations on integer matrices[†] are a particular case of transformations (1.2), we may suppose either that $\|b_{ij}\|$ or that $\|c_{ij}\|$ has diagonal form. But if $\|a_{ij}\|$ has an inverse, the determinant $|a_{ij}|$ has the value $\pm x^p$, so that at least one of $|b_{ij}|$, $|c_{ij}|$ is zero. If $|b_{ij}|$ is zero suppose that $\|b_{ij}\|$ is reduced to diagonal form; otherwise suppose that $\|c_{ij}\|$ is reduced to diagonal form. Then some column of $\|a_{ij}\|$, which we may take to be the last, consists either entirely of integers or entirely of integer multiples of x . In the latter case multiply it by x^{-1} . By a further manipulation of rows, this column can be made to take the form

$$(0, 0, \dots, 0, \lambda),$$

where, since $\|a_{ij}\|$ has an inverse, $\lambda = \pm 1$. Hence the order of the matrix can be reduced without destroying its linearity. By an induction on order the theorem follows.

Appendix.

The purpose of this appendix is to show that, if two groups are indicable throughout, so are their direct product and their free product.

LEMMA. *If a self-conjugate subgroup H of a group G and the corresponding factor group G/H are both indicable throughout, then G is indicable throughout.*

Let K be a subgroup of G not consisting of the identity alone. If K is contained in H , it can be indexed, since H is indicable throughout. If,

[†] See, for instance, those given by M. Bôcher, *Introduction to higher algebra* (New York, 1907), 268, ex. 2. The interchange of two rows, or columns, can be brought about thus: $(a, b) \rightarrow (a, a+b) \rightarrow (-b, a+b) \rightarrow (-b, a) \rightarrow (b, a)$, so that this type of transformation is superfluous.

however, K is not contained in H , let D be the intersection of K and H . Then D is a self-conjugate subgroup of K , and the factor group K/D is isomorphic to a subgroup of G/H not consisting of the identity alone†. It can, therefore, be indexed. We can, therefore, index K by giving each of its elements the degree of the element of K/D containing it.

THEOREM 1. *If two groups are indicable throughout, their direct product is indicable throughout.*

For the direct product $G \times H$ of two groups G and H , contains a self-conjugate subgroup isomorphic to G , whose factor group is isomorphic to H .

THEOREM 2. *If two groups are indicable throughout, their free product is indicable throughout.*

An element in the free product $G \circ H$ of two groups G and H can be written in the form

$$(1) \quad \gamma = g_1 h_1 g_2 h_2 \dots g_r h_r \quad (r \geq 1, \quad g_i \in G, \quad h_i \in H, \quad i = 1, 2, \dots, r).$$

Consider the self-conjugate subgroup U consisting of those elements for which

$$(2) \quad g_1 g_2 \dots g_r = 1; \quad h_1 h_2 \dots h_r = 1.$$

The factor group $(G \circ H)/U$ is isomorphic to the direct product $G \times H$ of G and H . Theorem 2 therefore follows from the lemma when we have proved the following theorem.

THEOREM 3. *The subgroup U of the free product $G \circ H$ of two groups G and H , defined as above, is a free group freely generated by the commutators $(g, h) = ghg^{-1}h^{-1}$, where g, h cover independently all elements other than the identity in G, H respectively.*

We have, identically, as can easily be shown by induction on r ,

$$g_1 h_1 g_2 h_2 \dots g_r h_r$$

$$= \left\{ \prod_{i=1}^{r-1} (g_1 g_2 \dots g_i, h_1 h_2 \dots h_i) (g_1 g_2 \dots g_{i+1}, h_1 h_2 \dots h_i)^{-1} \right\} g_1 g_2 \dots g_r h_1 h_2 \dots h_r.$$

Omitting on the right any commutators in which either term is 1, and the factor $g_1 g_2 \dots g_r h_1 h_2 \dots h_r$, we obtain an expression for γ , given by (1) and

† Speiser, *loc. cit.*, 36 (Theorem 25). Speiser's proof in no way depends on the fact that he is dealing with finite groups.

satisfying (2), in terms of the commutators (g, h) , $g \in G$, $h \in H$, $g \neq 1$, $h \neq 1$. These commutators do therefore generate U .

To prove that U is the free group in these generators, we must show that if the expression

$$(3) \quad \delta = (a_1, b_1)^{\eta_1} (a_2, b_2)^{\eta_2} \dots (a_q, b_q)^{\eta_q} \\ (a_i \in G, b_i \in H, a_i \neq 1, b_i \neq 1, \eta_i = \pm 1, i = 1, 2, \dots, q)$$

is equal to 1, then for some value of i

$$(4) \quad a_i = a_{i+1}, \quad b_i = b_{i+1}, \quad \eta_i + \eta_{i+1} = 0.$$

Now if an element δ of $G \circ H$ can be written as $x_1 x_2 \dots x_p$ ($p \geq 1$), where x_i is an element of G or H , $x_i \neq 1$, and no two consecutive elements x_i, x_{i+1} are in the same group G or H , then $\delta \neq 1$. It is sufficient therefore to show that if δ is given by (3) with (4) false for $i = 1, \dots, q-1$, then δ can be put in this form with $x_1 = a_1, x_2 = b_1$, or $x_1 = b_1, x_2 = a_1$, according as $\eta_1 = 1$ or $\eta_1 = -1$. This is certainly true for $q = 1$, and will be proved by induction on q . Since the rôles of G and H are identical there is no loss of generality in supposing that $\eta_1 = 1$. The element $(a_2, b_2)^{\eta_2} (a_3, b_3)^{\eta_3} \dots (a_q, b_q)^{\eta_q}$, by the hypothesis of the induction, can be written in the form $a_2 b_2 X$ or in the form $b_2 a_2 X$ according as $\eta_2 = 1$ or $\eta_2 = -1$. In the former case, $\delta = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 X$. In the latter, if $b_1 \neq b_2$, $\delta = a_1 b_1 a_1^{-1} b_1^{-1} b_2 a_2 X$, and if $b_1 = b_2$, $\delta = a_1 b_1 a_1^{-1} a_2 X$. We cannot also have $a_1 = a_2$, since (4) is false for $i = 1$. In all cases δ is reduced to the desired form.

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SUMMABILITY OF MULTIPLE SERIES

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[Received and read 23 March, 1939.]

1. Introduction and definition of notation*.

A double series $\sum_{k,l=1}^{\infty} c_{kl}$ may be classified according to the behaviour of the double sequence of its partial sums $s_{kl} = \sum_{i,j=1}^{k,l} c_{ij}$ as follows. The sequence $\{s_{kl}\}$ is *convergent* (abbreviated *c*) if $\lim_{k,l \rightarrow \infty} s_{kl}$ exists†; *boundedly convergent* (*bc*) if *c* and if s_{kl} is bounded for all k, l ; *ultimately regularly convergent* (*urc*) if *c*, and if there exists a number Q such that $\lim_{k \rightarrow \infty} s_{kl}$ and $\lim_{l \rightarrow \infty} s_{kl}$ both exist for all $l > Q$ and all $k > Q$ respectively; *regularly convergent* (*rc*) if in the preceding case Q can be taken to be zero; *boundedly ultimately regularly convergent* (*burc*) if both *b* and *urc*. We note that *rc* implies *burc*, which implies *bc* and *urc*. The series $\sum c_{kl}$ is transformed by an infinite matrix $(g_{kl}(a, b))$ into a double sequence $\{\gamma(a, b)\}$, where $g_{kl}(a, b)$ may be real or complex and $\gamma(a, b) = \sum_{k,l=1}^{\infty} g_{kl}(a, b) c_{kl}$; a, b are continuous real positive variables and, as a particular case, may be positive integers;

* For the most part the notation is that used by H. J. Hamilton, *Bull. American Math. Soc.*, 42 (1936), 275–283. This paper will be referred to as H. The present paper uses, as lemmas, only five of the large number of series-to-series transformations given in H. These five are, in the notation of H, $rc \rightarrow c$ (due to Hardy, used in Theorem 1), $bc \rightarrow c$ (due to Moore, used in Theorem 3), $c \rightarrow c$ (due to Kojima, used in Theorem 5), $urc \rightarrow c$ (required in Theorem 7), and $burc \rightarrow c$ (required in Theorem 9).

† k, l tend to ∞ simultaneously but independently. The expression “ $\lim \dots$ exists” is to be interpreted always as meaning that a *finite* limit exists.

the matrix may be row-finite (including semi-matrix) or infinite. We find conditions* which $g_{kl}(a, b)$ has to satisfy in order that, whenever $\sum_{k, l=1}^{\infty} c_{kl}$ converges to s in some sense, $\{\gamma(a, b)\}$ may be *convergent* to (i) a definite value bearing a known relation to s [Type 1], (ii) s [Type 2]. Also from the existing conditions we derive conditions for $\{\gamma(a, b)\}$ to be *bc*, *rc*, *urc*, and *burc*.

The process of transforming will be indicated by an arrow. Thus $NS\ rc \rightarrow c$ (Type 1) should be read as "necessary and sufficient conditions for the sequence $\{\gamma(a, b)\}$ to be *convergent* whenever the series $\sum c_{kl}$ is *regularly convergent*", while (Type 2) substituted for (Type 1) indicates that $\{\gamma(a, b)\}$ is *convergent* to s whenever $\sum c_{kl}$ is *regularly convergent* to s .

2. Convergence of $\{\gamma(a, b)\}$.

THEOREM 1. $NS\ rc \rightarrow c$ (Type 1) are†

$$(1.1) \quad \sum_{k, l=1}^{\infty} |\Delta_{11} g_{kl}(a, b)| \leq M \quad \text{independently of } a > a', b > b';$$

$$(1.21) \quad \sum_{k=1}^{\infty} |\Delta_{10} g_{k1}(a, b)| \leq M' \quad \text{independently of } a > a', b > b';$$

$$(1.22) \quad \sum_{l=1}^{\infty} |\Delta_{01} g_{1l}(a, b)| \leq M'' \quad \text{independently of } a > a', b > b';$$

$$(1.3) \quad \lim_{a, b \rightarrow \infty} g_{kl}(a, b) = \beta_{kl} \quad \text{for each fixed } k \text{ and } l.$$

Moreover

$$(1.4) \quad \lim_{a, b \rightarrow \infty} \gamma(a, b) = \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl})(s_{kl} - s_k'' - s_l' + s) \\ + \sum_{k=1}^{\infty} (\Delta_{10} \beta_{k1})(s_k'' - s) + \sum_{l=1}^{\infty} (\Delta_{01} \beta_{1l})(s_l' - s) + \beta_{11} s,$$

where

$$\lim_{k \rightarrow \infty} s_{kl} = s_l' \quad \text{and} \quad \lim_{l \rightarrow \infty} s_{kl} = s_k''.$$

* All transformations used in evaluating divergent simple series satisfy this test of consistency. See P. Dienes, *The Taylor series* (1931), 393-397.

† For brevity we write g_{kl} for $g_{kl}(a, b)$ and also

$$\Delta_{11} g_{kl} = g_{kl} - g_{k+1, l} - g_{k, l+1} + g_{k+1, l+1},$$

$$\Delta_{10} g_{kl} = g_{kl} - g_{k+1, l}; \quad \Delta_{01} g_{kl} = g_{kl} - g_{k, l+1}.$$

We assume the following lemma due to Hardy*:

Necessary and sufficient conditions for $\sum_{k,l} d_{kl} c_{kl}$ to be convergent whenever $\sum_{k,l} c_{kl}$ is regularly convergent are

$$(1) \quad \sum_{k,l=1}^{\infty} |\Delta_{11} d_{kl}| \leq N,$$

$$(2) \quad \sum_{k=1}^{\infty} |\Delta_{10} d_{k1}| \leq N',$$

$$(3) \quad \sum_{l=1}^{\infty} |\Delta_{01} d_{1l}| \leq N''.$$

These conditions are included in (1.1), (1.21), and (1.22) respectively.

We use the following decomposition of double sums.

$$\begin{aligned} \sigma_{pq} &= \sum_{k,l=1}^{p,q} d_{kl} c_{kl} \\ &= \sum_{k,l=1}^{p-1,q-1} (\Delta_{11} d_{kl})(s_{kl} - s_{kq} - s_{pl}) + \sum_{k=1}^{p-1} (\Delta_{10} d_{k1}) s_{kq} \\ &\quad + \sum_{l=1}^{q-1} (\Delta_{01} d_{1l}) s_{pl} + d_{pq} s_{pq} \\ &= \sum_{k,l=1}^{p-1,q-1} (\Delta_{11} d_{kl})(s_{kl} - s_{kq} - s_{pl} + s_{pq}) + \sum_{k=1}^{p-1} (\Delta_{10} d_{k1})(s_{kq} - s_{pq}) \\ &\quad + \sum_{l=1}^{q-1} (\Delta_{01} d_{1l})(s_{pl} - s_{pq}) + d_{11} s_{pq}. \end{aligned}$$

As p, q tend to ∞ , the first term of the right-hand side exists, by (1), since $s_{kl} - s_{kq} - s_{pl} + s_{pq}$ is bounded and $\lim_{p,q \rightarrow \infty} (s_{kl} - s_{kq} - s_{pl} + s_{pq})$ exists. The second and third terms similarly exist, by (2) and (3) respectively. The last term obviously exists. Thus the lemma shows that

$$\begin{aligned} \sigma \equiv \lim_{p,q \rightarrow \infty} \sigma_{pq} &= \sum_{k,l=1}^{\infty} (\Delta_{11} d_{kl})(s_{kl} - s_k'' - s_l' + s) \\ &\quad + \sum_{k=1}^{\infty} (\Delta_{10} d_{k1})(s_k'' - s) + \sum_{l=1}^{\infty} (\Delta_{01} d_{1l})(s_l' - s) + d_{11} s. \end{aligned}$$

* G. H. Hardy, *Proc. Cambridge Phil. Soc.*, 19 (1920), 86-95. The conditions are stated in the form given in H.

Accordingly for particular values of a, b we have

$$\begin{aligned}
 (4) \quad \gamma(a, b) &= \sum_{k, l=1}^{\infty} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s_k'' - s_l' + s) \\
 &\quad + \sum_{k=1}^{\infty} [\Delta_{10} g_{k1}(a, b)](s_k'' - s) + \sum_{l=1}^{\infty} [\Delta_{01} g_{1l}(a, b)](s_l' - s) \\
 &\quad + g_{11}(a, b)s.
 \end{aligned}$$

Proof of sufficiency. We have to show that $\lim_{a, b \rightarrow \infty} \gamma(a, b)$ exists. Let $H = \max(M, M', M'')$. Since Σc_{kl} is regularly convergent, given ϵ we can find m and n , depending only on ϵ , such that

$$(5) \quad |s_{kl} - s| < \epsilon/(9H) \quad (k > m, \quad l > n),$$

$$(6) \quad |s_{kl} - s_l'| < \epsilon/(9H) \quad \text{independently of } l; \quad (k > m),$$

$$(7) \quad |s_{kl} - s_k''| < \epsilon/(9H) \quad \text{independently of } k; \quad (l > n),$$

$$(8) \quad |s_k'' - s| < \epsilon/(9H) \quad (k > m),$$

$$(9) \quad |s_l' - s| < \epsilon/(9H) \quad (l > n).$$

Rewriting (4) we have

$$\begin{aligned}
 (10) \quad \gamma(a, b) &= \sum_{k, l=1}^{m, n} (\Delta_{11} g_{kl})(s_{kl} - s_k'' - s_l' + s) + \sum_{k=1}^m (\Delta_{10} g_{k1})(s_k'' - s) \\
 &\quad + \sum_{l=1}^n (\Delta_{01} g_{1l})(s_l' - s) + g_{11}s + \sum_{k=m+1, l=n+1}^{\infty} (\Delta_{11} g_{kl})(s_{kl} - s_k'' - s_l' + s) \\
 &\quad + \sum_{k=m+1, l=1}^{\infty, n} (\Delta_{11} g_{kl})(s_{kl} - s_k'' - s_l' + s) \\
 &\quad + \sum_{k=1, l=n+1}^{m, \infty} (\Delta_{11} g_{kl})(s_{kl} - s_k'' - s_l' + s) \\
 &\quad + \sum_{k=m+1}^{\infty} (\Delta_{10} g_{k1})(s_k'' - s) + \sum_{l=n+1}^{\infty} (\Delta_{01} g_{1l})(s_l' - s).
 \end{aligned}$$

We see from (1.3) that, as $a, b \rightarrow \infty$, the first term on the right tends to

$$\sum_{k, l=1}^{m, n} (\Delta_{11} \beta_{kl})(s_{kl} - s_k'' - s_l' + s),$$

the second to
$$\sum_{k=1}^m (\Delta_{10} \beta_{k1})(s_k'' - s),$$

the third to
$$\sum_{l=1}^n (\Delta_{01} \beta_{l1})(s_l' - s),$$

and the fourth to
$$\beta_{11} s.$$

By (1.1), for all $a > a'$, $b > b'$, we have

$$\left| \sum_{k=m+1, l=n+1}^{\infty} (\Delta_{11} g_{kl}) [(s_{kl} - s) - (s_k'' - s) - (s_l' - s)] \right| < \frac{1}{3} \epsilon$$

from (5), (8), and (9),

$$\left| \sum_{k=m+1, l=1}^{\infty, n} (\Delta_{11} g_{kl}) [(s_{kl} - s_l') - (s_k'' - s)] \right| < \frac{2}{9} \epsilon$$

from (6) and (8), and

$$\left| \sum_{k=1, l=n+1}^{m, \infty} (\Delta_{11} g_{kl}) [(s_{kl} - s_k'') - (s_l' - s)] \right| < \frac{2}{9} \epsilon$$

from (7) and (9).

By (8) and (1.21),

$$\left| \sum_{k=m+1}^{\infty} (\Delta_{10} g_{k1})(s_k'' - s) \right| < \frac{1}{9} \epsilon \quad \text{for all } a > a', b > b'.$$

By (9) and (1.22),

$$\left| \sum_{l=n+1}^{\infty} (\Delta_{01} g_{l1})(s_l' - s) \right| < \frac{1}{9} \epsilon \quad \text{for all } a > a', b > b'.$$

Accordingly from (10) we have

$$\begin{aligned} \lim_{a, b \rightarrow \infty} \gamma(a, b) &= \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl})(s_{kl} - s_k'' - s_l' + s) \\ &\quad + \sum_{k=1}^{\infty} (\Delta_{10} \beta_{k1})(s_k'' - s) + \sum_{l=1}^{\infty} (\Delta_{01} \beta_{l1})(s_l' - s) + \beta_{11} s. \end{aligned}$$

Proof of necessity.

We have $\gamma(a, b) = \sum_{k, l=1}^{\infty} g_{kl}(a, b) c_{kl}$. Suppose that $\gamma(a, b)$ tends to a limit when Σc_{kl} is regularly convergent. Then, taking $c_{k_0 l_0} = 1$ and $c_{kl} = 0$ ($k \neq k_0, l \neq l_0$), we have $\gamma(a, b) = g_{k_0 l_0}(a, b)$.

(1.3) is therefore necessary.

In order to prove the necessity of (1.1), (1.21), and (1.22) we shall use the equality (4) which has been obtained solely by means of the conditions of Hardy's lemma, the necessity of which we assume.

We prove the necessity of (1.1) by choosing a real double sequence for which $s'_l = s''_k = s = 0$. (4) is reduced to

$$(11) \quad \gamma(a, b) = \sum_{k, l=1}^{\infty} [\Delta_{11} g_{kl}(a, b)] s_{kl}.$$

$$(12) \quad \sum_{k, l=1}^{\infty} |\Delta_{11} g_{kl}(a, b)| = S(a, b)$$

is convergent for fixed a, b , by (1).

If (1.1) is not satisfied, $\lim_{a, b \rightarrow \infty} S(a, b) = \infty$; and, if we put $g_{kl}(a, b) = g'_{kl}(a, b) + i g''_{kl}(a, b)$, there is a pair of sequences $\{a_m\}, \{b_n\}$ for which one or both of $\sum_{k, l} |\Delta_{11} g'_{kl}(a_m, b_n)|$ and $\sum_{k, l} |\Delta_{11} g''_{kl}(a_m, b_n)|$ tends to ∞ .

Put $\Delta_{11} g'_{kl}(a_m, b_n) \equiv \mathfrak{L}_{mnkl}$ and suppose that

$$(13) \quad \sum_{k, l=1}^{\infty} |\mathfrak{L}_{mnkl}| \equiv G_{mn} \rightarrow \infty.$$

We now construct the sequence s_{kl} , already chosen, for which $|s_{kl}| \leq 1$, such that $T_{m, n} \rightarrow \infty$ for a certain sequence of values of m and n , where, from (11),

$$\Re\{\gamma(a_m, b_n)\} = \sum_{k, l=1}^{\infty} \mathfrak{L}_{mnkl} s_{kl} \equiv T_{mn}.$$

Let the maximum value of $\sum_{k, l=1}^{p, q} |\mathfrak{L}_{mnkl}|$ be denoted by C_{pq} , which is finite, by (1.3) [already established], for fixed p, q .

By (13), for conveniently chosen values m_1, n_1, p_1', q_1' of m, n, p, q respectively, we have

$$\sum_{k, l=1}^{p_1', q_1'} |\mathfrak{L}_{m_1 n_1 kl}| > \mathfrak{L}^2,$$

where $\mathfrak{L} > 1$ and is fixed. When m_1, n_1 are already chosen, we have, by (12), $p_1 > p_1', q_1 > q_1'$ such that

$$\sum_{k=p_1+1, l=q_1+1}^{\infty} |\mathfrak{L}_{m_1 n_1 kl}| < \frac{1}{3}\epsilon,$$

$$\sum_{k=p_1+1, l=1}^{\infty, q_1} |\mathfrak{L}_{m_1 n_1 kl}| < \frac{1}{3}\epsilon,$$

and

$$\sum_{k=1, l=q_1+1}^{p_1, \infty} |\mathfrak{L}_{m_1 n_1 kl}| < \frac{1}{3}\epsilon.$$

Let
$$s_{kl} = \frac{1}{\mathfrak{L}} \operatorname{sgn}(\mathfrak{L}_{m_1 n_1 kl}) \quad \begin{cases} 1 \leq k \leq p_1 \\ 1 \leq l \leq q_1 \end{cases}.$$

We have

$$\begin{aligned} T_{m_1 n_1} &= \sum_{k, l=1}^{\infty} \mathfrak{L}_{m_1 n_1 kl} s_{kl} \\ &= \frac{1}{\mathfrak{L}} \sum_{k, l=1}^{p_1, q_1} |\mathfrak{L}_{m_1 n_1 kl}| + \sum_{k=p_1+1, l=1}^{\infty, q_1} \mathfrak{L}_{m_1 n_1 kl} s_{kl} \\ &\quad + \sum_{k=1, l=q_1+1}^{p_1, \infty} \mathfrak{L}_{m_1 n_1 kl} s_{kl} + \sum_{k=p_1+1, l=q_1+1}^{\infty} \mathfrak{L}_{m_1 n_1 kl} s_{kl}. \end{aligned}$$

Therefore

$$(14) \quad |T_{m_1 n_1}| > \mathfrak{L} - \epsilon.$$

We can now determine $m_2 > m_1$, $n_2 > n_1$, $p_2' > p_1$, $q_2' > q_1$, so that

$$\begin{aligned} (15) \quad \frac{1}{\mathfrak{L}^2} \sum_{k=p_1+1, l=q_1+1}^{p_2', q_2'} |\mathfrak{L}_{m_2 n_2 kl}| + \frac{1}{\mathfrak{L}^2} \sum_{k=1, l=q_1+1}^{p_1, q_2'} |\mathfrak{L}_{m_2 n_2 kl}| \\ + \frac{1}{\mathfrak{L}^2} \sum_{k=p_1+1, l=1}^{p_2', q_1} |\mathfrak{L}_{m_2 n_2 kl}| > \frac{1}{\mathfrak{L}} C_{p_1 q_1} + \mathfrak{L}^2. \end{aligned}$$

When m_2 , n_2 are thus fixed, we determine $p_2 \geq p_2'$, $q_2 \geq q_2'$ so that

$$(16) \quad \sum_{k=p_2+1, l=q_2+1}^{\infty} |\mathfrak{L}_{m_2 n_2 kl}| < \frac{1}{3}\epsilon,$$

$$(17) \quad \sum_{k=p_2+1, l=1}^{\infty, q_2} |\mathfrak{L}_{m_2 n_2 kl}| < \frac{1}{3}\epsilon,$$

and

$$(18) \quad \sum_{k=1, l=q_2+1}^{p_2, \infty} |\mathfrak{L}_{m_2 n_2 kl}| < \frac{1}{3}\epsilon.$$

Now choose

$$\begin{aligned} s_{kl} &= \frac{1}{\mathfrak{L}^2} \operatorname{sgn}(\mathfrak{L}_{m_2 n_2 kl}) \quad \text{for} \quad \begin{cases} p_1 < k \leq p_2 \\ q_1 < l \leq q_2 \end{cases} \\ &\quad \begin{cases} 1 \leq k \leq p_1 \\ q_1 < l \leq q_2 \end{cases} \quad \text{and} \quad \begin{cases} p_1 < k \leq p_2 \\ 1 \leq l \leq q_1 \end{cases}. \end{aligned}$$

We have

$$\begin{aligned} T_{m_2 n_2} &= \frac{1}{\mathfrak{L}} \sum_{k, l=1}^{p_1, q_1} \mathfrak{L}_{m_2 n_2 kl} \operatorname{sgn}(\mathfrak{L}_{m_1 n_1 kl}) + \frac{1}{\mathfrak{L}^2} \sum_{k=p_1+1, l=q_1+1}^{p_2, q_2} |\mathfrak{L}_{m_2 n_2 kl}| \\ &\quad + \frac{1}{\mathfrak{L}^2} \sum_{k=1, l=q_1+1}^{p_1, q_2} |\mathfrak{L}_{m_2 n_2 kl}| + \frac{1}{\mathfrak{L}^2} \sum_{k=p_1+1, l=1}^{p_2, q_1} |\mathfrak{L}_{m_2 n_2 kl}| \\ &\quad + \sum_{k=p_2+1, l=1}^{\infty, q_2} \mathfrak{L}_{m_2 n_2 kl} s_{kl} + \sum_{k=1, l=q_2+1}^{p_2, \infty} \mathfrak{L}_{m_2 n_2 kl} s_{kl} \\ &\quad + \sum_{k=p_2+1, l=q_2+1}^{\infty} \mathfrak{L}_{m_2 n_2 kl} s_{kl}. \end{aligned}$$

By (15), (16), (17), and (18), we have

$$|T_{m_2 n_2}| > \varepsilon^2 - \epsilon.$$

Continuing in this way, we determine the sequence $\{s_{kl}\}$ so that, for a certain sequence of values of m and n , $T_{mn} \rightarrow \infty$.

Therefore (1.1) is necessary.

In order to prove the necessity of (1.21) we form a regularly convergent sequence s_{kl} such that $s'_l = s = 0$.

(4) is reduced to

$$\gamma(a, b) = \sum_{k, l=1}^{\infty} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s_k'') + \sum_{k=1}^{\infty} [\Delta_{10} g_{k1}(a, b)]s_k''.$$

As $a, b \rightarrow \infty$, the limit of the first term on the right-hand side exists by (1.1) and (1.3), both of which have already been established.

Therefore $\lim_{a, b \rightarrow \infty} \sum_{k=1}^{\infty} [\Delta_{10} g_{k1}(a, b)]s_k''$ must exist. The necessity of (1.21) can now be proved in a manner similar to that used for (1.1).

Similarly we prove the necessity of (1.22).

THEOREM 2. *NSrc $\rightarrow$ c (Type 2) are (1.1), (1.21), (1.22), and*

$$(2.3) \quad \lim_{a, b \rightarrow \infty} g_{kl}(a, b) = 1$$

for each fixed k and l ; i.e. $\beta_{kl} = 1$.

Proof. By Theorem 1, the conditions are sufficient and the first three are also necessary.

If $\beta_{kl} \neq 1$ for a particular k_0 and l_0 , consider the series for which

$$\begin{aligned} c_{k_0 l_0} &= 1, \\ c_{kl} &= 0 \quad (k \neq k_0, \quad l \neq l_0). \end{aligned}$$

Therefore

$$\begin{aligned} 1 &= \lim_{a, b \rightarrow \infty} \gamma(a, b) = \lim_{a, b \rightarrow \infty} \sum_{k, l=1}^{\infty} g_{kl}(a, b) c_{kl} \\ &= \lim_{a, b \rightarrow \infty} g_{k_0 l_0}(a, b), \end{aligned}$$

which contradicts $\beta_{k_0 l_0} \neq 1$. Thus

$$\lim_{a, b \rightarrow \infty} g_{kl}(a, b) = 1$$

for each fixed k and l , which proves the necessity of (2.3).

THEOREM 3. $NS\ bc \rightarrow c$ (Type 1) are (1.1), (1.3), and

$$(3.21) \quad \lim_{l \rightarrow \infty} \Delta_{10} g_{kl}(a, b) = 0 \\ (k = 1, 2, \dots; \text{ every particular } a, b),$$

$$(3.22) \quad \lim_{k \rightarrow \infty} \Delta_{01} g_{kl}(a, b) = 0 \\ (l = 1, 2, \dots; \text{ every particular } a, b),$$

$$(3.41) \quad \lim_{a, b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \beta_{kl}]| = 0 \quad (l = 1, 2, \dots),$$

$$(3.42) \quad \lim_{a, b \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \beta_{kl}]| = 0 \quad (k = 1, 2, \dots).$$

Moreover

$$(3.5) \quad \lim_{a, b \rightarrow \infty} \gamma(a, b) = \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl})(s_{kl} - s) + \beta_{11} s.$$

We assume the following lemma due to C. N. Moore*:

Necessary and sufficient conditions for $\sum_{k, l} d_{kl} c_{kl}$ to be convergent whenever $\sum_{k, l} c_{kl}$ is boundedly convergent are

$$(19) \quad \sum_{k, l=1}^{\infty} |\Delta_{11} d_{kl}| \leq N,$$

$$(20) \quad \lim_{l \rightarrow \infty} \Delta_{10} d_{kl} = 0 \quad (k = 1, 2, \dots),$$

$$(21) \quad \lim_{k \rightarrow \infty} \Delta_{01} d_{kl} = 0 \quad (l = 1, 2, \dots).$$

These conditions are included in (1.1), (3.21), and (3.22) respectively.

We use the following decomposition of double sums, due to Hardy†.

$$\begin{aligned} \sigma_{p, q} &= \sum_{k, l=1}^{p, q} d_{kl} c_{kl} \\ &= \sum_{k, l=1}^{p-1, q-1} s_{kl} \Delta_{11} d_{kl} + \sum_{k=1}^{p-1} s_{kq} \Delta_{10} d_{kq} + \sum_{l=1}^{q-1} s_{pl} \Delta_{01} d_{pl} + s_{pq} d_{pq} \\ &= \sum_{k, l=1}^{p-1, q-1} (\Delta_{11} d_{kl})(s_{kl} - s_{pq}) + \sum_{k=1}^{p-1} (\Delta_{10} d_{kq})(s_{kq} - s_{pq}) \\ &\quad + \sum_{l=1}^{q-1} (\Delta_{01} d_{pl})(s_{pl} - s_{pq}) + d_{11} s_{pq}. \end{aligned}$$

* C. N. Moore, *Trans. American Math. Soc.*, 29 (1927), 227-238. (See Theorem 1.) The conditions are stated in the form given in H.

† Hardy, *Proc. London Math. Soc.* (2), 1 (1903), 124-128.

As p, q tend to ∞ , the first term of the right-hand side exists by (19), since $s_{kl} - s_{pq}$ is bounded and $\lim_{p, q \rightarrow \infty} (s_{kl} - s_{pq})$ exists. The second term tends to zero, by (19) and (20); and the third tends to zero, by (19) and (21)*. The last term obviously exists. Thus Moore's lemma shows that

$$\sigma \equiv \lim_{p, q \rightarrow \infty} \sigma_{p, q} = \sum_{k, l=1}^{\infty} (\Delta_{11} d_{kl})(s_{kl} - s) + d_{11} s.$$

Accordingly, for particular values of a and b , we have

$$(22) \quad \gamma(a, b) = \sum_{k, l=1}^{\infty} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s) + g_{11}(a, b) s.$$

Proof of sufficiency. We have to show that $\lim_{a, b \rightarrow \infty} \gamma(a, b)$ exists.

Since $\lim_{k, l \rightarrow \infty} s_{kl}$ exists, we can choose m and n so that

$$(23) \quad |s_{kl} - s| < \epsilon/M \quad (k > m, \quad l > n).$$

Rewriting (22), we have

$$\begin{aligned} \gamma(a, b) = & \sum_{k, l=1}^{m, n} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s) + g_{11}(a, b) s + \sum_{k=m+1, l=1}^{\infty, n} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s) \\ & + \sum_{k=1, l=n+1}^{m, \infty} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s) + \sum_{k=m+1, l=n+1}^{\infty} [\Delta_{11} g_{kl}(a, b)](s_{kl} - s). \end{aligned}$$

As $a, b \rightarrow \infty$, the first term of the right-hand side tends to

$$\sum_{k, l=1}^{m, n} (\Delta_{11} \beta_{kl})(s_{kl} - s)$$

and the second to $\beta_{11} s$, by (1.3). The third term tends to a finite limit, for, if $\max |s_{kl} - s| = D$, we have†

$$\begin{aligned} & \left| \sum_{k, l=1}^{\infty, n} [\Delta_{11} \{g_{kl}(a, b) - \beta_{kl}\}](s_{kl} - s) \right| \\ & \leq D \sum_{k, l=1}^{\infty, n} |\Delta_{11} [g_{kl}(a, b) - \beta_{kl}]| \\ & = D \sum_{l=1}^n \left[\sum_{k=1}^{\infty} |\Delta_{11} \{g_{kl}(a, b) - \beta_{kl}\}| \right] \\ & \rightarrow 0 \quad \text{by (3.41).} \end{aligned}$$

Similarly the fourth term exists.

* See (11) of H.

† The second step uses the result that absolute convergence is independent of the order of summation.

Finally, by (23) and (1.1),

$$\left| \sum_{k=m+1, l=n+1}^{\infty} [\Delta_{11} g_{kl}(a, b)](s_{kl}-s) \right| < \epsilon \quad \text{for all } a > a', b > b'.$$

Accordingly we have

$$\lim_{a, b \rightarrow \infty} \gamma(a, b) = \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl})(s_{kl}-s) + \beta_{11}s.$$

Proof of necessity. Since *rc* implies *bc*, (1.1) and (1.3) are necessary* by Theorem 1. (3.21) and (3.22) are the same as (20) and (21), and are, therefore, necessary. We now prove the necessity of (3.41). If it is not satisfied,

$$\lim_{a, b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \beta_{kl}]| \neq 0$$

for a certain l , say l_0 , and, if we put

$$g_{kl_0}(a, b) = g'_{kl_0}(a, b) + i g''_{kl_0}(a, b), \quad \beta_{kl_0} = \beta'_{kl_0} + i \beta''_{kl_0},$$

there is a pair of sequences $\{a_m\}$ and $\{b_n\}$ for which one or both of

$$\sum_{k=1}^{\infty} |\Delta_{11}[g'_{kl_0}(a_m, b_n) - \beta'_{kl_0}]|$$

and

$$\sum_{k=1}^{\infty} |\Delta_{11}[g''_{kl_0}(a_m, b_n) - \beta''_{kl_0}]|$$

exceeds 7δ ($\neq 0$).

Put

$$\mathfrak{L}'_{mnkl_0} = \Delta_{11}[g'_{kl_0}(a_m, b_n) - \beta'_{kl_0}]$$

and suppose that

$$(24) \quad \sum_{k=1}^{\infty} |\mathfrak{L}'_{mnkl_0}| > 7\delta.$$

We assume (1.1) and (1.3). Then

$$(25) \quad \sum_{k=1}^{\infty} |\mathfrak{L}'_{mnkl_0}|$$

is convergent for fixed m, n by (1.1).

* If $u \subset u'$, $v \subset v'$, where u, u' and v, v' represent classes of series and sequence respectively, then $N.u \rightarrow v'$ (Type 1) is at once $N.(u, u') \rightarrow (v, v')$ (Type 1) [4 cases], and $S.u' \rightarrow v$ (Type 1) is at once $S.(u, u') \rightarrow (v, v')$ (Type 1).

We now construct a real sequence s_{kl} for which $s_{kl} = 0$ ($l \neq l_0$) and $|s_{kl_0}| \leq 1$.

(22) is thus reduced to

$$\begin{aligned}\gamma(a, b) &= \sum_{k=1}^{\infty} [\Delta_{11} g_{kl_0}(a, b)] s_{kl_0} \\ &= \sum_{k=1}^{\infty} (\Delta_{11} \beta_{kl_0}) s_{kl_0} + \sum_{k=1}^{\infty} [\Delta_{11} \{g_{kl_0}(a, b) - \beta_{kl_0}\}] s_{kl_0}.\end{aligned}$$

The first term on the right-hand side exists, by (1.1). As $a, b \rightarrow \infty$, the second term must therefore tend to a finite value.

Therefore, if

$$T'_{m,n} \equiv \sum_{k=1}^{\infty} (\mathfrak{L}'_{mnkl_0}) s_{kl_0},$$

(26)

$$\lim_{m, n \rightarrow \infty} T'_{m,n}$$

exists. Choose any values m_1, n_1 of m, n ; and then choose p_1 so that

$$\sum_{k=p_1+1}^{\infty} |\mathfrak{L}'_{m_1 n_1 k l_0}| < \delta, \quad \text{by (25).}$$

Put

$$s_{kl_0} = \operatorname{sgn} (\mathfrak{L}_{m_1 n_1 k l_0}) \quad (1 \leq k \leq p_1).$$

Then

$$\begin{aligned}|T'_{m_1 n_1}| &\geq \sum_{k=1}^{\infty} |\mathfrak{L}'_{m_1 n_1 k l_0}| - 2 \sum_{k=p_1+1}^{\infty} |\mathfrak{L}'_{m_1 n_1 k l_0}| \\ &> 5\delta, \quad \text{by (24).}\end{aligned}$$

Next choose $m_2 > m_1, n_2 > n_1$ so that

$$\sum_{k=1}^{p_1} |\mathfrak{L}'_{m_2 n_2 k l_0}| < \delta, \quad \text{by (1.3).}$$

Then choose, as before, $p_2 > p_1$ such that

$$\sum_{k=p_2+1}^{\infty} |\mathfrak{L}'_{m_2 n_2 k l_0}| < \delta.$$

Now put

$$s_{kl_0} = 0, \quad (p_1 < k \leq p_2).$$

Then

$$|T'_{m_2 n_2}| < 2\delta.$$

Choose $m_3 > m_2, n_3 > n_2$ such that

$$\sum_{k=1}^{p_2} |\mathfrak{L}'_{m_3 n_3 k l_0}| < \delta,$$

and a $p_3 > p_2$ such that

$$\sum_{k=p_3+1}^{\infty} |\mathfrak{L}'_{m_3 n_3 k l_0}| < \delta.$$

Put $s_{kl_0} = \text{sgn} (\mathfrak{L}'_{m_3 n_3 k l_0}), \quad (p_2 < k \leq p_3).$

Then $|T'_{m_3 n_3}| > 3\delta.$

Continuing in this way, we find that, for odd r ,

$$|T'_{m_r n_r}| > 3\delta,$$

and, for even r , $|T'_{m_r n_r}| < 2\delta.$

This contradiction of (26) establishes the necessity of (3.41). Similarly we prove the necessity of (3.42).

THEOREM 4. $NS\ bc \rightarrow c$ (Type 2) are (1.1), (3.21), (3.22), (2.3), and

$$(4.41) \quad \lim_{a, b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11} g_{kl}(a, b)| = 0 \quad (l = 1, 2, \dots),$$

$$(4.42) \quad \lim_{a, b \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11} g_{kl}(a, b)| = 0 \quad (k = 1, 2, \dots).$$

For real transformations this theorem, in another form, is due to C. N. Moore*.

The conditions are easily seen to be sufficient by Theorem 3. (2.3) is necessary by Theorem 2 and the remaining conditions by Theorem 3.

THEOREM 5. $NS\ c \rightarrow c$ (Type 1) are (1.1), (1.3), and

$$(5.11) \quad \Delta_{11} g_{kl}(a, b) = 0$$

for all $a, b, k > K_l \quad (l = 1, 2, \dots),$

$$(5.12) \quad \Delta_{11} g_{kl}(a, b) = 0$$

for all $a, b, l > L_k \quad (k = 1, 2, \dots),$

$$(5.21) \quad \Delta_{10} g_{kl}(a, b) = 0$$

for all $l > E_k(a, b) \quad (k = 1, 2, \dots; \text{every particular } a, b),$

$$(5.22) \quad \Delta_{01} g_{kl}(a, b) = 0$$

for all $k > F_l(a, b) \quad (l = 1, 2, \dots; \text{every particular } a, b).$

Moreover

$$(5.4) \quad \lim_{a, b \rightarrow \infty} \gamma(a, b) = \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl})(s_{kl} - s) + \beta_{11} s.$$

* C. N. Moore, *loc. cit.* See Theorem 3 with $r = 1$.

We assume the following lemma due to Kojima*:

Necessary and sufficient conditions for $\sum_{k,l} d_{kl} c_{kl}$ to be convergent whenever $\sum_{k,l} c_{kl}$ is convergent are

$$(27) \quad \sum_{k,l=1}^{\infty} |\Delta_{11} d_{kl}| \leq N,$$

$$(28) \quad \Delta_{10} d_{kl} = 0$$

for all $l > L_k'$ ($k = 1, 2, \dots$);

$$(29) \quad \Delta_{01} d_{kl} = 0$$

for all $k > K_l'$ ($l = 1, 2, \dots$).

These conditions are included in (1.1), (5.21), and (5.22) respectively.

As in Theorem 3, we have

$$\begin{aligned} \sigma_{pq} = & \sum_{k,l=1}^{p-1, q-1} (\Delta_{11} d_{kl}) (s_{kl} - s_{pq}) + \sum_{k=1}^{p-1} (\Delta_{10} d_{kq}) (s_{kq} - s_{pq}) \\ & + \sum_{l=1}^{q-1} (\Delta_{01} d_{pl}) (s_{pl} - s_{pq}) + d_{11} s_{pq}. \end{aligned}$$

As p, q tend to ∞ , the first term of the right-hand side exists by (27), (28), and (29); the second tends to zero by (27) and (28); and the third tends to zero by (27) and (29)†. The last term obviously exists. Thus Kojima's lemma shows that

$$\sigma \equiv \lim_{p, q \rightarrow \infty} \sigma_{pq} = \sum_{k,l=1}^{\infty} (\Delta_{11} d_{kl}) (s_{kl} - s) + d_{11} s.$$

Accordingly, for particular values of a and b , we have

$$(30) \quad \gamma(a, b) = \sum_{k,l=1}^{\infty} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) + g_{11}(a, b) s.$$

Proof of sufficiency. We have to show that $\lim_{a, b \rightarrow \infty} \gamma(a, b)$ exists.

Since $\lim_{k, l \rightarrow \infty} s_{kl}$ exists, we can choose m and n so that

$$(31) \quad |s_{kl} - s| < \epsilon/M \quad (k > m, l > n).$$

* T. Kojima, *Tôhoku Math. J.*, 17 (1920), 213-220. The conditions are stated in the form given in H.

† See Theorem 14 of H.

Let $K = \max K_l \quad (l = 1, 2, \dots, n)$

and $L = \max L_k \quad (k = 1, 2, \dots, m).$

Rewriting (30), we have

$$\begin{aligned} \gamma(a, b) = & \sum_{k, l=1}^{m, n} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) + g_{11}(a, b) s \\ & + \sum_{k=m+1, l=1}^{\infty, n} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) + \sum_{k=1, l=n+1}^{m, \infty} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) \\ & + \sum_{k=m+1, l=n+1}^{\infty} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s). \end{aligned}$$

As $a, b \rightarrow \infty$, the first term of the right-hand side tends to

$$\sum_{k, l=1}^{m, n} (\Delta_{11} \beta_{kl}) (s_{kl} - s)$$

and the second to $\beta_{11} s$, by (1.3).

By (5.11),

$$\begin{aligned} & \sum_{k=m+1, l=1}^{\infty, n} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) \\ & = 0 \quad \text{if } K \leq m \\ & = \sum_{k=m+1, l=1}^{K, n} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) \quad \text{if } K > m \\ & \rightarrow \sum_{k=m+1, l=1}^{K, n} (\Delta_{11} \beta_{kl}) (s_{kl} - s), \quad \text{by (1.3).} \end{aligned}$$

If $L \leq n$, the fourth term is zero; otherwise it tends to that value, as above. By (31) and (1.1),

$$\left| \sum_{k=m+1, l=n+1}^{\infty} [\Delta_{11} g_{kl}(a, b)] (s_{kl} - s) \right| < \epsilon \quad \text{for all } a > a', b > b'.$$

Accordingly we have

$$\lim_{a, b \rightarrow \infty} \gamma(a, b) = \sum_{k, l=1}^{\infty} (\Delta_{11} \beta_{kl}) (s_{kl} - s) + \beta_{11} s.$$

Proof of necessity. Since rc implies c , (1.1) and (1.3) are necessary by Theorem 1. (5.21) and (5.22) are the same as (28) and (29), and are, therefore, necessary. We now prove the necessity of (5.11). Assume

(5.22). If (5.11) is not satisfied, we can choose three increasing sequences $\{a_p\}$, $\{b_p\}$, $\{k_p\}$, where $k_p > F_{l_0}(a_q, b_q)$ ($q < p$), each tending to infinity, such that

$$\Delta_{11} g_{k_p l_0}(a_p, b_p) \neq 0 \quad (p = 1, 2, \dots);$$

l_0 being a fixed value of l .

Select a double sequence $\{s_{kl}\}$ such that

$$s_{kl} = 0 \quad (l \neq l_0).$$

(30) is thus reduced to

$$\gamma(a, b) = \sum_{k=1}^{\infty} [\Delta_{11} g_{kl_0}(a, b)] s_{kl_0}.$$

Choose s_{kl_0} such that

$$s_{kl_0} = 0 \quad (k \neq k_p; p = 1, 2, \dots)$$

and

$$|[\Delta_{11} g_{k_p l_0}(a_p, b_p)] s_{k_p l_0}| > \left| \sum_{r=1}^{p-1} [\Delta_{11} g_{k_r l_0}(a_p, b_p)] s_{k_r l_0} \right| + p.$$

Now*

$$\gamma(a_p, b_p) = \sum_{r=1}^p [\Delta_{11} g_{k_r l_0}(a_p, b_p)] s_{k_r l_0}.$$

Therefore

$$|\gamma(a_p, b_p)| \geq |[\Delta_{11} g_{k_p l_0}(a_p, b_p)] s_{k_p l_0}| - \left| \sum_{r=1}^{p-1} [\Delta_{11} g_{k_r l_0}(a_p, b_p)] s_{k_r l_0} \right| > p.$$

$\{\gamma(a_p, b_p)\}$ thus tends to $\pm \infty$, and this is inconsistent with $\{\gamma(a, b)\}$ being convergent when $\sum_{k, l} c_{kl}$ is convergent. Similarly we prove the necessity of (5.12).

THEOREM 6. *NS $c \rightarrow c$ (Type 2) are (1.1), (5.11), (5.12), (5.21), (5.22), and (2.3).*

See Theorems 5 and 2.

THEOREM 7. *NS $urc \rightarrow c$ (Type 1) are (1.1), (5.11), (5.12), (5.21), (5.22), and (1.3).*

Since $urc \subset c$, the conditions are sufficient by Theorem 5. Since $rc \subset urc$, (1.1) and (1.3) are necessary by Theorem 1. (5.21) and (5.22) are necessary by H ($urc \rightarrow c$). The remaining two can be proved necessary as in Theorem 5.

* Since $k_p > F_{l_0}(a_q, b_q)$ ($q < p$), $\Delta_{11} g_{k_r l_0}(a_p, b_p) = 0$, $r > p$, by (5.22). $\Delta_{11} = \Delta_{10}(\Delta_{01})$.

THEOREM 8. $NS\ urc \rightarrow c$ (Type 2) are (1.1), (5.11), (5.12), (5.21), (5.22), and (2.3).

THEOREM 9. $NS\ burc \rightarrow c$ (Type 1) are (1.1), (3.21), (3.22), (1.3), (3.41), and (3.42).

The conditions are sufficient by Theorem 3. (1.1) and (1.3) are necessary by Theorem 1. (3.21) and (3.22) are necessary by H ($burc \rightarrow c$). The last two can be proved to be necessary as in Theorem 3.

THEOREM 10. $NS\ burc \rightarrow c$ (Type 2) are (1.1), (3.21), (3.22), (2.3), (4.41), and (4.42).

To each of the subsequent theorems there corresponds a theorem of Type 2 which is obtained by inserting the conditions (2.3), (4.41), and (4.42) in place of (1.3), (3.41), and (3.42) respectively.

3. Bounded convergence of $\{\gamma(a, b)\}$.

In the theorems of §2 $\{\gamma(a, b)\}$ was ultimately bounded, *i.e.*, bounded for all $a, b > Q'$, say. We state without proof the following theorems respectively corresponding to those in §2.

THEOREM 11. $NS\ rc \rightarrow bc$ (Type 1) are (1.3) and

$$(11.1) \quad \sum_{k,l=1}^{\infty} |\Delta_{11} g_{kl}(a, b)| \leq M;$$

$$(11.21) \quad \sum_{k=1}^{\infty} |\Delta_{10} g_{k1}(a, b)| \leq M';$$

$$(11.22) \quad \sum_{l=1}^{\infty} |\Delta_{01} g_{1l}(a, b)| \leq M'';$$

where M, M', M'' are independent of a and b .

THEOREM 13. $NS\ bc \rightarrow bc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (3.41), and (3.42).

THEOREM 15. $NS\ c \rightarrow bc$ (Type 1) are (11.1), (5.21), (5.22), (1.3), and

$$(15.11) \quad \Delta_{11} g_{kl}(a, b) = 0 \quad \text{for all } k > K_l, \quad l = 1, 2, \dots; \quad \text{all } a, b;$$

$$(15.12) \quad \Delta_{11} g_{kl}(a, b) = 0 \quad \text{for all } l > L_k, \quad k = 1, 2, \dots; \quad \text{all } a, b.$$

THEOREM 17. $NS\ urc \rightarrow bc$ (Type 1) are (11.1), (15.11), (15.12), (5.21), (5.22), and (1.3).

THEOREM 19. $NS\ burc \rightarrow bc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (3.41), and (3.42).

4. Regular convergence of $\{\gamma(a, b)\}$.

Since $\{\gamma(a, b)\}$ being rc implies that it is bc , when the existence of $\gamma(a, b)$ is assumed, we derive the following theorems from those in § 3.

THEOREM 21. $NS\ rc \rightarrow rc$ (Type 1) are (11.1), (11.21), (11.22), (1.3), and

$$(21.31) \quad \lim_{a \rightarrow \infty} g_{kl}(a, b) = \phi_{kl}(b), \quad \text{all } b; \text{ for each fixed } k, l;$$

$$(21.32) \quad \lim_{b \rightarrow \infty} g_{kl}(a, b) = \psi_{kl}(a), \quad \text{all } a; \text{ for each fixed } k, l.$$

THEOREM 23. $NS\ bc \rightarrow rc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (21.31), (21.32), (3.41), (3.42), and

$$(23.41) \quad \begin{cases} \lim_{a \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, & \text{all } b; l = 1, 2, \dots; \\ \lim_{b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, & \text{all } a; l = 1, 2, \dots; \end{cases}$$

$$(23.42) \quad \begin{cases} \lim_{a \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, & \text{all } b; k = 1, 2, \dots; \\ \lim_{b \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, & \text{all } a; k = 1, 2, \dots. \end{cases}$$

THEOREM 25. $NS\ c \rightarrow rc$ (Type 1) are (11.1), (15.11), (15.12), (5.21), (5.22), (1.3), (21.31), and (21.32).

THEOREM 27. $NS\ urc \rightarrow rc$ (Type 1) are (11.1), (15.11), (15.12), (5.21), (5.22), (1.3), (21.31), and (21.32).

THEOREM 29. $NS\ burc \rightarrow rc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (21.31), (21.32), (3.41), (3.42), (23.41), and (23.42).

5. *Ultimately regular convergence of $\{\gamma(a, b)\}$.*

We state without proof the following theorems which are mainly derived from those in §2.

THEOREM 31. *$NS\ rc \rightarrow urc$ (Type 1) are (1.1), (1.21), (1.22), (1.3), and*

$$(31.31) \quad \lim_{a \rightarrow \infty} g_{kl}(a, b) = \phi_{kl}(b), \text{ all } b > U, \text{ for each fixed } k, l;$$

$$(31.32) \quad \lim_{b \rightarrow \infty} g_{kl}(a, b) = \psi_{kl}(a), \text{ all } a > U, \text{ for each fixed } k, l.$$

THEOREM 33. *$NS\ bc \rightarrow urc$ (Type 1) are (1.1), (3.21), (3.22), (1.3), (31.31) (31.32), (3.41), (3.42), and*

$$(33.41) \quad \left\{ \begin{array}{l} \lim_{a \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, \quad \text{all } b > V; \quad l = 1, 2, \dots, \\ \lim_{b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, \quad \text{all } a > V; \quad l = 1, 2, \dots; \end{array} \right.$$

$$(33.42) \quad \left\{ \begin{array}{l} \lim_{a \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, \quad \text{all } b > W; \quad k = 1, 2, \dots, \\ \lim_{b \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, \quad \text{all } a > W; \quad k = 1, 2, \dots. \end{array} \right.$$

THEOREM 35. *$NS\ c \rightarrow urc$ (Type 1) are (1.1), (5.11), (5.12), (5.21), (5.22), (1.3), (31.31), (31.32), (33.41), and (33.42).*

THEOREM 37. *$NS\ urc \rightarrow urc$ (Type 1) are (1.1), (5.11), (5.12), (5.21) (5.22), (1.3), (31.31), and (31.32).*

THEOREM 39. *$NS\ bure \rightarrow urc$ (Type 1) are (1.1), (3.21), (3.22), (1.3), (31.31), (31.32), (3.41), (3.42), and*

$$(39.41) \quad \left\{ \begin{array}{l} \lim_{a \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, \quad \text{all } b > A_l, \quad l = 1, 2, \dots; \\ \lim_{b \rightarrow \infty} \sum_{k=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, \quad \text{all } a > A_l, \quad l = 1, 2, \dots; \end{array} \right.$$

$$(39.42) \quad \left\{ \begin{array}{l} \lim_{a \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \phi_{kl}(b)]| = 0, \quad \text{all } b > B_k, \quad k = 1, 2, \dots; \\ \lim_{b \rightarrow \infty} \sum_{l=1}^{\infty} |\Delta_{11}[g_{kl}(a, b) - \psi_{kl}(a)]| = 0, \quad \text{all } a > B_k, \quad k = 1, 2, \dots. \end{array} \right.$$

6. *Bounded ultimately regular convergence of $\{\gamma(a, b)\}$.*

The theorems of the present section follow from those of §§ 3 and 5.

THEOREM 41. *$NS\ rc \rightarrow burc$ (Type 1) are (11.1), (11.21), (11.22), (1.3), (31.31), and (31.32).*

THEOREM 43. *$NS\ bc \rightarrow burc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (31.31), (31.32), (3.41), (3.42), (33.41), and (33.42).*

THEOREM 45. *$NS\ c \rightarrow burc$ (Type 1) are (11.1), (15.11), (15.12), (5.21), (5.22), (1.3), (31.31), and (31.32).*

THEOREM 47. *$NS\ urc \rightarrow burc$ (Type 1) are (11.1), (15.11), (15.12), (5.21), (5.22), (1.3), (31.31), and (31.32).*

THEOREM 49. *$NS\ burc \rightarrow burc$ (Type 1) are (11.1), (3.21), (3.22), (1.3), (31.31), (31.32), (3.41), (3.42), (39.41), and (39.42).*

7. *Multiple series and generalisation.*

Theorems have been stated and proofs given for double series owing to their simplicity. All the theorems are also valid for multiple series of order greater than 2. C. N. Moore* has given the analogue of Theorem 4 for multiple series. Moreover we have made no use of the equality of the order of the series and that of the sequence into which it is transformed. Accordingly a series of the m -th order can be transformed into a sequence of the n -th order. Further, for the sake of simplicity, the existence of $\gamma(a, b)$ for every particular a, b has been assumed. $\gamma(a, b)$ need not exist in case of the theorems of § 4 when $a, b \leq P$, say, and in case of those of §§ 2 and 5 when $a, b \not\leq Q$, say. Accordingly those conditions in these theorems which hold good for every particular a, b need to be true only when $a, b \leq P$ in the theorems of § 4 and when $a, b > Q$ in those of §§ 2 and 5.

8. *Conclusion.*

Since the main purpose of transformation is the evaluation of divergent series, $\gamma_6^\dagger$ may be used, if it applies, to evaluate any divergent double

* C. N. Moore, *loc. cit.* See Theorem 4 with $r=1$.

† We write γ_6 to denote transformation(s) satisfying the conditions of Theorem 6; and corresponding meanings for other suffixes. It is clear that $\gamma_2 \supset \gamma_4 \supset \gamma_6$, $\gamma_8 \equiv \gamma_6$, $\gamma_{10} \equiv \gamma_4$. We write γ_{4-6} to denote those transformations of γ_4 which are not contained in γ_6 and assign a similar meaning to γ_{2-6} .

series whatsoever. We can, with justification, evaluate divergent double series whose partial sums are bounded by γ_4 , and those whose terms are positive by γ_2 . There is, however, not much justification for evaluating divergent double series, whose partial sums are not bounded, by γ_{4-6} and those whose terms are not positive by γ_{2-6} . It is easily seen that γ_2 includes the scalar product of 2γ -matrices*, i.e., $\gamma_2 \supset g_k'(a) \cdot g_l''(b)$. Thus if we multiply one of the well-known summation processes for simple series, by itself or by another we get a γ_2 .

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* A γ -matrix transforms a convergent simple series into a sequence tending to the sum of the series. See P. Dienes, *loc. cit.*, 397.

A SOLUTION OF THE CESÀRO SUMMABILITY PROBLEM FOR SUCCESSIVELY DERIVED FOURIER SERIES

By L. S. BOSANQUET.

[Received and read 23 March, 1939.]

1. *Introduction.* I have proved elsewhere* that a necessary and sufficient condition for the series obtained by differentiating the terms of the Fourier series of a function $f(t)$, integrable in the Lebesgue sense in $(-\pi, \pi)$, to be summable $(C, \alpha+1)$ for $t=x$ to the sum s , where $\alpha \geq 0$, is that the even function $\{f(x+t)-f(x-t)\}/(2 \sin \frac{1}{2}t)$ is integrable in the Cesàro-Lebesgue† sense in $(0, \pi)$ and its Fourier series‡ is summable (C, α) to s for $t=0$. In other words the Cesàro summability problem for the derived Fourier series may be reduced to the Cesàro summability problem for the Fourier series of a function integrable in a more general sense. Thus it is only necessary to prove summability theorems about Cesàro-Lebesgue Fourier series§, and results about derived Lebesgue-Fourier series may be deduced as corollaries.

* Bosanquet, 3, Theorem 1. This paper will be referred to as D.F.S.

The corresponding result, under the assumption that $\{f(x+t)-f(x-t)\}/(2 \sin \frac{1}{2}t)$ is integrable L , was given by Zygmund, 28. The example $f(t) = \sin(1/t)$, $x=0$, shows that this assumption is not a necessary condition. See D.F.S.

The sufficiency part of Zygmund's theorem is due to Rajchman, 17, and to W. H. Young, 27.

† See § 3 below.

‡ It is to be understood that the function is defined by periodicity outside $(-\pi, \pi)$.

§ For example, if $g(t)$ is even, and integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, a necessary and sufficient condition that the Fourier series of $g(t)$ should be summable (C) to s at $t=0$ is that $g(t) \rightarrow s(C)$ as $t \rightarrow +0$.

The sufficiency part is due to Burkill, 7 (see Lemma 10 below). A proof of the necessity part is inserted below (p. 212, footnote *) in answer to a question raised by the Referee. The case when $g(t)$ is integrable L is due to Hardy and Littlewood, 11.

There are more precise sufficient conditions, and necessary conditions, but it is not known, even in the case of Lebesgue-Fourier series, whether there is a non-trivial necessary and sufficient condition for summability by a particular Cesàro mean.

I have also stated in D.F.S. a similar result for the series obtained by differentiating a Fourier series r -times, where r is a positive integer. The object of the present paper is to give a proof of this theorem. The same method may be used to solve the analogous problem for the series obtained by differentiating the allied series r -times.

In what follows § 2 contains a summary of the special notation employed, and also the relevant properties of Young's functions. § 3 contains a definition of the Cesàro-Lebesgue integral, and also a proof of the fact that it is included in the Cesàro-Perron integral (Theorem 1). § 4 contains some lemmas which are quoted in the main arguments. § 5 contains the main theorem (Theorem 2), the proof of which is divided into a number of lemmas. § 6 contains an alternative form of the main theorem (Theorem 3), and a statement of the analogous result for successively derived allied series (Theorem 4).

The reader may, if he wishes to do so, proceed straight to § 5, after reading the definitions in §§ 2 and 3.

2. *Notation.* We suppose throughout the paper that $f(t)$ is integrable in the Lebesgue sense in the interval $(-\pi, \pi)$, and is periodic with period 2π . We denote the Fourier series of $f(t)$ by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t). \quad (2.1)$$

Then the allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t). \quad (2.2)$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} \quad \text{and} \quad \psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \quad (2.3)$$

$$\text{so that} \quad \phi(t) \sim \sum_{n=0}^{\infty} A_n \cos nt, \quad (2.4)$$

where $A_n = A_n(x)$, and

$$\psi(t) \sim \sum_{n=1}^{\infty} B_n \sin nt, \quad (2.5)$$

where $B_n = B_n(x)$.

The series obtained by differentiating the Fourier series of $f(t)$ r -times at $t = x$ is*

$$\sum_{n=0}^{\infty} \left(\frac{d}{dx}\right)^r A_n(x) \begin{cases} = \Sigma(-1)^{\frac{1}{2}r} n^r A_n & (r \text{ even}), \\ = \Sigma(-1)^{\frac{1}{2}(r-1)} n^r B_n & (r \text{ odd}), \end{cases} \quad (2.6)$$

and the series obtained by differentiating the allied series r -times at $t = x$ is

$$\sum_{n=1}^{\infty} \left(\frac{d}{dx}\right)^r B_n(x) \begin{cases} = \Sigma(-1)^{\frac{1}{2}r} n^r B_n & (r \text{ even}), \\ = \Sigma(-1)^{\frac{1}{2}(r+1)} n^r A_n & (r \text{ odd}). \end{cases} \quad (2.7)$$

Fractional integrals. We write, for $t > 0$,

$$\left. \begin{aligned} \Phi_{\sigma}(t) &= \frac{1}{\Gamma(\sigma)} \int_0^t (t-u)^{\sigma-1} \phi(u) du \quad (\sigma > 0), \\ \Phi_0(t) &= \phi(t), \end{aligned} \right\} \quad (2.8)$$

and similar notation is used when we replace $\phi(u)$ by $\psi(u)$, $\chi(u)$, ...

$$\text{If} \quad \Phi_{\sigma}(t) \sim \frac{\Gamma(p+1)}{\Gamma(\sigma+p+1)} st^{p+\sigma} \quad (p > -1) \quad (2.9)$$

as $t \rightarrow +0$, we write $\phi(t) \sim st^p$ (C, σ), and the usual related notation is interpreted in the obvious way.

Cesàro sums and means. Then n -th Cesàro sum of order σ , S_n^{σ} , of a series Σu_n is defined formally by $(1-x)^{-\sigma-1} \Sigma u_n x^n = \Sigma S_n^{\sigma} x^n$.

The n -th Cesàro mean of order σ , s_n^{σ} , is $S_n^{\sigma}/A_n^{\sigma}$, where A_n^{σ} is defined formally by $(1-x)^{-\sigma-1} = \Sigma A_n^{\sigma} x^n$.

Differences. We write

$$\Delta u_n = u_n - u_{n+1}, \quad \Delta^0 u_n = u_n \quad \text{and} \quad \Delta^{\kappa} u_n = \Delta(\Delta^{\kappa-1} u_n),$$

where κ is a positive integer, so that

$$\Delta^{\kappa}(u_n v_n) = \sum_{\nu=0}^{\kappa} \binom{\kappa}{\nu} \Delta^{\kappa-\nu} u_{n+\nu} \Delta^{\nu} v_n. \quad (2.10)$$

Young's functions†. We employ the functions $\gamma_{\sigma}(x)$ and $\bar{\gamma}_{\sigma}(x)$, defined by

$$\gamma_{\sigma}(x) + i\bar{\gamma}_{\sigma}(x) = \int_0^1 (1-u)^{\sigma-1} e^{ixu} du \quad (\sigma > 0). \quad (2.11)$$

* Here and elsewhere Σ denotes $\sum_0^{\infty}$, for a series $u_0 + u_1 + u_2 + \dots$, while it denotes $\sum_1^{\infty}$, for a series $u_1 + u_2 + \dots$.

† Young, 26, defines the function $C_{\sigma}(t)$, where

$$C_{\sigma}(t) + iC_{\sigma+1}(t) = \{\gamma_{\sigma}(t) + i\bar{\gamma}_{\sigma}(t)\} t^{\sigma} / \Gamma(\sigma).$$

It is easily verified, by integration by parts, that, if $\sigma > 0$ and ρ is a positive integer,

$$\gamma_{\sigma}(x) = \sum_{\nu=0}^{\rho-1} (-1)^{\nu} \frac{\Gamma(\sigma)}{\Gamma(\sigma+2\nu+1)} x^{2\nu} + (-1)^{\rho} \frac{\Gamma(\sigma)}{\Gamma(\sigma+2\rho)} x^{2\rho} \gamma_{\sigma+2\rho}(x) \quad (2.12)$$

and

$$\bar{\gamma}_{\sigma}(x) = \sum_{\nu=1}^{\rho} (-1)^{\nu-1} \frac{\Gamma(\sigma)}{\Gamma(\sigma+2\nu)} x^{2\nu-1} + (-1)^{\rho} \frac{\Gamma(\sigma)}{\Gamma(\sigma+2\rho+1)} x^{2\rho+1} \gamma_{\sigma+2\rho+1}(x). \quad (2.13)$$

It is obvious that $\gamma_{\sigma}(x)$, $\bar{\gamma}_{\sigma}(x)$ and their successive derivatives are bounded for all real x , and it is known* that, for large positive x ,

$$\gamma_{\sigma}(x) + i\bar{\gamma}_{\sigma}(x) = \frac{i}{x} + \frac{1}{\Gamma(\sigma)} \frac{e^{i(x-\frac{1}{2}\pi\sigma)}}{x^{\sigma}} + O\left(\frac{1}{x^2}\right) + O\left(\frac{1}{x^{\sigma+1}}\right), \quad (2.14)$$

while asymptotic formulae for $\gamma_{\sigma}^{(r)}(x)$ and $\bar{\gamma}_{\sigma}^{(r)}(x)$ are obtained by formally differentiating (2.14).

It follows† that, if κ and ρ are positive integers and $\sigma > 0$, then, for $t \geq 0$ and $n = 0, 1, 2, \dots$,

$$\left| \Delta^{\kappa} \left(\frac{d}{dt} \right)^{\rho} \gamma_{\sigma}(nt) \right| \leq \begin{cases} At^{\kappa} n^{\rho} \\ At^{\kappa} n^{\rho} \{ (nt)^{-\kappa-\rho-2} + (nt)^{-\sigma} \} \end{cases} \quad (2.15)$$

and

$$\left| \Delta^{\kappa} \left(\frac{d}{dt} \right)^{\rho} \bar{\gamma}_{\sigma}(nt) \right| \leq \begin{cases} At^{\kappa} n^{\rho} \\ At^{\kappa} n^{\rho} \{ (nt)^{-\kappa-\rho-1} + (nt)^{-\sigma} \}, \end{cases} \quad (2.16)$$

where A is independent of t and n .

3. *The Cesàro-Lebesgue integral.* The Cesàro-Lebesgue integral‡, which is a particular case of the Cesàro-Perron integral§, may be defined inductively as follows.

* Stated by Hardy and Littlewood, **11** (a). See also Bosanquet and Linfoot, **4**, where a method of Hardy is used to prove a more general formula.

† Cf. Bosanquet, **2**.

‡ Defined for $\lambda = \text{integer}$ in D.F.S.

§ Defined by Burkhill, **5**, **6**, **8**.

Suppose that $g(t)$ is integrable L in every interval (ϵ, a) , where $0 < \epsilon < a$ (a fixed). Then if

$$\lim_{\epsilon \rightarrow +0} \int_{\epsilon}^a g(t) dt \quad (3.1)$$

exists we say that $g(t)$ is integrable $C_0 L$ in $(0, a)$ to the value (3.1). Next suppose that $\lambda > 0$, and that the $C_{\mu} L$ integral has been defined, where $\mu = \max(\lambda - 1, 0)$, and that the $C_{\mu} L$ integral of $h(t)$ over $(0, a)$ is denoted by

$$\int_{\rightarrow 0 (C, \mu)}^a h(t) dt. \quad (3.2)$$

Then, if (i) $g(t)$ is integrable L in every interval (ϵ, a) , where $0 < \epsilon < a$,

(ii) $\int_t^a g(u) du$ is integrable $C_{\mu} L$ in $(0, a)$, and

$$(iii) \quad \lim_{\epsilon \rightarrow +0} \frac{\lambda}{\epsilon^{\lambda}} \int_{\rightarrow 0 (C, \mu)}^{\epsilon} (\epsilon - t)^{\lambda-1} dt \int_t^a g(u) du \quad (3.3)$$

exists, we say that $g(t)$ is integrable $C_{\lambda} L$ in $(0, a)$ to the value (3.3), which we denote by

$$\int_{\rightarrow 0 (C, \lambda)}^a g(t) dt. \quad (3.4)$$

If $g(t)$ is integrable $C_{\lambda} L$ for some unspecified value of λ we say that $g(t)$ is integrable in the Cesàro-Lebesgue sense.

In the case where λ is an integer the above definition is equivalent to the one given in D.F.S. To show this we observe, using the old definition, that

$$\begin{aligned} & \frac{1}{\Gamma(\lambda)} \int_{\rightarrow 0 (C, \lambda-1)}^t (t-u)^{\lambda-1} h(u) du \\ &= \int_{\rightarrow 0 (C, 0)}^t du_1 \int_{\rightarrow 0 (C, 1)}^{u_1} du_2 \dots \int_{\rightarrow 0 (C, \lambda-1)}^{u_{\lambda-1}} h(u) du, \end{aligned} \quad (3.5)$$

whenever $h(u)$ is integrable $C_{\lambda-1} L$ in $(0, t)$. This formula is proved by an argument analogous to integration by parts*. Since the definitions are the same when $\lambda = 0$, it follows by induction that, when λ is an integer, the expression (3.4) has the same value with either definition.

If $\lambda > 0$, $p > -1$, and $g(t)$ is integrable $C_{\mu} L$, where $\mu = \max(\lambda - 1, 0)$, and if

$$\lim_{\epsilon \rightarrow +0} \frac{\Gamma(\lambda + p + 1)}{\Gamma(p + 1) \Gamma(\lambda)} \frac{1}{\epsilon^{\lambda + p}} \int_{\rightarrow 0 (C, \mu)}^{\epsilon} (\epsilon - t)^{\lambda-1} g(t) dt = l, \quad (3.6)$$

* Compare the proof of Lemma 2, D.F.S., where $e^{i\mu u}$ may be replaced by $(t-u)^{\lambda-1}$.

we write $g(t) \sim l^p(C, \lambda)$ as $t \rightarrow +0$. This is consistent with (2.9); the usual related notation is interpreted in the obvious way.

The $C_\lambda P$ integral of Burkill* is defined, by means of major and minor functions, by induction from the Perron integral, which is the case $\lambda = 0$. For completeness we show here that the $C_\lambda P$ integral includes the $C_\lambda L$ integral. We use the following lemma.

LEMMA 1. If $\lambda \geq 0$ and (i) $g(t)$ is integrable $C_\lambda P$ in every interval (ϵ, a) , where $0 < \epsilon < a$,

$$(ii) \quad (C_\lambda P) \int_t^a g(u) du \rightarrow l \quad (C, \lambda)^\dagger \quad (3.7)$$

as $t \rightarrow +0$, then $g(t)$ is integrable $C_\lambda P$ in $(0, a)$ to the value l .

The case $\lambda = 0$ is well known[‡], and the case $\lambda = 1$ has been proved by Miss Grimshaw§. The proof in the general case, which is the same in principle, will be published elsewhere||.

THEOREM 1. If $\lambda \geq 0$ and $g(t)$ is integrable $C_\lambda L$ in $(0, a)$, then it is integrable $C_\lambda P$ in $(0, a)$, and

$$\int_{\rightarrow 0(C, \lambda)}^a g(t) dt = (C_\lambda P) \int_0^a g(t) dt. \quad (3.8)$$

Proof. When $\lambda = 0$ the result is a particular case of Lemma 1. If $\lambda > 0$, and the result is assumed when λ is replaced by $\mu = \max(\lambda - 1, 0)$, then the hypotheses of Lemma 1 are satisfied, and hence the conclusion. The proof is now completed by induction.

We also remark that if $\lambda \geq 0$ and (i) $g(t)$ is integrable L in every interval (ϵ, a) , where $0 < \epsilon < a$ (a fixed), (ii) $g(t)$ is integrable $C_\lambda P$ in $(0, a)$, then $g(t)$ is integrable $C_\lambda L$ in $(0, a)$. This result is certainly true when $\lambda = 0$. If $\lambda > 0$ and we assume it to be true when λ is replaced by $\mu = \max(\lambda - 1, 0)$,

* Burkill, 5, 6, 8.

† This assumes that $(C_\lambda P) \int_t^a g(u) du$ is integrable $C_\mu P$ in $(0, a)$, where $\mu = \max(\lambda - 1, 0)$.

See Burkill, 5, 6, 8.

‡ See Saks, 18, pp. 247-8.

§ Grimshaw, 10.

|| Bosanquet, 3 (a).

it follows from the hypotheses that

$$\int_t^a g(u) du \quad \text{and} \quad (\epsilon - t)^{\lambda-1} \int_t^a g(u) du$$

are integrable L in (η, ϵ) , where $0 < \eta < \epsilon < a$, (ii) integrable $C_\mu P$ in $(0, \epsilon)$, and hence are integrable $C_\mu L$ in $(0, \epsilon)$. Therefore, by Theorem 1, for $0 < \epsilon < a$,

$$\frac{\lambda}{\epsilon^\lambda} \int_{\rightarrow 0 (C, \mu)}^\epsilon (\epsilon - t)^{\lambda-1} dt \int_t^a g(u) du = \frac{\lambda}{\epsilon^\lambda} \int_{0 (C_\mu P)}^\epsilon (\epsilon - t)^{\lambda-1} dt \int_t^a g(u) du,$$

which tends to a limit as $\epsilon \rightarrow +0$, by the C_λ -continuity property of $C_\lambda P$ integrals.

Thus *integrability $C_\lambda L$ is the particular case of integrability $C_\lambda P$ where the left-hand end point of the interval is the only point of non-absolute integrability*. The various relevant laws of consistency and manipulation of fractional integrals follow at once from the corresponding laws for $C_\lambda P$ integrals, though they may if desired be developed independently.

4. *Preliminary lemmas.* In this section we collect some results which it will be convenient to quote in the main arguments.

LEMMA 2. *If λ is a non-negative integer, and (i) $g(t)$ is integrable $C_\lambda L$ in $(0, a)$, (ii) $g(t) = o(t^r)$ ($C, \lambda+1$) as $t \rightarrow +0$, (iii) $h(t) = O(t^{-r})$ and $h^{(\rho)}(t) = O(t^{-r-\rho})$, for $\rho = 1, 2, \dots, \lambda+1$, then $g(t) h(t)$ is integrable $C_\lambda L$ in $(0, a)$ and is $o(1)$ ($C, \lambda+1$) as $t \rightarrow +0$.*

This has been proved in D.F.S.*

LEMMA 3. *If $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, then so is $g(t)e^{i\mu t}$ when μ is real.*

This has been proved in D.F.S.†

Definition. If $g(t)$ is even, and integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, then we define the Fourier series of $g(t)$ as $\frac{1}{2}\gamma_0 + \sum \gamma_n \cos nt$, where

$$\gamma_n = \frac{2}{\pi} \int_{\rightarrow 0 (C)}^\pi g(t) \cos nt dt. \quad (4.1)$$

* D.F.S., Lemma 1.

† D.F.S., Lemma 2.

LEMMA* 4. If $\kappa \geq 0$, $\sigma > \kappa + 1$, and $u_0 + u_1 + \dots$ is summable (C, κ) to sum s , then $\lim_{\rho \rightarrow 1-0} \sigma \sum u_n \gamma_\sigma(nt) \rho^n \rightarrow s$ as $t \rightarrow +0$.

If κ is an integer, and S_n^κ is the n -th Cesàro sum of order κ of the series $\sum u_n$, we have, for a fixed $t > 0$, after $\kappa + 1$ partial summations, supposing as we may that $s = 0$,

$$\begin{aligned} S(t) &= \lim_{\rho \rightarrow 1-0} \sum_{n=0}^{\infty} u_n \gamma_\sigma(nt) \rho^n \\ &= \lim_{\rho \rightarrow 1-0} \sum_{n=0}^{\infty} S_n^\kappa \Delta^{\kappa+1} \{ \gamma_\sigma(nt) \rho^n \} \\ &= \lim_{\rho \rightarrow 1-0} \sum_{\nu=0}^{\kappa+1} \binom{\kappa+1}{\nu} (1-\rho)^\nu \sum_{n=0}^{\infty} S_n^\kappa \Delta^{\kappa+1-\nu} \gamma_\sigma \{ (n+\nu)t \} \rho^n \\ &= \lim_{\rho \rightarrow 1-0} \sum_{n=0}^{\infty} S_n^\kappa \Delta^{\kappa+1} \gamma_\sigma(nt) \rho^n \\ &\quad + \lim_{\rho \rightarrow 1-0} \sum_{\nu=1}^{\kappa+1} \binom{\kappa+1}{\nu} (1-\rho)^\nu \sum_{n=0}^{\infty} o(n^\kappa) O(n^{-\kappa-1+\nu}) \rho^n \\ &= \sum_{n=0}^{\infty} S_n^\kappa \Delta^{\kappa+1} \gamma^\sigma(nt), \end{aligned}$$

by Abel's theorem, since this series converges for $\sigma > \kappa + 1$. Hence, for $t > 0$,

$$\begin{aligned} S(t) &= \sum_{n \leq t^{-1}} o(n^\kappa) O(t^{\kappa+1}) + \sum_{n > t^{-1}} o(n^\kappa) \{ O(t^{-2} n^{-\kappa-3}) + O(t^{\kappa+1-\sigma} n^{-\sigma}) \} \\ &= o(1) + o(1), \end{aligned}$$

as $t \rightarrow +0$, since $\sigma > \kappa + 1$.

The case where κ is not an integer may be proved by an elaboration of this argument†.

LEMMA‡ 5. If $u_1 + u_2 + \dots$ is summable (C, κ) , where $\kappa > 0$, then $\Sigma(n+\beta)^{-1} u_n$ is summable $(C, \kappa-1)$, where $\beta \geq 0$.

* Verblunsky, 23.

† Cf. Verblunsky, 23; Bosanquet, 2.

‡ Chapman, 9. The case of k an integer was given by H. Bohr and M. Riesz

If $\beta = 0$, the result is obtained from the identities*

$$\sigma_n^{\kappa+1} = (\kappa+1) \sum_{\nu=1}^n \frac{s_\nu^\kappa}{\nu(\nu+\kappa+1)} \quad (4.2)$$

$$\text{and} \quad \lambda \sigma_n^{\lambda-1} = (n+\lambda) \sigma_n^\lambda - n \sigma_{n-1}^\lambda \quad (4.3)$$

with $\lambda = \kappa+1$ and $\lambda = \kappa$, where s_n^λ and σ_n^λ are the n -th Cesàro means of order λ of Σu_n and $\Sigma n^{-1} u_n$ respectively.

The general case is easily deduced, or proved similarly.

LEMMA 6. *If r is a positive integer, there exist constants c_ν, d_ν ($\nu = 0, 1, \dots, r$) such that*

$$\cos \frac{1}{2}t = \sum_{\nu=0}^r c_\nu \cos \nu t + c(t) \quad (4.4)$$

$$\text{and} \quad \sin \frac{1}{2}t = \sum_{\nu=0}^r d_\nu \sin \nu t + d(t), \quad (4.5)$$

where $c(t)$ and $d(t)$ are $O(t^{r+1})$ as $t \rightarrow +0$.

It is easily seen that constants k_ν, l_ν, c_ν may be chosen so that

$$\begin{aligned} \cos \frac{1}{2}t &= \sum_{\nu=0}^{[\frac{1}{2}r]} k_\nu t^{2\nu} + O(t^{r+1}) \\ &= \sum_{\nu=0}^{[\frac{1}{2}r]} l_\nu \sin^{2\nu} t + O(t^{r+1}) \\ &= \sum_{\nu=0}^r c_\nu \cos \nu t + O(t^{r+1}), \end{aligned}$$

and (4.5) is proved similarly.

LEMMA 7. *If r is a positive integer, there exist constants p_ν, q_ν ($\nu = 0, 1, \dots, r$) such that*

$$\left(\frac{2 \sin \frac{1}{2}t}{t} \right)^r = \sum_{\nu=0}^r p_\nu \cos \nu t + p(t), \quad (4.6)$$

$$\left(\frac{t}{2 \sin \frac{1}{2}t} \right)^r = \sum_{\nu=0}^r q_\nu \cos \nu t + q(t), \quad (4.7)$$

where $p(t)$ and $q(t)$ are $O(t^{r+1})$ as $t \rightarrow +0$.

The proof is similar to that of Lemma 6.

* Kogbetliantz, 13.

5. *The main theorem.* In this section we prove the following theorem.

THEOREM 2*. *If $f(t)$ is integrable in the Lebesgue sense in $(-\pi, \pi)$, and periodic with period 2π , then a necessary and sufficient condition for the series $\Sigma \left(\frac{d}{dx}\right)^r (a_n \cos nx + b_n \sin nx)$ to be summable $(C, \alpha+r)$ to s , where $\alpha \geq 0$ and r is a positive integer, is that constants α_ν exist such that (i) $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$ and (ii) its Fourier series is summable (C, α) to s at $t=0$, where, for $-\pi < t < \pi$,*

$$g(t) \begin{cases} = \frac{r!}{(2 \sin \frac{1}{2}t)^r} \left\{ \phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{\alpha_{2\nu}}{(2\nu)!} t^{2\nu} \right\} & (r \text{ even}) \\ = \frac{r!}{(2 \sin \frac{1}{2}t)^r} \left\{ \psi(t) - \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu-1}}{(2\nu-1)!} t^{2\nu-1} \right\} & (r \text{ odd}). \end{cases} \quad (5.1)$$

Moreover, for $\nu = r-2, r-4, r-6, \dots$,

$$\alpha_\nu = \lim_{\rho \rightarrow 1-0} \Sigma \left(\frac{d}{dx}\right)^\nu (a_n \cos nx + b_n \sin nx) \rho^n \dagger. \quad (5.2)$$

We require a number of lemmas.

LEMMA ‡ 8. *If $f(t)$ is integrable L in $(-\pi, \pi)$ and $\Sigma \left(\frac{d}{dx}\right)^r A_n(x)$ is summable (C) to s , then there exist constants α_ν such that $\chi(t) \sim st^r/r!$ (C) as $t \rightarrow +0$, where*

$$\chi(t) \begin{cases} = \phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{\alpha_{2\nu}}{(2\nu)!} t^{2\nu} & (r \text{ even}) \\ = \psi(t) - \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu-1}}{(2\nu-1)!} t^{2\nu-1} & (r \text{ odd}). \end{cases} \quad (5.3)$$

* The sufficiency part, for the class of functions for which $g(t)$ is integrable L , was given by W. H. Young, **27**, for $\alpha = 0, 1, \dots$, and by Wang, **24**, for $\alpha > 0$. Actually Young's result is stated in terms of $G(2t)$, while Wang's is in terms of $\gamma(t)$. [See (5.5) and (6.1).]

The method adopted in this paper is an elaboration of the methods of Young, **27**, Rajchman, **17**, and Zygmund, **28**, and of the method in D.F.S. It was, however, an examination of Wang's method which led to the correct formulation of the result.

† We may also write

$$g(t) = \frac{\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}}{2(\sin \frac{1}{2}t)^r/r!},$$

where $P(t) = \sum_{\nu=0}^{r-1} \frac{\alpha_\nu}{\nu!} t^\nu$, but in this form certain of the α_ν are arbitrary.

‡ This lemma may be deduced from a theorem on trigonometrical series due to Plessner, **16**. His method (see Zygmund, **29**, 260-261) is to show that the λ -th integral of $f(t)$ has a de la Vallée Poussin generalized symmetric derivate at $t=0$ of order $\lambda+r$. This requires λ to be an integer; the proof given here leads to a more precise result.

Moreover, for $\nu = r-2, r-4, r-6, \dots$,

$$a_\nu = \lim_{\rho \rightarrow 1-0} \Sigma \left(\frac{d}{dx} \right)^\nu A_n(x) \rho^n. \quad (5.4)$$

Suppose r to be even. We first observe that we may suppose, without loss of generality, that $s = 0$. For if, in particular, $x = 0$ and

$$f(t) = \phi(t) = (-1)^{\frac{1}{2}r} s \cos t,$$

the r -th derived series consists of the single term $s \cos t$, so that it converges to s for $t = 0$. Moreover, if $a_\nu = (-1)^{\frac{1}{2}(r-\nu)}$, for $\nu = 0, 2, 4, \dots$, the constants a_ν satisfy (5.4) and

$$\chi(t) = (-1)^{\frac{1}{2}r} s \left\{ \cos t - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{(-1)^\nu}{(2\nu)!} t^{2\nu} \right\} \sim \frac{st^r}{r!}$$

as $t \rightarrow +0$. Thus, if $\phi(t)$ is replaced by $\phi(t) - (-1)^{\frac{1}{2}r} s \cos t$, s is replaced by 0.

Suppose now that $\Sigma(-1)^{\frac{1}{2}r} n^r A_n$ is summable (C, κ) to 0, where $\kappa \geq r$. Then, if $\lambda \geq 1$,

$$\begin{aligned} \Phi_\lambda(t) &= \frac{t^\lambda}{\Gamma(\lambda)} \int_0^1 (1-u)^{\lambda-1} \phi(tu) du \\ &= \frac{t^\lambda}{\Gamma(\lambda)} \sum_{n=0}^{\infty} A_n \gamma_\lambda(nt)^* \\ &= t^\lambda \sum_{n=0}^{\infty} A_n \left\{ \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{(-1)^\nu n^{2\nu} t^{2\nu}}{\Gamma(\lambda+2\nu+1)} + \frac{(-1)^{\frac{1}{2}r} n^r t^r}{\Gamma(\lambda+r)} \gamma_{\lambda+r}(nt) \right\}^\dagger \\ &= \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{t^{\lambda+2\nu}}{\Gamma(\lambda+2\nu+1)} \lim_{\rho \rightarrow 1-0} \sum_{n=0}^{\infty} (-1)^\nu n^{2\nu} A_n \rho^n \\ &\quad + \frac{t^{\lambda+r}}{\Gamma(\lambda+r)} \lim_{\rho \rightarrow 1-0} \sum_{n=0}^{\infty} (-1)^{\frac{1}{2}r} n^r A_n \gamma_{\lambda+r}(nt) \rho^n \ddagger \\ &= \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{a_{2\nu}}{\Gamma(\lambda+2\nu+1)} t^{\lambda+2\nu} + o(t^{\lambda+r}), \end{aligned}$$

by Lemma 4, provided that $\lambda+r > \kappa+1$. •

* Since $(1-u)^{\lambda-1}$ is of bounded variation in $(0, 1)$. Cf. Hobson, 12, 582.

† By (2.12).

‡ By Lemma 5 and the theorem of Frobenius-Hölder.

Hence, if $\lambda > \kappa - r + 1$,

$$\begin{aligned} X_\lambda(t) &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-u)^{\lambda-1} \left\{ \phi(u) - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{a_{2\nu}}{\Gamma(2\nu+1)} u^{2\nu} \right\} du \\ &= \Phi_\lambda(t) - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{a_{2\nu}}{\Gamma(\lambda+2\nu+1)} t^{\lambda+2\nu} = o(t^{\lambda+r}) \end{aligned}$$

as $t \rightarrow +0$.

Suppose r to be odd. It is easily verified that, if $\psi(t)$ is replaced by

$$\psi(t) - (-1)^{\frac{1}{2}(r-1)} s \sin t,$$

s is replaced by 0. Thus we may suppose, without loss of generality, that $\Sigma(-1)^{\frac{1}{2}(r-1)} n^r B_n$ is summable (C, κ) , $\kappa \geq r$, to 0. Then, if $\lambda > \kappa - r + 1$,

$$\begin{aligned} \Psi_\lambda(t) &= \frac{t^\lambda}{\Gamma(\lambda)} \sum_{n=1}^{\infty} B_n \bar{\gamma}_\lambda(nt) \\ &= t^\lambda \sum_{n=1}^{\infty} B_n \left\{ \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{(-1)^{\nu-1} n^{2\nu-1} t^{2\nu-1}}{\Gamma(\lambda+2\nu)} + \frac{(-1)^{\frac{1}{2}(r-1)} n^r t^r}{\Gamma(\lambda+r)} \gamma_{\lambda+r}(nt) \right\} \\ &= \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{t^{\lambda+2\nu-1}}{\Gamma(\lambda+2\nu)} \lim_{\rho \rightarrow 1-0} \sum_{n=1}^{\infty} (-1)^{\nu-1} n^{2\nu-1} B_n \rho^n \\ &\quad + \frac{t^{\lambda+r}}{\Gamma(\lambda+r)} \lim_{\rho \rightarrow 1-0} \sum_{n=1}^{\infty} (-1)^{\frac{1}{2}(r-1)} n^r B_n \gamma_{\lambda+r}(nt) \rho^n \\ &= \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{a_{2\nu-1}}{\Gamma(\lambda+2\nu)} t^{\lambda+2\nu-1} + o(t^{\lambda+r}) \end{aligned}$$

as $t \rightarrow +0$.

Hence, as before, $X_\lambda(t) = o(t^{\lambda+r})$ as $t \rightarrow +0$.

Since the constants a_ν satisfy (5.4), and since there cannot be another set of a_ν such that $\chi(t) \sim st^r/r!$ (C), the lemma is proved*.

LEMMA 9. If $\chi(t)$ is integrable L in $(0, \pi)$, and if $\chi(t) \sim st^r/r!$ (C) as $t \rightarrow +0$, then $r!(2 \sin \frac{1}{2}t)^{-r} \chi(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, and tends to s (C) as $t \rightarrow +0$.

This follows from Lemma 2, with

$$g(t) = \chi(t) - st^r/r! \quad \text{and} \quad h(t) = r!(2 \sin \tfrac{1}{2}t)^{-r}.$$

* This result, when combined with Lemma 9, may be compared with a result given by Hyslop for absolute summability, under the hypothesis that $\chi(t)/t^r$ is integrable L , Hyslop, 30, Theorem 3.

LEMMA 10. If $g(t)$ is even and integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, and if $g(t) \rightarrow s$ (C) as $t \rightarrow +0$, then the Fourier series of $g(t)$ is summable (C) to s for $t = 0$.

This is, by Theorem 1, a particular case of a result due to Burkill*, but it has been shown in D.F.S. that it may be proved independently of the Cesàro-Perron integral.

LEMMA 11. If constants α_ν exist such that $g(t)$, defined by (5.1), is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, then there exist constants β_ν such that $G(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, and the Fourier series of $g(t)$ and $G(t)$ are equi-summable (C, α)[†], $\alpha \geq 0$, at $t = 0$, where, for $-\pi < t < \pi$,

$$G(t) \begin{cases} = \frac{r!}{(2 \sin \frac{1}{2}t)^r} \left\{ \phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-2)} \beta_{2\nu} \sin^{2\nu} t \right\} & (r \text{ even}) \\ = \frac{r!}{(2 \sin \frac{1}{2}t)^r} \left\{ \psi(t) - \sum_{\nu=1}^{\frac{1}{2}(r-1)} \beta_{2\nu-1} \sin^{2\nu-1} t \right\} & (r \text{ odd}). \end{cases} \quad (5.5)$$

* Burkill, 7. It is worth noting that the converse of Lemma 10, and more generally of Burkill's theorem, is also true. If $g(t)$ is even, and integrable $C_r P$ in $(0, \pi)$, where r is a positive integer, and if its Fourier series, $\frac{1}{2}u_0 + \sum u_n \cos nt$, is summable (C, κ) to s at $t = 0$, where $\kappa \geq r$, then $g(t) \rightarrow s$ (C, γ) as $t \rightarrow +0$, for $\gamma > \kappa + 1$.

We observe that, if p is a positive even integer and

$$\beta_\mu = \lim_{\rho \rightarrow 1-0} \sum (-1)^{\frac{1}{2}\mu} \frac{u_n}{n^\mu} \rho^n \quad (\mu = 2, 4, \dots),$$

the function

$$G_p(t) - \frac{1}{2}u_0 t^p/p! + \sum_{\nu=1}^{\frac{1}{2}p} \beta_{2\nu} t^{p-2\nu}/(p-2\nu)!$$

is even, integrable $C_{r-p} P$, and has the Fourier series $\sum a_n \cos nt$, where $a_n = (-1)^{\frac{1}{2}p} u_n/n^p$ ($n \geq 1$). In the case $p = 2$ the a_n are easily obtained by the analogue of integration by parts for Cesàro-Perron integrals, and the value of β_2 checked by Burkill's theorem. The general case may be proved by a repetition of the same argument and induction.

If p is a positive odd integer, the function

$$G_p(t) - \frac{1}{2}u_0 t^p/p! + \sum_{\nu=1}^{\frac{1}{2}(p-1)} \beta_{2\nu} t^{p-2\nu}/(p-2\nu)!$$

is odd, integrable $C_{r-p} P$, and has the Fourier series $\sum b_n \sin nt$, where $b_n = (-1)^{\frac{1}{2}(p-1)} u_n/n^p$.

Taking $p = r$ and arguing as in Lemma 8, with $x = 0$ and $(d/dx)^r A_n(x) = u_n$ [the proof of Lemma 8 still holds when $f(t)$ is only integrable in the Perron sense] we find that

$$G_r(t) \sim st^r/r! \quad (C, \lambda) \text{ as } t \rightarrow +0, \text{ for } \lambda > \kappa - r + 1,$$

i.e.

$$G_{r+\lambda}(t) \sim st^{r+\lambda}/\Gamma(r+\lambda+1),$$

i.e.

$$g(t) \rightarrow s \quad (C, r+\lambda).$$

† I.e. either both series are summable (C, α) or neither is.

If r is even, constants β_ν may be chosen so that

$$\sum_{\nu=0}^{\frac{1}{2}(r-2)} \frac{\alpha_{2\nu}}{(2\nu)!} t^{2\nu} = \sum_{\nu=0}^{\frac{1}{2}r} \beta_{2\nu} \sin^{2\nu} t + O(t^{r+1}).$$

It follows that $g(t) - G(t) \rightarrow r! \beta_r$ as $t \rightarrow +0$, and hence that $G(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$.

If now s_n and σ_n are the n -th partial sums of the Fourier series of $g(t)$ and $G(t)$ respectively at $t = 0$, we have

$$s_n - \sigma_n - r! \beta_r = \frac{2}{\pi} \int_0^\pi (g(t) - G(t) - r! \beta_r) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt = o(1),$$

by the Riemann-Lebesgue theorem, since $\{g(t) - G(t) - r! \beta_r\} / (2 \sin \frac{1}{2}t)$ is $O(1)$, and so integrable L .

The proof when r is odd is similar.

LEMMA 12. If (i) $G(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, (ii) $t^r G(t)$ is integrable L in $(0, \pi)$, where r is a positive integer, and if

$$\alpha(\mu) = \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \cos \mu t dt, \quad (5.6)$$

then the series $\Sigma \alpha(n)$ and $\Sigma \alpha(n - \frac{1}{2}r)$ are equi-summable (C, α) , $\alpha \geq 0$.

The case when r is even is trivial, and it is clear that when r is odd we may replace $\Sigma \alpha(n - \frac{1}{2}r)$ by $\Sigma \alpha(n - \frac{1}{2})$.

By Lemma 3, $G(t) \cos \frac{1}{2}t$ and $G(t) \sin \frac{1}{2}t$ are both integrable in the Cesàro-Lebesgue sense in $(0, \pi)$. If then

$$s_n = \frac{1}{2} \alpha(0) + \sum_{\nu=1}^n \alpha(\nu) \quad \text{and} \quad \sigma_n = \sum_{\nu=1}^n \alpha(\nu - \tfrac{1}{2}),$$

we have, by Lemma 6, for $n \geq r$,

$$\begin{aligned} \sigma_n &= \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \frac{\sin nt}{2 \sin \frac{1}{2}t} dt \\ &= \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \cos \frac{1}{2}t \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt - \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \sin \frac{1}{2}t \frac{\cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &= \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \left\{ \sum_{\nu=0}^r c_\nu \cos \nu t + c(t) \right\} \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &\quad - \frac{2}{\pi} \int_{\rightarrow 0(C)}^\pi G(t) \left\{ \sum_{\nu=0}^r d_\nu \sin \nu t + d(t) \right\} \frac{\cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &= \frac{1}{2} \sum_{\nu=0}^r c_\nu (s_{n+\nu} + s_{n-\nu}) - \frac{1}{2} \sum_{\nu=0}^r d_\nu (s_{n+\nu} - s_{n-\nu}) \\ &\quad + \int_0^\pi G(t) O(t^r) \sin(n + \frac{1}{2})t dt + \int_0^\pi G(t) O(t^r) \cos(n + \frac{1}{2})t dt. \end{aligned}$$

Hence, by the Riemann-Lebesgue theorem,

$$\sigma_n = \frac{1}{2} \sum_{\nu=0}^r (c_\nu - d_\nu) s_{n+\nu} + \frac{1}{2} \sum_{\nu=0}^r (c_\nu + d_\nu) s_{n-\nu} + o(1). \quad (5.7)$$

Similarly, it may be shown that, for $n \geq r$,

$$s_n = \frac{1}{2} \sum_{\nu=0}^r (c_\nu + d_\nu) \sigma_{n+\nu} + \frac{1}{2} \sum_{\nu=0}^r (c_\nu - d_\nu) \sigma_{n-\nu} + o(1). \quad (5.8)$$

The result then follows from (5.7) and (5.8).

LEMMA 13. *If $G(t)$ is defined by (5.5), and $\alpha(\mu)$ by (5.6), then*

$$n^r \Delta^r \alpha(n - \tfrac{1}{2}r) = r! \left(\frac{d}{dx} \right)^r A_n(x),$$

for $n \geq r$.

If r is even, we have, for $n \geq r$,

$$\begin{aligned} n^r \Delta^r \alpha(n - \tfrac{1}{2}r) &= \frac{2n^r}{\pi} \int_{\rightarrow 0(C)}^{\pi} G(t) \Delta^r \cos(n - \tfrac{1}{2}r)t dt \\ &= \frac{2(-1)^{\frac{1}{2}r} n^r}{\pi} \int_{\rightarrow 0(C)}^{\pi} G(t) (2 \sin \tfrac{1}{2}t)^r \cos nt dt \\ &= \frac{2r! (-1)^{\frac{1}{2}r} n^r}{\pi} \int_0^{\pi} \phi(t) \cos nt dt \\ &= r! (-1)^{\frac{1}{2}r} n^r A_n, \end{aligned}$$

since

$$\int_0^{\pi} \sin^{2p} t \cos qt dt = 0$$

for $q > 2p \geq 0$, where p and q are integers.

If r is odd, we have, for $n \geq r$,

$$\begin{aligned} n^r \Delta^r \alpha(n - \tfrac{1}{2}r) &= \frac{2(-1)^{\frac{1}{2}(r-1)} n^r}{\pi} \int_{\rightarrow 0(C)}^{\pi} G(t) (2 \sin \tfrac{1}{2}t)^r \sin nt dt \\ &= r! (-1)^{\frac{1}{2}(r-1)} n^r B_n, \end{aligned}$$

since

$$\int_0^{\pi} \sin^{2p+1} t \sin qt dt = 0$$

for $q > 2p + 1 \geq 1$.

LEMMA* 14. *If Σu_n is summable (C) , then a necessary and sufficient condition for $\Sigma n^r \Delta^r u_n$ to be summable $(C, \alpha+r)$, where $\alpha \geq 0$, is that Σu_n should be summable (C, α) .*

We first observe that, if $A_n^r = \binom{n+r}{r}$, the series $\Sigma n^r \Delta^r u_n$ and $\Sigma A_n^r \Delta^r u_n$ are equi-summable $(C, \alpha+r)$. For there exist constants k_ν such that

$$A_n^r = \sum_{\nu=0}^r k_\nu n^\nu, \quad (5.9)$$

and hence constants l_ν such that

$$n^r = \sum_{\nu=0}^r l_\nu A_n^\nu. \quad (5.10)$$

The equi-summability of the two series then follows from (5.9) and (5.10), by Lemma 5.

On the other hand, if $t_{\rho,n}^\sigma$ denotes the n -th Cesàro mean of order σ of $\Sigma A_n^\rho \Delta^\sigma u_n$, we have†

$$\rho t_{\rho,n}^\sigma = \rho t_{\rho-1,n}^{\sigma-1} - \sigma (t_{\rho-1,n+1}^{\sigma-1} - t_{\rho-1,n}^{\sigma-1}). \quad (5.11)$$

The lemma may now be proved by successive applications of (5.11) and the consistency theorem for Cesàro limits.

Proof of Theorem 2. Necessity. Suppose that $\Sigma (d/dx)^r A_n(x)$ is summable $(C, \alpha+r)$ to s . Then, by Lemma 8, constants a_ν , defined by (5.2), exist such that $\chi(t) \sim st^r/r! (C)$ as $t \rightarrow +0$, where $\chi(t)$ is defined by (5.3). Hence, by Lemma 9, $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$ and $g(t) \rightarrow s (C)$ as $t \rightarrow +0$, where $g(t)$ is defined by (5.1). It follows, by Lemma 10, that the Fourier series of $g(t)$ is summable (C) to s at $t=0$.

Now, by Lemma 11, $g(t)$ defines a function $G(t)$, of the form (5.5), which is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$ and its Fourier series is also summable (C) at $t=0$, i.e. the series $\frac{1}{2}a(0) + \Sigma \alpha(n)$ is summable (C) , where $\alpha(\mu)$ is defined by (5.6). It follows, by Lemma 12, that $\Sigma \alpha(n - \frac{1}{2}r)$ is also summable (C) . Now, by Lemma 13,

$$n^r \Delta^r \alpha(n - \tfrac{1}{2}r) = r! \left(\frac{d}{dx} \right)^r A_n(x),$$

* This is a modification of a particular case of a theorem of Andersen, 1.

† Winn, 25.

for $n \geq r$, and so $\Sigma n^r \Delta^r a(n - \frac{1}{2}r)$ is summable $(C, \alpha + r)$. Hence, by Lemma 14, $\Sigma a(n - \frac{1}{2}r)$ is summable (C, α) , and, by Lemma 12, so also is $\Sigma a(n)$. It follows, by Lemma 11, that the Fourier series of $g(t)$ is summable (C, α) at $t = 0$, and we have already proved that the sum is s .

Sufficiency. Suppose that constants α_ν exist such that $g(t)$, defined by (5.1), is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, and that its Fourier series is summable (C, α) to s at $t = 0$. Then, by Lemma 11, the function $G(t)$, defined by (5.5), is determined, is integrable in the Cesàro-Lebesgue sense and its Fourier series is summable (C, α) at $t = 0$, i.e. $\frac{1}{2}\alpha(0) + \Sigma a(n)$ is summable (C, α) . It follows, by Lemma 12, that $\Sigma a(n - \frac{1}{2}r)$ is summable (C, α) , and hence, by Lemma 14, $\Sigma n^r \Delta^r a(n - \frac{1}{2}r)$ is summable $(C, \alpha + r)$, i.e., by Lemma 13, that $\Sigma (d/dx)^r A_n(x)$ is summable $(C, \alpha + r)$. Finally, by Lemmas 8 and 9, the constants α_ν , which are unique, satisfy (5.2), and, by Lemmas 8, 9, and 10, the Cesàro sum of $\Sigma (d/dx)^r A_n(x)$ is s .

Thus the theorem is proved.

6. In this section we give an alternative form of Theorem 2, and also state an analogous theorem for allied series.

THEOREM 3. *In Theorem 2 the function $g(t)$, defined by (5.1), may be replaced by $\gamma(t)$, where, for $-\pi < t < \pi$,*

$$\gamma(t) \begin{cases} = \frac{r!}{t^r} \left\{ \phi(t) - \sum_{\nu=1}^{\frac{1}{2}(r-2)} \frac{\alpha_{2\nu}}{(2\nu)!} t^{2\nu} \right\} & (r \text{ even}) \\ = \frac{r!}{t^r} \left\{ \psi(t) - \sum_{\nu=1}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu-1}}{(2\nu-1)!} t^{2\nu-1} \right\} & (r \text{ odd}), \end{cases} \quad (6.1)$$

the constants α_ν being given by (5.2).

This is an immediate consequence of Theorem 2 and the following lemma.

LEMMA 15. *If $\chi(t)$, defined by (5.3), is integrable L , and if either of the even functions $g(t) = r! \chi(t) / (2 \sin \frac{1}{2}t)^r$ or $\gamma(t) = r! \chi(t) / t^r$ is integrable in the Cesàro-Lebesgue sense, then so is the other, and their Fourier series are equi-summable (C, α) , $\alpha \geq 0$, at $t = 0$ with the same sum*.*

* I.e. the series are either both summable (C, α) with the same sum, or neither is summable (C, α) .

Suppose that $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, and that its Fourier series is summable (C, α) to s at $t=0$. Then, since $\gamma(t) = g(t)\{(2 \sin \frac{1}{2}t)/t\}^r$ and $t^r g(t)$ is integrable L , the integrability* of $\gamma(t)$ in the Cesàro-Lebesgue sense follows from Lemmas 7 and 3.

If s_n and σ_n are the n -th partial sums the Fourier series of $g(t)$ and $\gamma(t)$ respectively, we have, for $n \geq r$,

$$\begin{aligned}\sigma_n &= \frac{2}{\pi} \int_{\rightarrow 0 (C)}^{\pi} g(t) \left(\frac{2 \sin \frac{1}{2}t}{t} \right)^r \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &= \frac{2}{\pi} \int_{\rightarrow 0 (C)}^{\pi} g(t) \left\{ \sum_{\nu=0}^r p_{\nu} \cos \nu t + p(t) \right\} \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &= \frac{1}{2} \sum_{\nu=0}^r p_{\nu} (s_{n+\nu} + s_{n-\nu}) + o(1),\end{aligned}$$

by the Riemann-Lebesgue theorem, since $p(t) = O(t^{r+1})$ and $t^r g(t)$ is integrable L . It is easily seen, from (4.6), that $\sum_{\nu=0}^r p_{\nu} = 1$, and thus it follows that σ_n tends to s (C, α) .

Again, if $\gamma(t)$ is integrable in the Cesàro-Lebesgue sense, it may be shown that, for $n \geq r$,

$$s_n = \frac{1}{2} \sum_{\nu=0}^r q_{\nu} (\sigma_{n+\nu} + \sigma_{n-\nu}) + o(1) \quad \left(\sum_{\nu=0}^r q_{\nu} = 1 \right),$$

and the converse result follows.

Finally we state without proof the analogue of Theorems 2 and 3 for successively derived allied series†.

THEOREM 4. *If $f(t)$ is integrable L in $(-\pi, \pi)$, then a necessary and sufficient condition for $\Sigma (d/dx)^r (b_n \cos nx - a_n \sin nx)$ to be summable $(C, \alpha+r)$ to s , where $\alpha \geq 0$ and r is a positive integer, is that constants a_{ν} exist such that (i) the odd function $\bar{\gamma}(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$, (ii) its allied series is summable (C, α) at $t=0$, and (iii)*

$$\frac{2}{\pi} \int_t^{\infty} \frac{\bar{\gamma}(u)}{u} du \rightarrow s \quad (C)^{\ddagger} \quad (6.2)$$

* Or we may argue directly as in the proof of Lemma 3, with $\{(2 \sin \frac{1}{2}t)/t\}^r$ in place of $e^{i\mu t}$, and avoid using the integrability L of $t^r g(t)$ at this stage.

† For results concerning the Cesàro summability of derived and successively derived allied series see Sayers, 19, Takahashi, 22, Smith, 20, 21, Moursund, 14, 15.

‡ Since $\bar{\gamma}(t)$ is not periodic the limit in (6.2) is not in general the Cesàro sum of the allied series of $\bar{\gamma}(t)$. Incidentally, the existence of this limit shows that $\bar{\gamma}(t)/t$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$.

as $t \rightarrow +0$, where, for $-\pi < t < \pi$,

$$\bar{\gamma}(t) \begin{cases} = \frac{r!}{t^r} \left\{ \psi(t) - \sum_{\nu=1}^{\frac{1}{2}r} \frac{\alpha_{2\nu-1}}{(2\nu-1)!} t^{2\nu-1} \right\} & (r \text{ even}) \\ = \frac{r!}{t^r} \left\{ \phi(t) - \sum_{\nu=0}^{\frac{1}{2}(r-1)} \frac{\alpha_{2\nu}}{(2\nu)!} t^{2\nu} \right\} & (r \text{ odd}). \end{cases} \quad (6.3)$$

Moreover, for $\nu = r-1, r-3, r-5, \dots$,

$$\alpha_\nu = \lim_{\rho \rightarrow 1-0} \sum \left(\frac{d}{dx} \right)^\nu (a_n \cos nx + b_n \sin nx) \rho^n. \quad (6.4)$$

List of references*.

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ON THE TOTAL VARIATION OF A FUNCTION OF TWO VARIABLES

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[Received 12 April, 1939.—Read 20 April, 1939.]

1. Definition of the total variation.

The object of this paper is to draw attention to what seems a natural definition for the total variation of a function $f(x, y)$ that is of bounded variation (in Tonelli's sense*), and to show that this definition permits the generalization of two important formulae for the total variation of a continuous function of one variable.

Let $f(x, y)$ be defined in a closed convex region R in the x - y -plane, and consider its total variations when one of its arguments remains constant:

$$(1.11) \quad T_x[y; f; R] = \int |df(t, y)| \quad [(t, y) \text{ in } R],$$

$$(1.12) \quad T_y[x; f; R] = \int |df(x, t)| \quad [(x, t) \text{ in } R],$$

where t is the variable of integration. The function $f(x, y)$ is said to be of *bounded variation over R in Tonelli's sense* if the (Lebesgue) integrals

$$(1.2) \quad \int T_x[y; f; R] dy, \quad \int T_y[x; f; R] dx$$

both exist and are both finite†.

The sum of these integrals might serve to measure the total variation of $f(x, y)$; but it would have the disadvantage of not being invariant under a rotation of axes

$$(1.3) \quad \xi = x \cos \theta + y \sin \theta, \quad \eta = -x \sin \theta + y \cos \theta.$$

* For this and other kinds of bounded variation of $f(x, y)$ see Clarkson and Adams, **1, 2**; also Saks, **1**, V, § 4.

† Here and elsewhere the range of integration is $(-\infty, \infty)$ unless otherwise stated: actually the integrands vanish outside finite intervals.

This suggests that we should consider the expressions (1.2) corresponding to the function

$$(1.4) \quad f^\theta(x, y) = f(\xi, \eta) \quad [(x, y) \text{ in } R^\theta],$$

where R^θ is the region obtained by rotating R about the origin through the angle θ , and integrate them with respect to θ . We write

$$(1.5) \quad W(\theta) = W[\theta; f; R] = \int T_x[y; f^\theta; R^\theta] dy.$$

Noting that the integrals (1.2) are simply $W(0)$ and $W(\frac{1}{2}\pi)$, we see that they do not need separate treatment: thus we put

$$(1.6) \quad V_R[f] = \frac{1}{2} \int_0^\pi W[\theta; f; R] d\theta.$$

The factor $\frac{1}{2}$ is preferred to $1/\pi$ since in the case where f depends only on one variable and R is a rectangle with sides parallel to the axes it leads to

$$(1.7) \quad V_R[f] = \int |df(x)| \int dy \quad [(x, y) \text{ in } R].$$

It will be shown in § 6 that, when f is continuous, $V_R[f]$ is finite when and only when both the integrals (1.2) are finite.

2. Formulae for the total variation.

Henceforth we assume that $f(x, y)$ is continuous in R . Indeed without some such limitation the bounded variation is not independent of the directions of the axes*.

We proceed to consider the generalization of Banach's theorem† according to which, if $f(x)$ is a continuous function of one variable, its total variation is given by

$$(2.1) \quad \int_a^b |df(x)| = \int N(y) dy,$$

where $N(y)$ is the number of values of x in (a, b) for which $f(x) = y$. We may, in fact, use this theorem in obtaining its generalization. For consider the surface (or superficial set)

$$(2.2) \quad \Sigma^\theta(E): z = f(\xi, \eta) \quad [(x, y) \text{ in } E],$$

* Clarkson and Adams, 2, § 9.

† Banach, 1, Theorem 2: also given in Saks, 1, IX, (6.4).

where ξ, η are defined by (1.3) and E is any Borel set in the (x, y) -plane that is contained in R^θ ; and write

$$(2.31) \quad N_x[y, z; \theta; E] = \text{number of values of } x \text{ for which } (x, y, z) \text{ is on } \Sigma^\theta(E),$$

$$(2.32) \quad N_y[x, z; \theta; E] = \text{number of values of } y \text{ for which } (x, y, z) \text{ is on } \Sigma^\theta(E).$$

Then Banach's theorem yields

$$(2.4) \quad T_x[y; f^\theta; R^\theta] = \int N_x[y, z; \theta; R^\theta] dz.$$

Hence we have, subject to later justification in §4,

$$(2.5) \quad W(\theta) = \int dy \int N_x dz = \iint N_x dy dz = \int dz \int N_x dy.$$

Thus, if we write

$$(2.6) \quad \begin{cases} P_x[z; \theta; E] = \int N_x[y, z; \theta; E] dy, \\ P_y[z; \theta; E] = \int N_y[x, y; \theta; E] dx, \end{cases}$$

we have, again subject to justification,

$$(2.7) \quad \begin{aligned} V_R[f] &= \frac{1}{2} \int_0^\pi d\theta \int P_x[z; \theta; R^\theta] dz \\ &= \frac{1}{2} \int dz \int_0^\pi P_x[z; \theta; R^\theta] d\theta. \end{aligned}$$

Now consider the following sets in the (ξ, η) -plane*.

$$(2.81) \quad S_z: f(\xi, \eta) = z, \quad \Omega_z^+: f(\xi, \eta) > z, \quad \Omega_z^-: f(\xi, \eta) < z.$$

The set S_z is the projection on to this plane of the section of $\Sigma^\theta(R^\theta)$ by the parallel plane at height z above it. We may call Ω_z^+ , Ω_z^- the "positive region" and the "negative region" respectively: they are clearly separated in R by S_z .

If we suppose, for the moment, that S_z consists simply of a Jordan arc J_z it is clear that Banach's theorem enables us to interpret $P_x[z; \theta; R^\theta]$ as a total variation along J_z , namely

$$P_x[z; \theta; R^\theta] = \int N_x[y, z; \theta; R^\theta] dy = \int_{J_z} |dy|.$$

* It is to be noted that relative to the axes of ξ and η these sets are constant as θ varies. So also is R , the set on which $f(\xi, \eta)$ is defined.

Substituting $\xi \sin \theta + \eta \cos \theta$ for y and integrating we have

$$\begin{aligned}
 (2.82) \quad \frac{1}{2} \int_0^\pi P_x[z; \theta; R^\theta] d\theta &= \frac{1}{2} \int_0^\pi d\theta \int_{J_z} |d\xi \sin \theta + d\eta \cos \theta| \\
 &= \frac{1}{2} \int_{J_z} \sqrt{(d\xi^2 + d\eta^2)} \int_0^\pi |\sin(\theta + \psi)| d\theta \\
 &= \int_{J_z} \sqrt{(d\xi^2 + d\eta^2)} = L(J_z),
 \end{aligned}$$

where $\psi = \arctan d\eta/d\xi$ and $L(J_z)$ stands for the length of J_z . The last chain of equations is, in fact, the proof, in a summary form, of a well-known formula due to Cauchy*.

The extensions to rectifiable continua of Banach's theorem and of this formula are given, as Theorems 2 and 3, in the next paragraph. It will be shown later that we have, still more generally,

$$(2.83) \quad \frac{1}{2} \int_0^\pi P_x[z; \theta; R^\theta] d\theta = L(S_z) \quad (\text{almost all } z),$$

where $L(S_z)$ stands for the length of S_z according either to the definition of Carathéodory or to that of Gross†. The required generalization of (2.1), given below in Theorem 1, is now obtained by using (2.83) to simplify the formula (2.7) for $V_R[f]$.

In Theorem 4 (§11) we give the incidental results that, for almost all z , $L(S_z)$ is equal to the total length of the continua in S_z and is lower semi-continuous.

The second result in Theorem 1 is the generalization of the formula‡

$$(2.9) \quad \int_a^b |df(x)| = \lim_{h \rightarrow +0} \int_a^{b-h} \left| \frac{f(x+h) - f(x)}{h} \right| dx,$$

valid for a function $f(x)$ of bounded variation in (a, b) that is continuous at a and b .

THEOREM 1. *Let $f(x, y)$ be a continuous function of (x, y) in a closed convex region R , and let $V_R[f]$ be its total variation over R as defined in (1.6);*

* The special case where J_z is closed and convex is the most familiar: the formula then states that the average projection of the curve is $1/\pi$ times its length.

† Cf. Saks, 1, II, §8; also Gross, 1.

‡ Cf. Saks, 1, V. (8.10) and (8.11). For details and more general results, see Hardy and Littlewood, 1, especially Theorem 22.

also let the length of the section of the surface

$$\Sigma(R): z = f(x, y) \quad [(x, y) \text{ in } R]$$

made by the plane $z = u$ be denoted by $L(S_u)$. Then

$$(2.91) \quad V_R[f] = \int_{-\infty}^{\infty} L(S_z) dz,$$

$$(2.92) \quad V_R[f] = \lim_{\alpha, \beta \rightarrow 0} \iint_{R_{\alpha\beta}} \left\{ \left(\frac{f(x+\alpha, y) - f(x, y)}{\alpha} \right)^2 + \left(\frac{f(x, y+\beta) - f(x, y)}{\beta} \right)^2 \right\}^{\frac{1}{2}} dx dy,$$

where $R_{\alpha\beta}$ is the region consisting of those points (x, y) of R such that $(x+\alpha, y)$ and $(x, y+\beta)$ also lie in R . In each equation the two sides are finite or infinite together; and f is of bounded variation over R in Tonelli's sense when and only when $V_R[f]$ is finite.

We may note that in the case where f has continuous first derivatives (2.92) reduces to

$$(2.93) \quad V_R[f] = \iint_R \left\{ \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right\}^{\frac{1}{2}} dx dy,$$

which can be obtained from the definition of $V_R[f]$. The more general expression in (2.92) is similar in form to L. C. Young's formula* for the area of $\Sigma(R)$, and yields in the same way his necessary and sufficient condition for bounded variation*; it is deducible from the special case by his methods. These arguments will be developed in §12.

Since bounded variation of f in Tonelli's sense is a necessary and sufficient condition for the area of $\Sigma(R)$ to be finite† the theorem provides us with a further result.

COROLLARY. *A necessary and sufficient condition for the surface $\Sigma(R)$ to have a finite area is that*

$$(2.94) \quad \int_{-\infty}^{\infty} L(S_z) dz < \infty.$$

Lastly it will be shown in §13 that $V_R[f]$ is identical with the total variation over R of a certain vector function of rectangles.

* Cf. Saks, **1**, V, (8.4) and (8.5).

† Cf. Saks, **1**, V, (8.1); also Banach, **1**, Theorem 3, where the condition is stated in terms of the double integral expressions (2.5) for $W(0)$ and $W(\frac{1}{2}\pi)$.

List of notations occurring in more than one paragraph.

§ 1.	$f(x, y), R$		T_x, T_y	(1.11), (1.12)
	$\xi, \eta; \theta$	(1.3)	f^θ, R^θ	(1.4)
	$W(\theta)$	(1.5)	$V_R[f]$	(1.6)
§ 2.	$\Sigma^\theta(E)$	(2.2)	N_x, N_y	(2.31), (2.32)
	P_x, P_y	(2.6)	$S_z, \Omega_z^+, \Omega_z^-$	(2.81)
	L	(2.83)	$R_{\alpha\beta}$	(2.92)
§ 3.	L_G	(3.33)		
§ 5.	L_J^θ, L_C	(5.1)		
§ 6.	$U_1[f]$	Lemma 4.		
§ 7.	$U_2[f]$		$U_3[f]$	Lemma 6
§ 8.	C_z, I_z, c_z^i	(8.1), (8.2)	C_z^0, S_z^+, S_z^-	(8.7), (8.81)
§ 9.	$\epsilon, \delta; C_z^m$	(9.11)	m	(9.12)
	B_z	(9.22)		
§ 10.	R_ω, A_z	(10.11)		

3. Generalization of Banach's theorem and of Cauchy's formula.

We give first, although it is not strictly necessary for the proof of Theorem 1, the generalization of Banach's theorem to the case of a rectifiable continuum. We recall that, for a continuum c , the various different definitions (Carathéodory, Gross, Jordan, Lebesgue-Favard, etc.) of its length $L(c)$ are all equivalent. We note also that the total variation of y over a set in the (x, y) -plane can be defined in the same way as the Carathéodory length, which it will not exceed: it is then an additive function for all sets that are measurable with respect to it, in particular for continua and for (denumerable) sums and differences of continua*.

THEOREM 2. *If c is a rectifiable continuum in the (x, y) -plane and $N(u; E)$ is the number of intersections of the set E with the line $y = u$, then we have, for the total variation of y over c ,*

$$(3.1) \quad \int_c |dy| = \int_{-\infty}^{\infty} N(y; c) dy.$$

* Cf. Saks, **1**, II, §§ 4 and 8.

It is known* that a sequence $\{J_r\}$ of Jordan arcs can be constructed in such a way that

$$(3.21) \quad J_r \subset c \text{ (all } r), \quad L(c) = \sum_{r=1}^{\infty} L(J_r),$$

and each J_r has exactly one point in common with K_{r-1} , where

$$(3.22) \quad K_n = \sum_{r=1}^n J_r, \quad H_n = c - K_n.$$

The relation (3.1) holds if J_r replaces c , by Banach's theorem (2.1); hence also

$$(3.31) \quad \int_{K_n} |dy| = \sum_{r=1}^n \int_{J_r} |dy| = \sum_{r=1}^n \int N(y; J_r) dy = \int N(y; K_n) dy.$$

But the difference between the left-hand side and that of (3.1) is equal to

$$(3.32) \quad \int_{H_n} |dy| \leq L(H_n) = L(c) - L(K_n) = \sum_{r=n+1}^{\infty} L(J_r).$$

And the difference between the right-hand sides equal to

$$(3.33) \quad \int N(y; H_n) dy \leq L_G(H_n) = L(H_n),$$

where $L_G(H_n)$ is the length of H_n in the sense of Gross. In fact, since H_n is the difference of two rectifiable continua, it is linearly measurable, whence the inequality†, and $L_G(H_n)$ is the same as the length of H_n according to the other definitions. It is clear, from (3.32) and (3.33), that the two sides of (3.31) differ from the corresponding sides of (3.1) by arbitrarily small amounts if n is taken large enough; hence (3.1) follows from (3.31).

COROLLARY. *The equation (3.1) remains valid if c is replaced by a denumerable set of rectifiable continua whose total length is finite.*

In fact, K_n now consists of n Jordan arcs grouped in a finite number of continua, and the argument proceeds as before.

THEOREM 3. *If c is a rectifiable continuum in the (x, y) -plane and c^θ is the continuum obtained by rotating c about the origin through an angle θ , we have*

$$(3.4) \quad L(c) = \frac{1}{2} \int_0^\pi d\theta \int_{c^\theta} |dy|.$$

* Favard, 1, § 6.

† A proof of the corresponding inequality for sets superficially measurable in the sense of Gross is given by Schauder, 1, Theorem 3.

COROLLARY. *The equality (3.4) remains valid when c is replaced by a denumerable set of rectifiable continua whose total length is finite.*

The proofs of these results are on the same lines as those for Theorem 2 and its corollary. The two corollaries taken together yield the following.

If C is a finite or denumerably infinite set of rectifiable continua in the (x, y) -plane, and if their total length is finite, then

$$(3.5) \quad L(C) = \frac{1}{2} \int_0^\pi d\theta \int_{-\infty}^{\infty} N(y; C^\theta) dy.$$

4. Justification of the inversions (2.5) and (2.7).

It is sufficient, by Fubini's theorem*, to establish the following.

LEMMA 1. *The function† $N_x[y, z; \theta; R^\theta]$ is measurable in (y, z) -space for every θ , and also in (y, z, θ) -space.*

Banach's argument showing the measurability of the function $N(y)$ that appears in (2.1) can be adapted to prove this result. We give the proof for measurability in (y, z, θ) -space, the case when θ is constant being similar but easier‡.

Let us write $r = \sqrt{(x^2 + y^2)} = \sqrt{(\xi^2 + \eta^2)}$ and consider the annular regions

$$A_k^n: (k-1)2^{-n} \leq r < k \cdot 2^{-n}$$

in the plane $z = 0$, and the corresponding portions $\Sigma^\theta(A_k^n)$ of the surface $\Sigma^\theta(R^\theta)$. Let $L_k^n(y, z; \theta)$ be the characteristic function of the set $\alpha_k^n(\theta)$ in the (y, z) -plane obtained by projecting $\Sigma^\theta(A_k^n)$ upon it; and write§

$$(4.1) \quad N^n(y, z; \theta) = \sum_{k=1}^{\infty} L_k^n(y, z; \theta).$$

Then this function is equal to the number of intervals

$$\sqrt{\{(k-1)^2 2^{-2n} - y^2\}} \leq |x| < \sqrt{\{k^2 2^{-2n} - y^2\}}$$

containing values of x such that (x, y, z) lies on $\Sigma^\theta(R^\theta)$. Hence, as in

* Cf. Saks, **1**, III, (8.1).

† The function N_x was defined in (2.31) and R^θ below (1.4).

‡ This case has been dealt with by Banach, **1**, Theorem 3.

§ The series is actually terminating, since R^θ is a finite region.

Banach's argument, $N^n(y, z; \theta)$ cannot decrease as n increases; and further

$$N^n(y, z; \theta) \rightarrow N_x[y, z; \theta; R^0] \quad (n \rightarrow \infty).$$

It is consequently sufficient to show that $N^n(y, z; \theta)$ is measurable.

Now let B_k^n be the *closed* annulus corresponding to A_k^n ; then $\Sigma^0(B_k^n)$ is a closed set, and so also is its projection $\beta_k^n(\theta)$ on to the y - z -plane: further, as θ varies from 0 to π (inclusive), this projection moves continuously and clearly must trace out a closed set β_k^n in (y, z, θ) -space. Hence β_k^n is a measurable set in (y, z, θ) -space.

The same property must be possessed also by the set α_k^n traced out by $\alpha_k^n(\theta)$: for we can show that $\beta_k^n - \alpha_k^n$ is contained in a closed set of zero measure. In fact, $\beta_k^n(\theta) - \alpha_k^n(\theta)$ is contained in the projection $\gamma_k^n(\theta)$ of the set $r = k \cdot 2^{-n}$ on $\Sigma^0(R^3)$: the projection is thus a closed set in the y - z -plane that is met by each parallel to the z -axis in at most two points. The (superficial) set traced out by $\gamma_k^n(\theta)$ as θ varies therefore has the same property, and hence it must have zero measure.

We have shown that each term in (3.1) is the characteristic function of a measurable set: it follows that $N^n(y, z; \theta)$ is itself a measurable function.

5. Inequalities for the length of certain sets.

LEMMA 2. *For any Borel set E in the (ξ, η) -plane we have*

$$(5.11) \quad P_x[z; \theta; E] \leq L(ES_z),$$

$$(5.12) \quad P_x[z; \theta; E] \leq L_{J^0}(ES_z) \leq P_x[z; \theta; E] + P_y[z; \theta; E].$$

We employ L_G , L_{J^0} , L_C to denote the length of a set in the senses of Gross*, Jansen†, and Carathéodory respectively, and L as in § 2 to denote either L_G or L_C . The Jansen length depends upon the axes of x and y , and thus, for sets in the (ξ, η) -plane, upon θ . It is known† that

$$(5.2) \quad \frac{1}{3}L_{J^0} \leq L_G \leq L_C,$$

for all sets. Further, since $N_x[y, z; \theta; E]$ is equal to the number of points (x, y) in ES_z , and P_x is its integral with respect to y , (5.11) and the first inequality in (5.12) follow from a known result‡.

To prove the rest of (5.12) we note first that, if H is any set in the (x, y) -plane, the Lebesgue measures of the projections H_x , H_y (supposed

* Gross, 1.

† Cf. Schauder, 1, Introduction, and III, § 1.

‡ See foot-note †, p. 296.

measurable) of HS_z on to the x and y axes, respectively, satisfy the relations

$$\begin{aligned}\sqrt{(|H_x|^2 + |H_y|^2)} &\leq |H_x| + |H_y| \\ &\leq \int N_y[x, z; \theta; H] dx + \int N_x[y, z; \theta; H] dy \\ &= P_y[z; \theta; H] + P_x[z; \theta; H].\end{aligned}$$

Hence, if the (x, y) -plane is subdivided by lines $x = i \cdot 2^{-n}$, $y = k \cdot 2^{-n}$, into half-open squares p_{ik} , and if $H^{ik} = Ep_{ik}$, since this set clearly has measurable projections, we have,

$$(5.3) \quad \sum_{i, k} \sqrt{(|H_x^{ik}|^2 + |H_y^{ik}|^2)} \leq P_y[z; \theta; E] + P_x[z; \theta; E].$$

For E is the sum of the disjoint sets H^{ik} , so that

$$N_y[x, z; \theta; E] = \sum_{i, k} N_y[x, z; \theta; H^{ik}],$$

and similarly for N_x . But $L_J^\theta(ES_z)$ is defined* to be the limit, as $n \rightarrow \infty$, of the sum in (5.3). Thus $L_J^\theta(ES_z)$ cannot exceed the right-hand side of (5.1).

6. Finiteness of the various integrals.

Since, by (2.5) and (2.6),

$$W(\theta) = \int P_x[z; \theta; R^\theta] dz, \quad W(\theta + \tfrac{1}{2}\pi) = \int P_y[z; \theta; R^\theta] dz,$$

we have, as an immediate consequence of (5.1),

$$(6.1) \quad W(\theta) \leq \int L(S_z) dz, \quad \int L_J^\theta(S_z) dz \leq W(\theta) + W(\theta + \tfrac{1}{2}\pi);$$

and so, integrating with respect to θ and remembering that

$$W(\theta + \pi) = W(\theta),$$

we get, writing L_J for the mean value of L_J^θ ,

$$(6.2) \quad 2V_R[f] \leq \pi \int L(S_z) dz, \quad \pi \int L_J(S_z) dz \leq 4V_R[f].$$

Now the right-hand side of (6.1) is the sum of a pair of integrals such as that in (1.2), whose finiteness (for some θ) constitutes the bounded variation property for f : hence we have†

* It is to be noted that S_z is closed, and so ES_z is a Borel set.

† We have assumed the above integrands to be measurable: in fact we shall show in § 11 (Theorem 4), by arguments independent of the present ones, that in every case covered by Lemma 3 all four integrands are equivalent to a lower semi-continuous function of z .

LEMMA 3. *If $f(x, y)$ is continuous in R the following statements are equivalent, provided that $L_f^{\theta}(S_z) = L(S_z)$ for almost all z and all θ :*

(6.31) *$f(x, y)$ is of bounded variation in Tonelli's sense in R ;*

(6.32) *$L(S_z)$ is a summable function of z over $(-\infty, \infty)$;*

(6.33)
$$V_R[f] < \infty.$$

With this, subject to verification of the proviso by Theorem 4, the concluding assertion of Theorem 1 has been proved.

We note the following consequence of (5.12), which is rather weaker than (6.1) and (6.2), and which requires no assumptions about measurability.

LEMMA 4. *When either side of (2.91) is finite, or when (6.31) holds, if ES_z is a sum of continua, then $L(ES_z)$ is finite for all values of z outside a set $U_1 = U_1[f]$ of zero measure, and for such values $P_x[z; \theta; E]$ is finite for all θ .*

7. Other exceptional sets of values of z .

Let $U_2 = U_2[f]$ be the set of values assumed by $f(x, y)$ at points where it has a maximum or minimum (in the broad sense). Then U_2 is at most denumerably infinite.

For if we divide the (x, y) -plane into squares of side 2^{-n} as in §5 and take the upper and lower bounds of $f(x, y)$ on these squares we get a set of numbers that will clearly contain any assigned maximum (or minimum) value of $f(x, y)$ if n is sufficiently large: and the sum of these sets (for all values of n) is plainly a denumerable set.

LEMMA 5. *If S_z , Ω_z^+ , and Ω_z^- are defined as in (2.81) and z does not belong to the set $U_2[f]$ defined above, every point of S_z is a common boundary point of the two sets Ω_z^+ , Ω_z^- .*

For, if $\xi = \alpha$, $\eta = \beta$ gives a point on S_z , $f(\xi, \eta)$ does not have a maximum or minimum at (α, β) and hence takes in any neighbourhood of (α, β) both values greater than $z = f(\alpha, \beta)$ and values less than z .

Lastly we consider a third exceptional set.

LEMMA 6. *If we write $F(R)$ for the boundary of the convex region R , the length of the set $S_z F(R)$ is zero unless z belongs to a set $U_3[f]$ that is at most denumerably infinite.*

To prove this we observe that $F(R)$, being a convex curve, has a finite length, and that it is the sum of the sets $S_z F(R)$, which are disjoint. It follows that only a finite number of these sets can have lengths exceeding an assigned magnitude: hence the assertion in the lemma.

8. The structure of S_z .

Throughout this and the following three paragraphs we make the assumption that z does not belong to $U_1[f] + U_2[f] + U_3[f]$.

We write first

$$(8.1) \quad S_z = C_z + I_z + Q_z,$$

where C_z is the sum of the continua in S_z , I_z consists of those points of S_z that are enclosed by an infinite number of its continua, and Q_z is the remainder.

Since z does not belong to U_1 , $L(C_z)$ is finite. Now let c_z^i ($i = 1, 2, \dots, \nu$) be continua belonging to C_z . Then

$$(8.2) \quad L(C_z) \geq L(\Sigma c_z^i) = \Sigma L(c_z^i) \quad (1 \leq i \leq \nu),$$

since the continua c_z^i must be separated by positive distances.

From (8.2) it follows that only a finite number of continua in C_z can have length greater than an assigned number; hence C_z consists of a denumerable set of continua c_z^i and we may write*

$$(8.3) \quad C_z = \sum_{i=1}^{\infty} c_z^i; \quad L(C_z) = \sum_{i=1}^{\infty} L(c_z^i).$$

As for I_z , we note that for any n it is enclosed by the sum of the continua c_z^i ($i \geq n$). Clearly, if we select a suitable subsequence, each continuum encloses one of a number of non-overlapping parts of I_z : and the diameter of each part is less than the length of the continuum enclosing it. But, for sufficiently large n , we have

$$(8.4) \quad \sum_{i=n}^{\infty} L(c_z^i) < \epsilon.$$

Hence the parts have diameters less than ϵ , and the sum of these diameters does not exceed ϵ . Hence, since L_G never exceeds L_C , we must have

$$(8.5) \quad L_G(I_z) = L_C(I_z) = 0.$$

* The second equation in (8.3) holds for any outer linear measure: cf. Saks, I, II, (4.6).

Coming to Q_z , we see from Lemma 5 in § 7 that every point on it is a common boundary point of Ω_z^+ and Ω_z^- . Now if P is a point of Q_z it cannot belong to the common boundary of a domain of Ω_z^+ and a domain of Ω_z^- ; for in that case it would lie on a continuum bounding one of these domains* and so would belong to C_z . Again, since P is not in C_z , we can find arbitrarily small polygons $\pi(P)$ that enclose it without meeting S_z ; thus each polygon must lie in a domain† belonging to one or other of the regions Ω_z^+ , Ω_z^- . Two polygons lying one in each of two domains must be separated by a continuum of S_z . Since P is not enclosed by an infinite number of such continua all sufficiently small polygons $\pi(P)$ must lie in a single domain of Ω_z^+ or Ω_z^- . Clearly P itself belongs to the boundary of this domain; if the domain is in Ω_z^+ we shall say that P belongs to Q_z^+ , otherwise to Q_z^- . Thus

$$(8.6) \quad Q_z = Q_z^+ + Q_z^-; \quad Q_z^+ Q_z^- = 0.$$

We observe, lastly, that, since $L(I_z) = 0$, no continuum of C_z can be contained in I_z . Hence the sub-set C_z^0 of C_z that consists of continua not meeting the boundary of R can be resolved in the same way as Q_z :

$$(8.7) \quad C_z^0 = C_z^+ + C_z^-, \quad C_z^+ C_z^- = 0,$$

where every polygon enclosing a continuum of C_z^+ (or C_z^-) without meeting S_z must be in Ω_z^+ (or Ω_z^-) if its points all lie within a sufficiently small distance from the continuum. We now write

$$(8.81) \quad S_z^+ = C_z^+ + Q_z^+, \quad S_z^- = C_z^- + Q_z^-,$$

so that

$$(8.82) \quad S_z^+ + S_z^- = C_z^0 + Q_z, \quad S_z^+ S_z^- = 0.$$

9. A semi-continuity property for part of C_z .

Now let ϵ and δ be (small) positive numbers, to be kept fixed until § 11 except at (9.23), let

$$(9.11) \quad C_z^m = c_z^1 + c_z^2 + \dots + c_z^m,$$

and take m equal to $m(\epsilon)$, the smallest integer such that

$$(9.12) \quad L(c_z^i) < \epsilon \quad [i > m(\epsilon)].$$

* Cf. v. Kerékjártó, 1, 37.

† If P is on the boundary of R , $\pi(P)$ does not lie wholly in R ; in such a case it will be said to lie in Ω_z^+ (or Ω_z^-) if $R\pi(P)$ so lies.

We now consider the sub-set of S_z whose points lie at distances not less than δ from C_z^m ; since $c_z^1, c_z^2, \dots, c_z^m$ are maximal continua of S_z they can be separated in R from this sub-set and from each other by a finite number of polygons lying in the set $R - S_z$, which is open relative to R . We may further suppose that these polygons do not meet and that their sides are segments of the lines

$$\xi = i \cdot 2^{-n}, \quad \eta = k \cdot 2^{-n} \quad (i, k = 0, \pm 1, \pm 2, \dots),$$

for some n ; δ and m being already fixed, we take n to be the least integer for which this is possible. The polygons then divide R into domains*: for $i \leq m$ let B_z^i be formed by adding its boundary to the domain† containing c_z^i . We now have

$$(9.21) \quad B_z^i \text{ is closed}; \quad B_z^i B_z^j = 0 \quad (i \neq j); \quad c_z^i \subset B_z^i; \quad (i, j \leq m).$$

Moreover, the boundary of B_z^i does not meet S_z .

If we now define

$$(9.22) \quad B_z = B_z(m, \delta) = B_z^1 + B_z^2 + \dots + B_z^m,$$

it is clear from the above construction that no point of B_z is at a greater distance than δ from C_z^m and that B_z contracts as δ decreases. Thus

$$(9.23) \quad C_z^m \subset B_z(m, \delta) \rightarrow C_z^m \text{ monotonely as } \delta \rightarrow 0.$$

We proceed to establish a kind of lower semi-continuity for $L(C_z^m)$ as a function of z , namely

$$(9.3) \quad L(C_z^m) \leq \frac{1}{2} \liminf L(C_{z+h} B_z) + \frac{1}{2} \liminf L(C_{z-h} B_z) \quad (h \rightarrow +0).$$

In virtue of (9.11), (9.21), and (9.22) this follows at once from the inequalities

$$(9.4) \quad L(c_z^i) \leq \frac{1}{2} \liminf L(C_{z+h} B_z^i) + \frac{1}{2} \liminf L(C_{z-h} B_z^i) \quad (1 \leq i \leq m),$$

which we now prove.

First we take‡

$$(9.5) \quad h < h_0 = h_0(z, m) = \min |f(\xi, \eta) - z| \quad [(\xi, \eta) \text{ on boundary of } B_z].$$

* For convenience the word *domain* will be used for any connected set that is open relative to R .

† In order that B_z^i may be uniquely defined we suppose that it is the *minimal* region obtainable by the procedure specified.

‡ From the remark below (9.21) it follows that $h_0 > 0$.

Then C_{z+h} can have no points on this boundary. Since z is outside the set U_1 , we have, by Lemma 4,

$$\int N_x[y, z; \theta; C_z] dy = P_x[z; \theta; C_z] \leq L(C_z) < \infty \quad (\text{all } \theta).$$

Let us choose any particular value of θ , to be kept fixed until (9.7).

Then $N_x[y, z; \theta; c_z^i]$ is finite for almost every value of y . Taking such a value, we shall have N_x values of x such that the point* (x, y) lies in the continuum c_z^i ; let these points be

$$(9.61) \quad H_r: (x_r, y), \quad x_r < x_{r+1} \quad (1 \leq r \leq N_x);$$

so that $z = f^\theta(x_r, y)$ for every r concerned. If c_z^i has points on the boundary $F(R)$ of R we ignore them; this is legitimate since by Lemma 6 we have $L[S_z F(R)] = 0$.

Now choose points

$$(9.62) \quad H'_r: (x'_r, y), \quad x'_{2r} < x_r < x'_{2r+1} \quad (1 \leq r \leq N_x),$$

such that the segments $H'_{2r}H_r$ and $H_rH'_{2r+1}$ lie in B_z^i , and write

$$(9.63) \quad h_i = h_i(y, z) = \min |f^\theta(x'_s, y) - z| \quad (2 \leq s \leq 2N_x + 1).$$

Then, if

$$(9.64) \quad 0 < h < h_i(y, z), \quad 0 < h' < h_i(y, z),$$

every segment $H'_{2r}H_r$ or $H_rH'_{2r+1}$ must meet the set $(C_{z+h} + C_{z-h'})B_z^i$ at least once. For, if H'_{2r} is in Ω_z^+ and therefore, by (9.63) and (9.64), in Ω_{z+h}^+ also, while H_r is in S_z and, *a fortiori*, in Ω_{z+h}^- also, then the two points are in domains separated in R by S_{z+h} and hence† also by C_{z+h} . Similarly for $H_rH'_{2r+1}$ and for the cases where H'_{2r} or H'_{2r+1} is in Ω_z^- . But the statement at (9.64) implies that, for almost all y ,

$$2N_x[y, z; \theta; c_z^i] \leq N_x[y, z+h; \theta; C_{z+h}B_z^i] + N_x[y, z-h'; \theta; C_{z-h'}B_z^i]$$

when $0 < h, h' < h_i(y, z)$, and it follows that, as $h \rightarrow +0$,

$$N_x[y, z; \theta; c_z^i] \leq \frac{1}{2} \lim N_x[y, z+h; \theta; C_{z+h}B_z^i] + \frac{1}{2} \lim N_x[y, z-h; \theta; C_{z-h}B_z^i].$$

* I.e. the point $\xi = x \cos \theta + y \sin \theta$, $\eta = -x \sin \theta + y \cos \theta$ in the (ξ, η) -plane.

† For a set of points that separates (in R) two domains must contain a continuum that separates them (in R).

On integrating with respect to y and using Fatou's lemma*, we get

$$(9.7) \quad P_x[z; \theta; c_z^i] \leq \frac{1}{2} \liminf P_x[z+h; \theta; C_{z+h} B_z^i] \\ + \frac{1}{2} \liminf P_x[z-h; \theta; C_{z-h} B_z^i].$$

But, by (3.5),

$$\frac{1}{2} \int_0^\pi P_x[z; \theta; c_z^i] d\theta = L(c_z^i).$$

Similarly

$$\frac{1}{2} \int_0^\pi P_x[z \pm h; \theta; C_{z \pm h} B_z^i] d\theta = L(C_{z \pm h} B_z^i) \quad [h < \min(h_0, h_i)],$$

since, by (9.5), (9.63), and (9.64), the boundary of B_z^i does not dissect any of the continua of $C_{z \pm h}$. Hence, again by Fatou's lemma, (9.7), which holds for all θ , yields the desired inequality (9.4).

10. An approximation for the length of another part of S_z .

Let R_ω be the closed convex region consisting of all points of R whose distances from its boundary $F(R)$ are not less than ω , where ω is a number not less than ϵ , that will be kept fixed until after (11.2): and write

$$(10.11) \quad A_z = A_z(m, \delta) = R - B_z(m, \delta).$$

The object of the present paragraph is to show that

$$(10.12) \quad L^\epsilon(S_z A_z R_\omega) \leq \frac{1}{2} \liminf L(C_{z+h} A_z) + \frac{1}{2} \liminf L(C_{z-h} A_z) \quad (h \rightarrow +0),$$

where, for any set E , $L^\epsilon(E)$ is the ϵ -approximation† to $L_G(E)$.

Clearly A_z is open relative to R and R_ω ; moreover, no boundary-point of A_z interior to R belongs to S_z . Hence $S_z A_z R_\omega$ is closed. Again, by (9.12), A_z contains only such continua of C_z as are less than ϵ in length: hence $A_z R_\omega$ contains only such as do not meet the boundary of R , or, in the notation of § 8, we have $C_z A_z R_\omega = C_z^0 A_z R_\omega$. Thus, using (8.1) and (8.82), we see that

$$(10.2) \quad S_z A_z R_\omega = S_z^+ A_z R_\omega + S_z^- A_z R_\omega + I_z A_z R_\omega.$$

Now, in virtue of the construction of S_z^+ , every component of $S_z^+ A_z R_\omega$ can be enclosed by a polygon arbitrarily close to it lying in

* Cf. Saks, 1, I, (12.10).

† I.e. the (exact) lower bound of the sum of diameters, less than ϵ , of sets that together cover E .

$\Omega_z^+ A_z$; similarly with $S_z^- A_z R_\omega$ and $\Omega_z^- A_z$, and also with $I_z A_z R_\omega$ and $(\Omega_z^+ + \Omega_z^-) A_z$. But all components of $S_z A_z R_\omega$ have diameters less than ϵ , and so these enclosing polygons can be chosen to have the same property. The interiors of such polygons are open sets that together cover the closed set $S_z A_z R_\omega$; hence, by the Heine-Borel theorem, the latter can be covered by a finite number of them.

In particular, $S_z^+ A_z R_\omega$ can be enclosed by a finite set Π of polygons, lying in $\Omega_z^+ A_z$, with diameters less than ϵ . Now let us write

$$(10.3) \quad k = k(z; \epsilon, \delta, \omega) = \min |f(\xi, \eta) - z| \quad [(\xi, \eta) \text{ on } \Pi].$$

Then for $0 < h < k$ we have Π still lying in $\Omega_{z+h}^+ A_z$, while $S_z^+ A_z R_\omega$ lies in $\Omega_{z+h}^- A_z$. Hence the sub-set of $S_z^+ A_z R_\omega$ lying inside each polygon of Π must be separated from Π by S_{z+h} and hence must be enclosed by continua belonging to $C_{z+h} A_z$. Suppose that c_{z+h}^i is one of these continua when i is in the increasing sequence $\{i_s^+\}$; and let K_{z+h}^i be the part of $S_z^+ A_z R_\omega$ enclosed by c_{z+h}^i . Then for the diameters we have, successively,

$$\text{diam } K_{z+h}^i \leq \frac{1}{2} L(c_{z+h}^i);$$

$$(10.41) \quad \Sigma^+ = \Sigma \text{diam } K_{z+h}^i < \frac{1}{2} \Sigma L(c_{z+h}^i) \leq \frac{1}{2} L(C_{z+h} A_z),$$

where the summation is for $i = i_1^+, i_2^+, \dots$, and $0 < h < k$. Similarly for $S^- A_z R_\omega$ we get, when h' is sufficiently small,

$$(10.42) \quad \Sigma^- = \Sigma \text{diam } K_{z-h'} \leq \frac{1}{2} L(C_{z-h'} A_z) \quad (i = i_1^-, i_2^-, \dots).$$

Since their terms do not exceed ϵ we have*, for the sums Σ^+ and Σ^- ,

$$(10.43) \quad \Sigma^+ + \Sigma^- \geq L^e[(S_z^+ + S_z^-) A_z R_\omega] \geq L^e(S_z A_z R_\omega) - L^e(I_z A_z R_\omega)$$

in virtue of (10.2). But

$$L^e(I_z A_z R_\omega) \leq L_C(I_z A_z R_\omega) \leq L_C(I_z) = 0,$$

by (8.5). Thus, letting h and h' tend to 0, we get, from (10.41), (10.42), and (10.43),

$$L^e(S_z A_z R_\omega) \leq \frac{1}{2} \underline{\lim} L(C_{z+h} A_z) + \frac{1}{2} \underline{\lim} L(C_{z-h} A_z),$$

which is (10.12).

* We use here, besides the definition of L^e , the fact that $L^e(X) + L^e(Y) \geq L^e(X + Y)$ for any sets X, Y : cf. Saks, **1**, II, § 8.

11. Equality of $L(S_z)$ and $L(C_z)$ for almost all z .

Combining the inequalities (9.3) and (10.12), we get

$$\begin{aligned} L(C_z^m) + L^\epsilon(S_z A_z R_\omega) &\leq \frac{1}{2} \underline{\lim} L(C_{z+h} B_z) + \frac{1}{2} \underline{\lim} L(C_{z-h} B_z) \\ &\quad + \frac{1}{2} \underline{\lim} L(C_{z+h} A_z) + \frac{1}{2} \underline{\lim} L(C_{z-h} A_z) \\ (11.1) \quad &\leq \frac{1}{2} \underline{\lim} L(C_{z+h}) + \frac{1}{2} \underline{\lim} L(C_{z-h}). \end{aligned}$$

For, in virtue of (10.11) and of the fact that B_z is closed,

$$L(C_{z \pm h} A_z) + L(C_{z \pm h} B_z) = L(C_{z \pm h}).$$

Now, by (9.23), the set $A_z = A_z(m, \delta)$, which is open relative to R , expands to $R - C_z^m$ as limit when δ decreases to 0; hence*

$$L^\epsilon(S_z A_z R_\omega) \rightarrow L^\epsilon[S_z(R - C_z^m) R_\omega] = L^\epsilon[(S_z - C_z^m) R_\omega] \quad (\delta \rightarrow 0).$$

Thus (11.1) yields

$$(11.2) \quad L(C_z^m) + L^\epsilon[(S_z - C_z^m) R_\omega] \leq \frac{1}{2} \underline{\lim} L(C_{z+h}) + \frac{1}{2} \underline{\lim} L(C_{z-h}).$$

Now the left-hand side is not less than

$$L^\epsilon(C_z^m R_\omega) + L^\epsilon[(S_z - C_z^m) R_\omega] \geq L^\epsilon(S_z R_\omega),$$

$$\text{and}^\dagger \quad \lim_{\omega \rightarrow 0} \lim_{\epsilon \rightarrow 0} L^\epsilon(S_z R_\omega) = \lim_{\omega \rightarrow 0} L_C(S_z R_\omega) = L_C(S_z),$$

since $S_z R_\omega$ increases to limit S_z as ω decreases to 0. We thus get from (11.2), since its right-hand side involves neither ϵ nor ω ,

$$(11.3) \quad L_C(S_z) \leq \frac{1}{2} \underline{\lim} L(C_{z+h}) + \frac{1}{2} \underline{\lim} L(C_{z-h}) \quad (h \rightarrow +0).$$

This has been proved under the assumption, made at the beginning of §8, that z does not belong to the set $U_1 + U_2 + U_3$; thus it is valid for almost all z . But the two lower limits in (11.3) can only differ‡ for a denumerable set of values of z . Thus we arrive at the following:

$$(11.4) \quad L(C_z) \leq L_C(S_z) \leq \underline{\lim}_{h \rightarrow 0} L(C_{z+h}) \quad (\text{almost all } z).$$

* We have, in fact, for any expanding sequence of Borel sets $\{X_r\}$ with limit X , and for any set E , $L^\epsilon(E X_r) \rightarrow L^\epsilon(E X)$ ($r \rightarrow \infty$). For plainly

$$0 \leq L^\epsilon(E X) - L^\epsilon(E X_r) \leq L^\epsilon[E(X - X_r)] \leq L_C[E(X - X_r)] \rightarrow 0.$$

Cf. Saks, **1**, II; §8, (7.4), and (4.6).

† The condition $\omega \geq \epsilon$ laid down in §10 is here respected.

‡ Saks, **1**, IX, (1.1).

But the extreme terms here can differ only* for a denumerable set of values of z , and so, since $L(C_z) \leq L_G(S_z) \leq L_C(S_z)$, and similarly for L_J^θ , it is clear that

$$(11.5) \quad L(C_z) = L_G(S_z) = L_J^\theta(S_z) = L_C(S_z) \quad (\text{almost all } z).$$

The results (11.4) and (11.5) may be summed up as follows.

THEOREM 4. *If either side of (2.91) is finite, or if $f(x, y)$ is of bounded variation over R , then, for almost all z ,*

(i) *the length of S_z , in the sense either of Gross, of Jansen, or of Carathéodory, is equal to the length of the set C_z of continua contained in S_z ;*

(ii) *this length is a lower semi-continuous function of z .*

We are now in a position to complete the proof of (2.91); for this it only remains to establish the equality (2.83).

In virtue of (11.5) we have, for almost all z , $L(S_z - C_z) = 0$, so that, by Lemma 2, $P_x[z; \theta; S_z - C_z] = 0$ for all θ , and hence

$$(11.6) \quad \int_0^\pi P_x[z; \theta; S_z] d\theta - \int_0^\pi P_x[z; \theta; C_z] d\theta = \int_0^\pi P_x[z; \theta; S_z - C_z] d\theta = 0.$$

But, by (3.5),

$$L(C_z) = \frac{1}{2} \int_0^\pi d\theta \int N_x[y, z; \theta; C_z] dy = \frac{1}{2} \int_0^\pi P_x[z; \theta; C_z] d\theta.$$

Hence, by (11.5) and (11.6),

$$\begin{aligned} L(S_z) &= \frac{1}{2} \int_0^\pi P_x[z; \theta; S_z] d\theta \\ &= \frac{1}{2} \int_0^\pi P_x[z; \theta; R^\theta] d\theta \quad (\text{almost all } z), \end{aligned}$$

and this is the required relation (2.83).

12. Proof of (2.92).

In order to prove (2.92) we observe that†, from (1.11),

$$\begin{aligned} (12.1) \quad T_x[y; f^\theta; R^\theta] &= \int |df^\theta(t, y)| \quad [(t, y) \text{ in } R^\theta] \\ &= \lim_{r \rightarrow 0} \frac{1}{r} \int |f^\theta(t+r, y) - f^\theta(t, y)| dt, \end{aligned}$$

* See the preceding footnote.

† We use here the relation (2.9).

where it is supposed that $r > 0$ and that $(t+r, y)$ is also in R^g . Moreover, it is easily shown that the limit must be approached from below; hence, recalling (1.5), we have

$$\begin{aligned}(12.2) \quad W(\theta) &= \int T_x[y; f^\theta; R^g] dy \\ &= \lim_{r \rightarrow 0} \frac{1}{r} \int dy \int |f^\theta(t+r, y) - f^\theta(t, y)| dt \\ &= \lim_{r \rightarrow 0} \frac{1}{r} \iint_{R_{r,0}^g} |f^\theta(t+r, y) - f^\theta(t, y)| dt dy.\end{aligned}$$

With the same justification we may write

$$\begin{aligned}(12.31) \quad V_R[f] &= \frac{1}{2} \int_0^\pi W(\theta) d\theta = \frac{1}{4} \int_0^{2\pi} W(\theta) d\theta \\ &= \lim_{r \rightarrow 0} \frac{1}{4r} \int_0^{2\pi} d\theta \iint_{R_{r,0}^g} |f^\theta(t+r, y) - f(t, y)| dt dy \\ &= \lim_{r \rightarrow 0} \frac{1}{4r} \int_0^{2\pi} d\theta \iint_{R_r} |f(\xi+r \cos \theta, \eta-r \sin \theta) - f(\xi, \eta)| d\xi d\eta \\ &= \lim_{r \rightarrow 0} \frac{1}{2} \pi \iint_{R_r} m_r(\xi, \eta) d\xi d\eta,\end{aligned}$$

where R_r is defined like R_ω in § 10, so that $R_{r,0}^g - R_r^g \rightarrow 0$ with r , and

$$(12.32) \quad m_r(\xi, \eta) = \frac{1}{2\pi r} \int_0^{2\pi} |f(\xi+r \cos \theta, \eta-r \sin \theta) - f(\xi, \eta)| d\theta.$$

The function $m_r(\xi, \eta)$ is clearly *the average slope of a chord drawn from the point (ξ, η, z) and having its base* of length r .*

When $f(x, y)$ has continuous derivatives we obtain, by an elementary argument,

$$(12.4) \quad \pi m_r \rightarrow 2 \left\{ \left(\frac{\partial f}{\partial \xi} \right)^2 + \left(\frac{\partial f}{\partial \eta} \right)^2 \right\}^{\frac{1}{2}} \quad (r \rightarrow 0)$$

uniformly in (ξ, η) , *i.e.* the limit of the average slope as the base of the chord tends to 0 is equal to $2/\pi$ times the maximum slope. We then have, from (12.31) and (12.4),

$$V_R[f] = \iint_R \left\{ \left(\frac{\partial f}{\partial \xi} \right)^2 + \left(\frac{\partial f}{\partial \eta} \right)^2 \right\}^{\frac{1}{2}} d\xi d\eta,$$

* *I.e.* its projection on to the plane $z = 0$,

which is the same as (2.93). As already indicated in § 2, the more general formula can be deduced from this if integral means of $f(x, y)$ are employed to yield approximations to it that have continuous derivatives. The arguments can be derived from those relating to the area of the surface* by simply omitting the third term under the square root.

13. $V_R[f]$ as the total variation of a function of rectangles.

The total variation of a function of one variable is employed to yield the decomposition of a function of bounded variation into the sum of two monotonic functions. Such properties as this must find their two-dimensional analogues, so far as Tonelli's definition of bounded variation is concerned†, in properties of additive functions of *rectangles* associated with a function of points $f(x, y)$.

In what follows I stands for a rectangle with sides parallel to the axes of x and y .

For the rectangle

$$(13.1) \quad I: a_1 \leq x \leq a_2, b_1 \leq y \leq b_2,$$

we define

$$(13.2) \quad \eta_x(I) = \int_{b_1}^{b_2} \{f(a_2, y) - f(a_1, y)\} dy, \quad \eta_y(I) = \int_{a_1}^{a_2} \{f(x, b_2) - f(x, b_1)\} dx.$$

We now consider the vector $\eta(I)$, in the (x, y) -plane, whose components are $\eta_x(I)$ and $\eta_y(I)$: $\eta(I)$ is clearly the resolved part perpendicular to the z -axis of the vector area‡ of the circuit traced out by the point $[x, y, f(x, y)]$ as (x, y) describes the boundary of I . If $|\eta(I)|$ is taken to denote the magnitude of $\eta(I)$ the usual formulae defining the total variation of a function of rectangles§ can be applied to the vector function $\eta(I)$. We establish

THEOREM 5. *If $f(x, y)$ is a continuous function of (x, y) in the closed convex region R and if the vector-function $\eta(I)$ is defined as above, then $V_R[f]$ is the total variation over R of $\eta(I)$.*

It is sufficient to remark that, if the integrands in (13.2) are replaced by their absolute values, we get the expressions $G_2(f; I)$ and $G_1(f; I)$ of

* Cf. Saks, 1, V, §8.

† For properties following from other definitions see Clarkson and Adams, 2.

‡ In the sense of W. H. Young.

§ Cf. Saks, 1, III, §4.

Geöcze, and the arguments employed by Radó, Tonelli, and L. C. Young* to deal with these expressions and with the derived expression

$$G(f; I) = \{G_1^2 + G_2^2 + 1\}^{\frac{1}{2}}$$

can be adapted without trouble to show that the total variation of $\eta(I)$ is equal to the expression on the right-hand side of (2.92) and hence also to $V_R[f]$.

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* Cf. Saks, **1**, V, §§ 5-8.

ON THE ISOLATED UNITED POINTS OF A CYCLIC INVOLUTION ON AN ALGEBRAIC SURFACE

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[Received and read, 20 April, 1939.]

This paper is concerned with the solution of certain problems connected with the isolated united points of a cyclic involution, of order n say, on an algebraic surface. Almost all previous work on this subject has been done by Godeaux in a large number of papers and in particular in a monograph† to which we shall make frequent reference.

A cyclic involution I_n , of order n on an algebraic surface F , is, by definition, generated by repetitions of a cyclic self-transformation T , of period n of F . If O is an isolated united point of I_n , T sets up a cyclic self-transformation of the first neighbourhood of O whose period, δ say, may be n or any factor of n . We call O a δ -perfect or *imperfect* united point of I_n according as $\delta < n$ or $\delta = n$ (and we speak of a 1-perfect point as *perfect*). If O is not perfect, there are two isolated united points of I_n in its first neighbourhood, which may be classified in the same way; and if either of these is not perfect, the same applies to its first neighbourhood, and so on. Since every point in the first neighbourhood of a perfect united point is united, there is no *particular* point in its neighbourhood to be considered. But we find it convenient, when a perfect united point occurs, to choose a particular united point in its neighbourhood by certain conventions. The first problem is to determine the positions of the united points of I_n in the neighbourhoods of O and the numbers δ associated with them. The solution is as follows:

If $O, O_1, O_2, \dots, O_m$, is a sequence of united points of I_n in the neighbourhoods of O , O_i being in the first neighbourhood of O_{i-1} , it is found that only the first few points of the sequence are free and after that all are satellites.

† Godeaux (1).

Furthermore, the free points all belong to one of two particular sequences. With a suitable choice of local parameters u, v , in the neighbourhood of O , the equations of a branch of least order through these points may be written in the form

$$u = t^X, \quad v = t^Y P(t),$$

where $P(t)$ is a power series in t and $P(0) \neq 0$. The numbers X, Y completely determine the position of the point O_m in the neighbourhood of O ; we therefore denote O_m by (X, Y) . In this paper X, Y are found in the first place as geometrical characters of the branch of least order through O_m .

The transformation T generating the involution has in the neighbourhood of O the form

$$\left. \begin{aligned} u_1 &= \epsilon^\beta u + P(u, v), \\ v_1 &= \epsilon^\alpha v + Q(u, v), \end{aligned} \right\}$$

where ϵ is a primitive n -th root of unity, P and Q are power series having zeros of order two at $u = v = 0$, and α, β , and n do not have a factor common to all three. The integers α, β are found to characterise the whole neighbourhood of O with respect to the involution and the solution of the first problem is given by the fact that the point (X, Y) is a δ -perfect united point of I_n , where δ is the least integral solution of the congruence

$$\delta(\alpha X - \beta Y) \equiv 0 \pmod{n}.$$

By means of this congruence we define certain matrices characterising the neighbourhood of O and in terms of these matrices we solve the second problem, which is:

Given a linear system $|C|$ of curves on F , satisfying certain conditions, compounded of I_n and not having a base-point at O , to discover the nature of the (virtual) base-points in the neighbourhoods of O of sub-systems of $|C|$ which have base-points at O .

To describe the solution of this problem here would require too much space. It is sufficient to say that it is given in the form of a matrix equation whose sets of integral solutions give the multiplicities of the sub-systems at the various united points of I_n .

The third problem is to determine the nature of the image point S of the point O on a projective model Φ of the system $|C|$. And this problem is also completely solved in that we are able to give a complete description of

the singularity of Φ at S in terms of numbers deduced by an algorithmic process from α , β , and n . The solution is, however, dependent on the assumption that a certain sub-system of $|C|$, whose *virtual* existence we prove in solving the second problem, is an *actual* system. The division of the paper into three parts corresponds to these three problems.

Just as this paper owes much to Godeaux's researches, so also it makes great use of the work of Dr. Du Val on the neighbourhoods of both simple and singular points of surfaces. The greater part of this work is unfortunately not yet published, but I have been privileged to see the essay† in which much of it is at present embodied. Some of Du Val's earlier work on the neighbourhoods of simple points of surfaces has been published‡, but even this work is expressed in a much better way in the unpublished essay. Whenever possible, I have indicated where I have used theorems and ideas of Du Val, while at the same time proving the particular cases which I require by other methods more easily applicable to the present work; but it must be stated that a very large part of this paper has been influenced by his work in a way that cannot be indicated by quotations.

PART I.

The neighbourhoods of an isolated united point.

1. Let T be a cyclic self-transformation of period n of an algebraic surface F . Then, if P is any point of F , the points $T^i(P)$, $i = 0, 1, 2, \dots, n-1$, form a group of n points which is completely determined by any one of its members. By varying P , we obtain on F a cyclic involution I_n , of order n .

There are, in general, some points P on F such that $T(P) \equiv P$. Then $T^i(P) \equiv P$, by induction, and so P counted n times forms a group of I_n . Then P is said to be a *united point* of I_n (and of T). Since $T(P) \equiv P$ implies two conditions on P , there are, in general, only a finite number of united points of I_n . There may, however, be a single infinity of such points, lying on a curve called a *united curve* of I_n , together with a finite number not lying on this curve.

We shall be concerned only with united points of I_n which do not lie on any united curve, that is, points O such that $T(O) \equiv O$, but through which there is no curve of F whose generic point P enjoys the property $T(P) \equiv P$; we call these *isolated united points* of I_n .

† Du Val (2).

‡ Du Val (1).

Let O be an isolated united point of I_n and let F_1 be the surface obtained from F by resolving the point O (that is, F_1 is to be a projective model of a sufficiently general linear system of curves having a simple base-point at O). The neighbourhood of O is represented on F_1 by an exceptional curve, $E_0^{(1)}$ say. Let $T^{(1)}$ and $I_n^{(1)}$ be the images on F_1 of T and I_n . Then, since $T(O) \equiv O$, the transformation $T^{(1)}$ transforms every point of $E_0^{(1)}$ into a point of $E_0^{(1)}$ and therefore sets up a cyclic transformation, t_1 say, of $E_0^{(1)}$ of period δ , where δ is n or some factor of n .

If $\delta = n$, we call O an *imperfect united point* of I_n . If $\delta < n$, we call O a δ -*perfect united point* of I_n , and, in particular, if $\delta = 1$ (so that t_1 is the identical transformation on $E_0^{(1)}$), we call O a 1-perfect or simply a *perfect united point*.

If O is not perfect, repetitions of the transformation t_1 generate on $E_0^{(1)}$ a cyclic involution of order δ and, since $E_0^{(1)}$ is rational, this involution must have two distinct united points, which are clearly isolated united points of the involution $I_n^{(1)}$ on F_1 . We choose one of these points, calling it $O_1^{(1)}$. If, on the other hand, O is perfect, $E_0^{(1)}$ is a united curve of $I_n^{(1)}$ and, in general, it is not necessary to consider any further its neighbourhoods. But for certain purposes we find it convenient to define the point $O_1^{(1)}$, in this case, to be any point of $E_0^{(1)}$ (and in a similar way we shall select certain points, according to a convention, in the neighbourhoods of any perfect united points which occur in our considerations). The fictitious† image on F of $O_1^{(1)}$ we denote by O_1 . The point $O_1^{(1)}$ may be classified as imperfect, δ -perfect, or perfect in the same way as O .

We resolve the point $O_1^{(1)}$ of F_1 , obtaining a surface F_2 , on which the first neighbourhood of $O_1^{(1)}$ is represented by an exceptional curve $E_1^{(2)}$; we denote the images on F_2 of $E_0^{(1)}$ (diminished by the first neighbourhood of $O_1^{(1)}$)‡, $T^{(1)}$ and $I_n^{(1)}$ by $E_0^{(2)}$, $T^{(2)}$ and $I_n^{(2)}$, respectively. Then $E_0^{(2)}$ and $E_1^{(2)}$ have a point in common.

If $O_1^{(1)}$ is not perfect, the involution $I_n^{(2)}$ has on $E_1^{(2)}$ two isolated united points, one of which we call $O_2^{(2)}$. But if $O_1^{(1)}$ is perfect, every point of $E_1^{(2)}$ is a united point of $I_n^{(2)}$ and we choose the point $O_2^{(2)}$ to be one of these in a conventional way that will be explained in § 2. We denote the

† This is a convenient term introduced by Du Val in (2). An *actual* point of the surface is a point in the ordinary sense; a *fictitious* point is what is commonly called an "infinitely near" point, in any neighbourhood of an actual point, that is, the image on F of an actual point of some model of F obtained by a series of resolutions such as we are considering.

‡ Such phrases as this we shall usually omit, unless emphasis is required; by "the image on F' of the curve E of F ", unless the term "total image" is used, we shall mean the image of E diminished by the neighbourhoods of any points of E resolved in transforming F into F' .

fictitious image on F of $O_2^{(2)}$ by O_2 . We then resolve the point $O_2^{(2)}$, obtaining a surface F_3 on which the first neighbourhood of $O_2^{(2)}$ is represented by an exceptional curve $E_2^{(3)}$; we denote the images on F_3 of $E_0^{(2)}$, $E_1^{(2)}$, $T^{(2)}$, $I_n^{(2)}$ by $E_0^{(3)}$, $E_1^{(3)}$, $T^{(3)}$, $I_n^{(3)}$, respectively.

Continuing in this way, we define a sequence of surfaces

$$F, F_1, F_2, \dots, F_m,$$

in which are the actual points,

$$O, O_1^{(1)}, O_2^{(2)}, \dots, O_m^{(m)}.$$

On F_i , $i = 1, 2, \dots, m$, the images of T and I_n are denoted by $T^{(i)}$ and $I_n^{(i)}$, respectively, and the point $O_i^{(i)}$ is an actual united point of $I_n^{(i)}$ (which is isolated unless $O_{i-1}^{(i-1)}$ is perfect). The surface F_{i+1} is obtained from F_i by resolving the point $O_i^{(i)}$, and the image on F_{i+1} of $O_i^{(i)}$ is a curve $E_i^{(i+1)}$. If $O_i^{(i)}$ is not perfect, $I_n^{(i+1)}$ has two isolated united points on $E_i^{(i+1)}$, one of which is taken to be $O_{i+1}^{(i+1)}$; but if $O_i^{(i)}$ is perfect, the point $O_{i+1}^{(i+1)}$ is a point of $E_i^{(i+1)}$, chosen in a way to be explained in § 2.

The images on F of

$$O, O_1^{(1)}, O_2^{(2)}, \dots, O_m^{(m)}$$

are denoted by $O, O_1, O_2, \dots, O_m$, (1.1)

and they form a sequence of points, O actual and the rest fictitious, such that O_i is in the first neighbourhood of O_{i-1} , $i = 1, 2, \dots, m$ ($O_0 \equiv O$). We call O_i a perfect, δ -perfect, or imperfect united point of I_n according as $O_i^{(i)}$ is a perfect, δ -perfect, or imperfect united point of $I_n^{(i)}$.

The image on F_j , $j > i + 1$, of the curve $E_i^{(i+1)}$ (diminished by the neighbourhoods of any points of it resolved in transforming F_{i+1} into F_j) is denoted by $E_i^{(j)}$.

2. The work of this paragraph is based on the theorem†:

O_j is proximate to O_i , if, and only if, on the surface F_j , the actual point $O_j^{(j)}$ lies on the curve $E_i^{(j)}$.

† Du Val in (2) takes this property of proximate points to define them and proves the property (in terms of the multiplicities of curves through them) by which they are usually defined, e.g. in Enriques-Chisini (1), 381. Du Val's approach seems to be the better but, because his work is not yet generally available, we prove the theorem here from the generally accepted point of view, deducing it from a theorem of Barber and Zariski, whose work starts from the basis laid by Enriques and Chisini. See also Semple (1), 149.

To prove this, let O_j be proximate to O_i and consider only the sequence of points, $O, O_1, O_2, \dots, O_i, \dots, O_j$. Then on F_{j+1} there is a reducible exceptional curve consisting of the components, $E_0^{(j+1)}, E_1^{(j+1)}, \dots, E_j^{(j+1)}$, representing the neighbourhoods of these points.

O_j is the last of the proximate points of O_i in this sequence and therefore, by a theorem of Barber and Zariski†, $E_j^{(j+1)}$ meets $E_i^{(j+1)}$. Now $E_j^{(j+1)}$ and $E_i^{(j+1)}$ are the images on F_{j+1} , respectively of the actual point $O_j^{(j)}$ and the curve $E_i^{(j)}$ (diminished by the neighbourhood of $O_j^{(j)}$) of F_j ; hence on F_j the point $O_j^{(j)}$ lies on $E_i^{(j)}$. The converse is proved by retracing the steps.

Using this property, we discuss the proximity relations of the sequence (1.1) on F . We consider first the proximity relations of a point O_{i+1} when O_i is not perfect; later we shall define the point O_{i+1} , when O_i is perfect, in such a way that the proximity relations of O_{i+1} are the same in both cases. We suppose that O_i is a free point, that is to say, that it is proximate only to its immediate ancestor O_{i-1} . Then on F_i the point $O_i^{(i)}$ lies on $E_{i-1}^{(i)}$, but on no curve $E_j^{(i)}$, $j < i-1$. Thus on the surface F_{i+1} obtained by resolving $O_i^{(i)}$, the curve $E_i^{(i+1)}$ has one point in common with $E_{i-1}^{(i+1)}$ but meets no other curve $E_j^{(i+1)}$, $j < i-1$. On $E_i^{(i+1)}$ there are two isolated united points of $I_n^{(i+1)}$, one of which is $O_{i+1}^{(i+1)}$. Now the transformation $T^{(i+1)}$ transforms every point of $E_i^{(i+1)}$ into a point of $E_i^{(i+1)}$ and every point of $E_{i-1}^{(i+1)}$ into a point of $E_{i-1}^{(i+1)}$, and therefore transforms their common point into itself. Thus, one of the two united points of $I_n^{(i+1)}$ on $E_i^{(i+1)}$ is the point of intersection of this curve with $E_{i-1}^{(i+1)}$. And, by the theorem at the beginning of this paragraph, the image of this point on F is the point in the first neighbourhood of O_i proximate to O_{i-1} . The other united point of $I_n^{(i+1)}$ on $E_i^{(i+1)}$ does not lie on any curve $E_j^{(i+1)}$, $j < i$, and therefore its image on F is a free point in the first neighbourhood of O_i . Hence:

(i) *If the point O_i of the sequence (1.1) is a free point, its immediate successor O_{i+1} is either*

(a) *a free point, proximate only to O_i , or*

(b) *proximate to O_{i-1} as well as to O_i ($O_0 \equiv O$).*

Next let O_i be proximate not only to its immediate ancestor O_{i-1} , but also to some more remote ancestor, O_j , $j < i-1$ ($O_0 \equiv O$). Then, by the theorem at the beginning of this paragraph, on the surface F_i the point $O_i^{(i)}$ lies on the curves $E_{i-1}^{(i)}$ and $E_j^{(i)}$, but on no other curve, $E_k^{(i)}$. And so, on F_{i+1} , the curve $E_i^{(i+1)}$ meets $E_{i-1}^{(i+1)}$ and $E_j^{(i+1)}$ in one point each and meets no other

† Barber and Zariski (1), Theorem 5.

curve, $E_k^{(i+1)}$. Thus, by the same reasoning as before, $O_{i+1}^{(i+1)}$, being a united point of $I_n^{(i+1)}$ on $E_i^{(i+1)}$, must lie at the intersection of this curve with $E_{i-1}^{(i+1)}$ or $E_j^{(i+1)}$, and so on. F the point O_{i+1} is a point of the first neighbourhood of O_i proximate either to O_{i-1} or to O_j . Hence:

(ii) *If the point O_i of the sequence (1.1) is proximate to O_j ($O_0 \equiv O$), as well as to its immediate ancestor O_{i-1} , then its immediate successor O_{i+1} is either*

(a) *proximate to O_{i-1} as well as to O_i , or*

(b) *proximate to O_j as well as to O_i .*

Since a point in the neighbourhood of a non-singular point of an algebraic surface cannot be proximate to more than two points, this exhausts the possible cases.

So far, we have not completely defined the point O_{i+1} when O_i is perfect. We now do so in such a way that the relations described in (i) and (ii) above are always satisfied whether O_i is perfect or not; in fact, we make the definition:

If O_i is a perfect united point proximate only to its immediate ancestor O_{i-1} , then O_{i+1} is any point of the first neighbourhood of O_i ; if O_i is perfect and is proximate to O_j , $j < i-1$, then O_{i+1} is a point of the first neighbourhood of O_i , proximate either to O_{i-1} or to O_j .

With this definition, it is clear that (i) and (ii) are always true. It will be seen that, though we consider points in the neighbourhoods of perfect united points, these are, in fact, very special points and our reason for doing so is merely to allow certain conventional solutions of some of the problems which will occur and thus to avoid certain particular restrictions. The definition of the point $O_{i+1}^{(i+1)}$, when $O_i^{(i)}$ is perfect, which is required in order formally to complete § 1, is, of course, just another form of the above definition. In the sequel the term "united point" will, unless otherwise stated, mean either "isolated united point" or one of the points which we have just defined in the neighbourhood of a perfect united point.

Now the point O_1 of the sequence is certainly a free point. Hence, by (i), O_2 may be either a free point or proximate to O ; in the former case, O_3 may be either a free point or proximate to O_1 ; in the former case, O_4 may be either a free point or proximate to O_2 , and so on. Let us suppose that $O_1, O_2, \dots, O_{s_1}$ are free points and that O_{s_1+1} is not free but is proximate to O_{s_1-1} .

Then, by (ii), O_{s_1+2} may be proximate either to O_{s_1-1} or to O_{s_1} ; in the former case, O_{s_1+3} may be proximate either to O_{s_1-1} or to O_{s_1+1} ; in the former case, O_{s_1+4} may be proximate either to O_{s_1-1} or to O_{s_1+2} , and so on.

Let us suppose that $O_{s_1+1}, O_{s_1+2}, \dots, O_{s_1+s_2}$ are all proximate to O_{s_1-1} , and $O_{s_1+s_2+1}$ is proximate, not to O_{s_1-1} , but to $O_{s_1+s_2-1}$.

Continuing in this way, we see that the sequence (1.1) has proximity relations of the following type:

O is an actual point of F ;

$O_1, O_2, \dots, O_{s_1}$ are free points, each proximate only to its immediate ancestor;

$O_{s_1+1}, O_{s_1+2}, \dots, O_{s_1+s_2}$ are proximate to O_{s_1-1} as well as to their immediate ancestors,

$O_{s_1+s_2+1}, O_{s_1+s_2+2}, \dots, O_{s_1+s_2+s_3}$ are proximate to $O_{s_1+s_2-1}$ as well as to their immediate ancestors;

... ..

$O_{s_1+s_2+\dots+s_{\mu-1}+1}, O_{s_1+s_2+\dots+s_{\mu-1}+2}, \dots, O_{s_1+s_2+\dots+s_{\mu-1}+s} \equiv O_m$ are proximate to $O_{s_1+s_2+\dots+s_{\mu-1}}$ as well as to their immediate ancestors

$$(s_i \geq 1, \quad i = 1, 2, \dots, \mu; \quad s_1 + s_2 + \dots + s_{\mu} = m).$$

We note that only the first few points of the sequence are free and that, if O_i is not free, then none of its successors is free; in fact, the points, $O_{s_1+1}, O_{s_1+2}, \dots, O_m$ are all satellites of O_{s_1} .

Also, from the fact that there are only two united points of I_n in the neighbourhood of any united point, we see that all the united points of I_n which are free points, that is all the free points of any sequence of the type (1.1), lie on one of two linear branches through O , each of which contains one of the two united points in the first neighbourhood of O .

The numbers $s_1, s_2, \dots, s_{\mu}$ completely determine the proximity relations of the sequence (1.1), and conversely. Thus, all the free points being fixed in view of a remark made above, these numbers completely determine the positions of the points of the sequence (1.1) and, in particular, completely determine the point O_m , apart from the ambiguity arising from the two possible choices of the point O_1 in the first neighbourhood of O . We name these two points arbitrarily the *upper* and the *lower* united points in the first neighbourhood of O and we say that (1.1) is an *upper* or a *lower* sequence and each of its points is an *upper* or a *lower* point, according as the point O_1 of (1.1) is the upper or lower point in the first neighbourhood of O .

Thus the point O_m is completely determined if it is known whether it is an upper or a lower point and if the numbers $s_1, s_2, \dots, s_{\mu}$ of the sequence of the type (1.1), of which O_m is the last point, are known.

A slight modification of the well-known Enriques-Chisini diagram† for representing sequences of fictitious points provides a useful schematic representation of the sequence (1.1) having the proximity relations described above. It is hoped that the diagram will be almost self-explanatory, when compared with the description given above of the proximity relations of the sequence. A complete description of it in words would be somewhat lengthy.

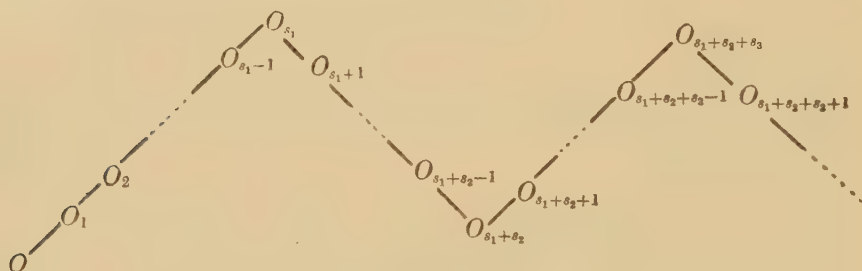


Fig. 1.

The figure is drawn in such a way that all the points lying along a straight line are proximate to the point immediately preceding the first point on that line, as well as to their immediate ancestors, except that those on the straight line nearest to the left are free points. If (1.1) is an upper sequence, the figure is drawn as printed; if it is a lower sequence, the figure is inverted, so that the free points $O, O_1, \dots, O_{s_1}$ lie along a diagonal downwards and to the right, the points $O_{s_1}, O_{s_1+1}, \dots, O_{s_1+s_2}$, which are proximate to O_{s_1-1} , lie along a diagonal upwards and to the right, and so on.

A representation of the arrangement of any set of united points in the neighbourhoods of O is given by combining the diagrams for any number of such sequences, points common to two sequences being superimposed, to give a figure in the form of a "tree" turned on its side. An example of such a figure occurs towards the end of § 5.

3. In this paragraph we find a pair of integers, defining the positions of the united points of I_n in the neighbourhoods of O , which may be thought of as lattice coordinates of the points. In fact, we define for each sequence of the type (1.1) a pair of integers X, Y , such that there is a (1, 1) correspondence between the ordered pairs (X, Y) and the sequences

† Enriques and Chisini (1), 379-381.

(1.1) and therefore between these ordered pairs and the points O_m at the end of these sequences.

Let γ be a primitive branch (branch of least order) through the points of the sequence (1.1). Since the only point of the sequence proximate to O_j is O_{j+1} , it follows that the multiplicities of γ at the points $O, O_1, \dots, O_{s_1-1}$ are all equal and equal, say, to ν . Similarly, since the only point of the sequence proximate to $O_{s_1+s_2+\dots+s_i+j}$ is $O_{s_1+s_2+\dots+s_i+j+1}$, it follows that the multiplicities of γ at each of the points $O_{s_1+s_2+\dots+s_i+j}$, where i is fixed and $j = 0, 1, 2, \dots, s_{i+1}-1$, are all equal and equal, say, to ν_i . Then, applying to the points $O_{s_1+s_2+\dots+s_{i-1}}$ the theorem that the multiplicity of γ at any point is equal to the sum of its multiplicities at points proximate to that point, we have

$$\nu_i = s_{i+2} \nu_{i+1} + \nu_{i+2}, \quad (3.1)$$

($\nu_0 \equiv \nu$), and, since γ is a primitive branch, $\nu_\mu = \nu_{\mu-1} = 1$.

Now let τ be the sum of the multiplicities of γ at the free points of the sequence (1.1). Then

$$\tau = s_1 \nu + \nu_1. \quad (3.2)$$

Clearly, if the numbers, $s_1, s_2, \dots, s_\mu$, are given, the equations (3.1) and (3.2) determine ν and τ uniquely. Conversely, the result, proved in §2, that, except for the initial points $O, O_1, O_2, \dots, O_{s_1}$, the sequence (1.1) has no free points, implies that ν_i is definitely less than ν_{i-1} (not merely $\nu_i \leq \nu_{i-1}$) and so the numbers $s_1, s_2, \dots, s_\mu$ are uniquely determined from ν and τ by the Euclidean algorithm given by (3.1) and (3.2). It also follows that ν and τ are mutually prime.

Thus the numbers ν and τ fix uniquely the point O_m at the end of the sequence (1.1) provided that it is known whether O_m is an upper or a lower point. If O_m is an upper point, we denote it by (ν, τ) ; if a lower point, by (τ, ν) . These two symbols are immediately distinguishable since $\nu < \tau$ (except that, for the point O itself, $\nu = \tau = 1$).

Hence any united point of I_n in any neighbourhood of O is denoted by a symbol of the form (X, Y) , where, if $X < Y$, the point is an upper point and $\nu = X, \tau = Y$; if $X > Y$, the point is a lower point and $\nu = Y, \tau = X$.

We can choose on F a pair of local parameters u, v , in the neighbourhood of O , such that $v = 0$ is a curve through as many as we please of the upper free points $(1, Y)$ and $u = 0$ is a curve through as many as we please of the lower free points $(X, 1)$. If γ is a primitive branch through (X, Y) , since all the multiple points of γ except $O, O_1, \dots, O_{s_1}$ are satellites of O_{s_1} , it follows that the equation of γ has only one characteristic term and it is

easily verified† that γ may therefore be written in the form

$$v = au^{Y/X} + \dots \quad (3.3)$$

Similarly, a primitive branch through (X', Y') is represented by an equation of the form

$$v = bu^{Y'/X'} + \dots$$

We can now use the theory of Enriques-Chisini‡ to evaluate the number of intersections of these branches. If (X, Y) and (X', Y') are different upper points, then, since the pairs X, Y and X', Y' are mutually prime, we have $XY' \neq X'Y$ and the intersection number is $\min(XY', X'Y)$. If $(X', Y') \equiv (X, Y)$ we are dealing with two branches having in common all their points as far as their last satellite (X, Y) , but having distinct free points following (X, Y) . Thus§ $a \neq b$ and again the number of intersections is $\min(XY', X'Y)$ which in this case is simply XY . It is easy to extend this work to cover the cases where one or both of (X, Y) , (X', Y') are lower points. Hence, with a change of notation, we have

THEOREM I. *The number ρ_{ij} of intersections of primitive branches through the united points (X, Y) and (X', Y') in the neighbourhoods of O is given by*

$$\rho_{ij} = \min(X_i Y_j, X_j Y_i).$$

4. We next consider the analytical representation in the neighbourhood of O of the transformation T generating the involution, obtaining as a result two integers α, β , which express the properties of T in the neighbourhood of O , and in terms of which all our future results are stated.

The considerations of §1 show that T has either two or an infinite number of united directions through O . We take a pair of local parameters u, v for the surface in the neighbourhood of O , such that O is $u = v = 0$ and the two (or two of the) united directions through O are those of the tangents to $u = 0$ and $v = 0$ (the parameters of the last paragraph form a special case of this). Then, if T changes (u, v) into (u_1, v_1) , its equations must be of the form

$$\left. \begin{aligned} u_1 &= lu + P(u, v), \\ v_1 &= mv + Q(u, v), \end{aligned} \right\}$$

† Enriques-Chisini (1), 361-371.

‡ *Loc. cit.*, 352-353.

§ Enriques-Chisini (1), 370.

where $P(u, v)$ and $Q(u, v)$ are power series in u, v having zeros of order two at $u = v = 0$.

Now, since the involution is of order n , n successive applications of T must transform (u, v) into itself; hence $l^n = m^n = 1$ and so we can write $l = \epsilon^\beta$, $m = \epsilon^\alpha$, where ϵ is a primitive n -th root of unity. Thus the equations representing T are of the form

$$\left. \begin{aligned} u_1 &= \epsilon^\beta u + P(u, v), \\ v_1 &= \epsilon^\alpha v + Q(u, v), \end{aligned} \right\} \quad (4.1)$$

where ϵ is a primitive n -th root of unity. Clearly α, β are only defined modulo n and we can take $0 \leq \alpha, \beta < n$.

But we can restrict the nature of this analytical representation even further, namely to the case in which there is no factor common to all three of the integers n, α, β . We show, in fact, that, if $\alpha = k\gamma$, $\beta = k\delta$, $n = km$, where k is the highest common factor of α, β, n , then T is really of period m , whence the result stated follows at once.

Let us suppose that T transforms (u, v) into (u_1, v_1) and (u_i, v_i) into (u_{i+1}, v_{i+1}) , $i = 1, 2, \dots, n-1$, (u_n, v_n) being identical with (u, v) . Thus

$$\left. \begin{aligned} u_{i+1} &= \epsilon^\beta u_i + P(u_i, v_i), \\ v_{i+1} &= \epsilon^\alpha v_i + Q(u_i, v_i), \end{aligned} \right\}$$

and by successive substitution we obtain

$$u_m = \epsilon^{m\beta} u + P'(u, v),$$

where $P'(u, v)$ is another power series in u, v , having a zero of order two at $u = v = 0$. But

$$\epsilon^{m\beta} = \epsilon^{mk\delta} = \epsilon^{n\delta} = 1, \quad \text{since } \epsilon^n = 1.$$

Therefore

$$\left. \begin{aligned} u_m &= u + P'(u, v), \\ v_m &= v + Q'(u, v), \end{aligned} \right\}$$

and similarly

where $Q'(u, v)$ is also a power series in u, v , having a zero of order two at $u = v = 0$.

Now let us assume that T is not of period m . Then one, at least, of P' or Q' is not identically zero; let the terms of least degree in P' or Q' be those of order r , so that

$$\left. \begin{aligned} u_m &= u + \Sigma A_{ij} u^i v^j + \dots, \\ v_m &= v + \Sigma B_{ij} u^i v^j + \dots, \end{aligned} \right\}$$

where Σ , as in the rest of this paragraph, denotes summation over all values of i, j , such that $i+j=r$ and where at least one of the coefficients $A_{ij}, B_{ij}, i+j=r$, is non-zero. Then

$$\begin{aligned} u_{2m} &= u_m + \Sigma A_{ij} u_m^i v_m^j + \dots \\ &= u + \Sigma A_{ij} u^i v^j + \dots + \Sigma A_{ij} \{u + \Sigma A_{ij} u^i v^j + \dots\} \{v + \Sigma B_{ij} u^i v^j + \dots\} + \dots \\ &= u + 2 \Sigma A_{ij} u^i v^j + \dots, \end{aligned}$$

and, similarly,
$$v_{2m} = v + 2 \Sigma B_{ij} u^i v^j + \dots$$

Again
$$\begin{aligned} u_{3m} &= u_{2m} + \Sigma A_{ij} u_{2m}^i v_{2m}^j + \dots \\ &= u + 3 \Sigma A_{ij} u^i v^j + \dots, \end{aligned}$$

and
$$v_{3m} = v + 3 \Sigma B_{ij} u^i v^j + \dots$$

Thus, by induction, we obtain

$$\left. \begin{aligned} u_n &= u_{km} = u + k \Sigma A_{ij} u^i v^j + \dots, \\ v_n &= v_{km} = v + k \Sigma B_{ij} u^i v^j + \dots \end{aligned} \right\}$$

But $u_n = u, v_n = v$ (at least for all u and v in a finite neighbourhood of $u = v = 0$) and this requires $A_{ij} = B_{ij} = 0$, for all i, j , such that $i+j=r$, which is a contradiction of our initial assumption; thus T is of period m . Hence the equations for T can be written in a form similar to (4.1), but with ϵ as a primitive m -th instead of n -th root of unity. If in this new form m and the numbers corresponding to α, β again have a common factor, we can once more reduce the period of T , and so on; and so we deduce that

The transformation T can always be represented by equations of the form (4.1) where there is no factor common to all three integers n, α, β .

By a similar method it can be shown that α and β cannot both be zero. For brevity we shall call O an (α, β) united point of I_n , meaning that α and β are the integers obtained by writing the transformation T generating I_n in this form.

Now, preserving the same meaning for u_i and v_i , we see that a condition for the transformation of directions through O set up by T to be of period δ is

$$v_\delta / u_\delta = v / u,$$

when terms of order greater than one are neglected. And by (4.1) this is equivalent to $\epsilon^{\delta(\alpha-\beta)} = 1$, i.e.,

$$\delta(\alpha-\beta) \equiv 0 \pmod{n}. \quad (4.2)$$

Thus, a condition for O to be a δ -perfect united point of I_n is that δ is the least integral solution of the congruence (4.2).

(If $\delta = 1$, O is perfect; if $\delta = n$, O is imperfect.)

5. This paragraph contains the most important results of Part I of this paper: these results determine the number δ_i (expressing that (X_i, Y_i) is a δ_i -perfect united point of I_n) in terms of the integers α, β of the last paragraph.

We can attach to each point O_i of the sequence (1.1) a pair of symbols α_i, β_i , analogous to α, β , namely those which arise when the transformation $T^{(i)}$, on the surface F_i is expressed in the form (4.1) in the neighbourhood of $O_i^{(i)}$. And O_i is a δ_i -perfect united point, where δ_i is the least integral solution of the congruence,

$$\delta_i(\alpha_i - \beta_i) \equiv 0 \pmod{n}.$$

If O is not perfect, then, when T is expressed in the form (4.1), the two united points in the first neighbourhood of O are those which lie in the directions of $u = 0$ and $v = 0$ at O . The surface F_1 of §1, on which the images of the two united points of I_n in the neighbourhoods of O are actual points, is obtained from F by any simple resolution of the point O . Such a resolution is given by a quadratic transformation of the parameters u, v , having a fundamental point at $u = v = 0$. We make use here of two such quadratic transformations, the first being

$$\left. \begin{aligned} u &= u', \\ v &= u'v', \end{aligned} \right\} \quad (5.1)$$

which carries the origin $u = v = 0$ into the curve $u' = 0$ and the point lying in its first neighbourhood in the direction $v = 0$ into the new origin, $u' = v' = 0$.

Applying this transformation to the equations (4.1), we obtain

$$u_1' = \epsilon^\beta u' + P'(u', v'),$$

$$v_1' = \epsilon^{\alpha-\beta} v' + Q'(u'v'),$$

where $P'(u', v')$, $Q'(u', v')$ are power series in u', v' , having zeros of order two at $u' = v' = 0$. Hence we see that, if O is an (α, β) united point of I_n , then one of the two united points in its first neighbourhood, say the upper, is an $(\alpha - \beta, \beta)$ united point; while the other, as may be seen by considering

the transformation

$$\left. \begin{aligned} u &= u'v', \\ v &= v', \end{aligned} \right\} \quad (5.2)$$

is an $(\alpha, \beta - \alpha)$ united point.

Now, taking u and v to be the specialized parameters of § 3, we prove that we can take a succession of the transformations (5.1) and (5.2) to be those which produce the surfaces $F_1, F_2, \dots, F_m$ of § 1, on which the actual images $O_1^{(1)}, O_2^{(2)}, \dots, O_m^{(m)}$ of $O_1, O_2, \dots, O_m$ are revealed. In fact, if (1.1) is an upper sequence, a succession of s_1 transformations of the type (5.1) gives the surfaces $F_1, F_2, \dots, F_{s_1}$, the points $O_1^{(1)}, O_2^{(2)}, \dots, O_{s_1}^{(s_1)}$ being at the origins of the parameter-systems on the respective surfaces; next a succession of s_2 transformations of the type (5.2) give the surfaces $F_{s_1+1}, F_{s_1+2}, \dots, F_{s_1+s_2}$, the points $O_{s_1+1}^{(s_1+1)}, O_{s_1+2}^{(s_1+2)}, \dots, O_{s_1+s_2}^{(s_1+s_2)}$ being again the origins of the parameter-systems; and again $O_{s_1+s_2+1}^{(s_1+s_2+1)}, \dots, O_{s_1+s_2+s_3}^{(s_1+s_2+s_3)}$ are the origins of the parameter-systems on the surfaces obtained by a further succession of s_3 transformations of the type (5.1), and so on. To verify this we need note only that, with the parameter-system which we have chosen on F , the branch of least order through the (upper) point O_m , say the point (ν, τ) , is, by (3.3), of the form

$$u = t^\nu, \quad v = t^\tau P(t)$$

(with a notation similar to that of § 3), and that our statement amounts to no more than this: that, if we resolve the singularity at $u = v = 0$ on this branch by means of the succession of transformations described, the new branch obtained at each stage is always a branch through the origin of the new parameter-system, so that the next resolution can again be carried out by means of one of the transformations (5.1), (5.2); and the verification of this is a simple and familiar process.

Analogous results hold for a lower sequence, the transformations (5.1) and (5.2) being interchanged throughout.

Now let the point O_i of the sequence (1.1) be an (α_i, β_i) united point of I_n , that is, let the corresponding actual point $O_i^{(i)}$ of F_i be an (α_i, β_i) united point of $I_n^{(i)}$. The results obtained above for the two points in the first neighbourhood of O can be generalized immediately to give the following:

If F_{i+1} is obtained from F_i by a transformation of the type (5.1), the points $O_i^{(i)}$ and $O_{i+1}^{(i+1)}$ being at the origins of the parameter-systems on the respective surfaces, then $\alpha_{i+1} = \alpha_i - \beta_i$, $\beta_{i+1} = \beta_i$; if it is obtained by a transformation of the type (5.2), then $\alpha_{i+1} = \alpha_i$, $\beta_{i+1} = \beta_i - \alpha_i$. Hence, in view of the description given above of the set of transformations

producing the sequence of surfaces F_i , we have

$$\alpha_i = \alpha_{i-1} - \beta_{i-1}, \quad \beta_i = \beta_{i-1}, \quad (i = 1, 2, \dots, s_1),$$

$$\alpha_{s_1+i} = \alpha_{s_1+i-1}, \quad \beta_{s_1+i} = \beta_{s_1+i-1} - \alpha_{s_1+i-1} \quad (i = 1, 2, \dots, s_2),$$

and so on, whence $\alpha_{s_1} = \alpha - s_1 \beta, \quad \beta_{s_1} = \beta, \quad (5.3)$

$$\alpha_{s_1+s_2} = \alpha_{s_1}, \quad \beta_{s_1+s_2} = \beta_{s_1} - s_2 \alpha_{s_1}, \quad (5.4)$$

and so on. And, by induction,

$$\left. \begin{aligned} \alpha_{s_1+s_2+\dots+s_{2i+1}} &= \alpha_{s_1+s_2+\dots+s_{2i}} - s_{2i+1} \beta_{s_1+s_2+\dots+s_{2i}}, \\ \beta_{s_1+s_2+\dots+s_{2i+1}} &= \beta_{s_1+s_2+\dots+s_{2i}}, \\ \alpha_{s_1+s_2+\dots+s_{2i}} &= \alpha_{s_1+s_2+\dots+s_{2i-1}}, \\ \beta_{s_1+s_2+\dots+s_{2i}} &= \beta_{s_1+s_2+\dots+s_{2i-1}} - s_{2i} \alpha_{s_1+s_2+\dots+s_{2i-1}}. \end{aligned} \right\} \quad (5.5)$$

Now suppose that the point O_m of the sequence is (X, Y) and suppose that

$$\alpha X \equiv \beta Y \pmod{\lambda}, \quad (5.6)$$

that is, since we are taking (1.1) to be an upper sequence,

$$\alpha \nu \equiv \beta \tau \pmod{\lambda}.$$

Then, by (3.2), $(\alpha - s_1 \beta) \nu \equiv \beta \nu_1 \pmod{\lambda},$

and therefore, by (5.3),

$$\alpha_{s_1} \nu \equiv \beta_{s_1} \nu_1 \pmod{\lambda}.$$

Thus, using (3.1), we have

$$\alpha_{s_1} \nu_2 \equiv (\beta_{s_1} - s_2 \alpha_{s_1}) \nu_1 \pmod{\lambda},$$

and therefore, by (5.4),

$$\alpha_{s_1+s_2} \nu_2 \equiv \beta_{s_1+s_2} \nu_1 \pmod{\lambda}.$$

In a similar way, using (3.1) and (5.5), we prove by induction that

$$\alpha_{s_1+s_2+\dots+s_{2i+1}} \nu_{2i} \equiv \beta_{s_1+s_2+\dots+s_{2i+1}} \nu_{2i+1} \pmod{\lambda},$$

$$\alpha_{s_1+s_2+\dots+s_{2i}} \nu_{2i} \equiv \beta_{s_1+s_2+\dots+s_{2i}} \nu_{2i-1} \pmod{\lambda}.$$

But $\nu_\mu = \nu_{\mu-1} = 1$, since the numbers ν_i are defined by a primitive branch through the points (1.1). Hence

$$\alpha_{s_1+s_2+\dots+s_\mu} \equiv \beta_{s_1+s_2+\dots+s_\mu} \pmod{\lambda},$$

i.e.,

$$\alpha_m \equiv \beta_m \pmod{\lambda}. \quad (5.7)$$

Putting $\lambda = n/\delta$, we see from the equivalence just proved of (5.6) and (5.7) that the congruence

$$\delta(aX - \beta Y) \equiv 0 \pmod{n}$$

implies

$$\delta(a_m - \beta_m) \equiv 0 \pmod{n}.$$

But we have shown in § 4 that the condition for O_m to be a δ -perfect united point of I_n is that δ should be the least solution of the last congruence.

It is clear that the argument can be reversed and so we have

THEOREM II. *A necessary and sufficient condition the point (X, Y) , in some neighbourhood of O , to be a δ -perfect united point of I_n is that δ is the least integral solution of the congruence*

$$\delta(aX - \beta Y) \equiv 0 \pmod{n}. \quad (5.8)$$

From Theorem II we deduce the following convenient way of cataloguing any number of the united points of I_n in the neighbourhoods of O . Let

$$\mathfrak{X} = \begin{pmatrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_r \end{pmatrix}$$

be a matrix of integers defined by the conditions that

$$(a) \quad ax_i \equiv \beta y_i \pmod{n}, \quad (5.9)$$

(b) x_i, y_i are not admitted as the elements of a column if $x_i = kx'$, $y_i = ky'$, where $k (> 1)$ is an integer and x', y' are integral solutions of the congruence $ax' \equiv \beta y' \pmod{n}$.

Let

$$\mathfrak{Y} = \begin{pmatrix} X_1, X_2, \dots, X_r \\ Y_1, Y_2, \dots, Y_r \\ \delta_1, \delta_2, \dots, \delta_r \end{pmatrix}$$

be the matrix of integers whose element in the last row and i -th column is the highest common factor δ_i of x_i, y_i , and n , and whose elements in the i -th column and first and second rows, respectively, are $X_i = x_i/\delta_i$ and $Y_i = y_i/\delta_i$. Then clearly X_i and Y_i are mutually prime and δ_i is the least integer such that

$$\delta_i(aX_i - \beta Y_i) \equiv 0 \pmod{n},$$

so that the i -th column of $\mathfrak{Y}$ gives the information that the point (X_i, Y_i) is a δ_i -perfect united point of I_n . Thus the matrix $\mathfrak{Y}$ gives a list of as many united points of I_n as we please. We describe the i -th column of $\mathfrak{X}$

or of $\mathcal{Y}$ as the column "representing" the δ_i -perfect united point (X_i, Y_i) of I_n in the neighbourhood of O .

For example, let $n = 6$, $\alpha = 2$, $\beta = 3$. Then

$$\mathfrak{X} = \begin{pmatrix} 3, & 3, & 3, & 6, & 6, & 9, & 9, & 12, & 12 \\ 2, & 4, & 6, & 2, & 6, & 2, & 4, & 2, & 6 \end{pmatrix},$$

the columns whose elements are 6, 4, and 9, 6 being excluded by condition (b). Hence

$$\mathcal{Y} = \begin{pmatrix} 3, & 3, & 1, & 3, & 1, & 9, & 9, & 6, & 2 \\ 2, & 4, & 2, & 1, & 1, & 2, & 4, & 1, & 1 \\ 1, & 1, & 3, & 2, & 6, & 1, & 1, & 2, & 6 \end{pmatrix}.$$

Placing the appropriate numbers δ_i at the points of our diagram representing these points, we see that the structure of O in so far as it is revealed by those columns which we have included in $\mathfrak{X}$ and $\mathcal{Y}$ is that shown in Fig. 2.

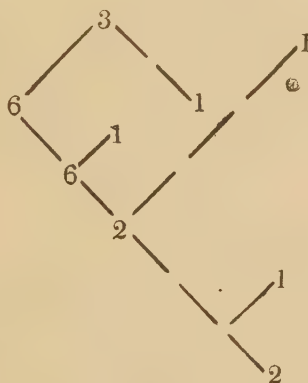


Fig. 2.

We have shown in § 4 that the numbers n, α, β do not have a factor common to all three. Any two of them may, however, have a common factor, so that the most general form is that in which

$$n = gh n', \quad \alpha = gk \alpha', \quad \beta = hk \beta',$$

where the sets $g, h, k; n', \alpha', \beta'; g, \beta'; h, \alpha'; k, n'$, each consist of mutually prime integers. We prove that the problem of classifying the united points of I_n in the neighbourhoods of O may be reduced to that of classifying the united points of an involution of order n' in the neighbourhoods of an (α', β') united point, O' . In fact, we show that the matrix $\mathfrak{X}$, which leads,

as we have seen, to a catalogue of a selection of the united points of I_n , is simply related to the corresponding matrix for I_n .

The numbers x_i, y_i of $\mathfrak{X}$ are defined by (a) and (b) above. Now, with the values assumed for n, α, β , (5.9) gives

$$gk\alpha' x_i \equiv hk\beta' y_i \pmod{ghn'},$$

and, since k is prime to g, h, n' , this implies that

$$g\alpha' x_i \equiv h\beta' y_i \pmod{ghn'}.$$

The last can be satisfied by integral values of x_i, y_i only if these are multiples of h, g respectively, say

$$x_i = hx'_i, \quad y_i = gy'_i, \quad (5.10)$$

and then we deduce that

$$\alpha' x'_i \equiv \beta' y'_i \pmod{n'}.$$

It is also easily verified that x'_i, y'_i satisfy a condition analogous to condition (b) above.

Thus x'_i, y'_i are elements of a matrix, $\mathfrak{X}'$, say, which describes the neighbourhood of an (α', β') united point of $I_{n'}$ in the same way as $\mathfrak{X}$ describes the neighbourhood of O . The equations (5.10) can then be written

$$\mathfrak{X} = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix} \mathfrak{X}'. \quad (5.11)$$

The matrices $\mathfrak{Y}$ and $\mathfrak{Y}'$ are not quite so simply related, since if x'_i and g (or y'_i and h , or both) have a common factor, this is a factor of x_i, y_i, n , but not of x'_i, y'_i, n' ; thus X_i, Y_i, δ_i are different from the corresponding numbers, X'_i, Y'_i, δ'_i say, derived from x'_i and y'_i .

Thus the investigation of the united points in the neighbourhoods of O is reduced to the investigation of a case in which n, α, β are all mutually prime, followed by the application of the equation (5.11).

But a further simplification is possible. Taking n, α, β to be all mutually prime, we can find an integer k , prime to n , such that $k\beta \equiv 1 \pmod{n}$; then, if $k\alpha \equiv a_0 \pmod{n}$, $a_0 < n$, a_0 is prime to n and the congruences

$$\alpha x_i \equiv \beta y_i \pmod{n} \quad (5.12)$$

and

$$a_0 x_i \equiv y_i \pmod{n} \quad (5.13)$$

are equivalent. Thus the matrices $\mathfrak{X}$ and $\mathfrak{Y}$ obtained for the neighbourhood of an (α, β) united point of an involution of order n (where n, α, β are mutually prime) are the same as those for the neighbourhood of an $(a_0, 1)$

point. This merely amounts to taking the transformation T^k , instead of T , to define the involution, it being clear, since k is prime to n , that these two transformations define the same involution.

It is clear that the argument can be reversed and so we have the result :

To investigate the united points of an involution I_n of order n in the neighbourhoods of an isolated (α, β) united point, we may first divide any pair of n, α, β by their highest common factor and repeat the process till the resulting numbers n', α', β' are all mutually prime. Then if $k\beta' \equiv 1 \pmod{n'}$ and $k\alpha' \equiv \alpha_0 \pmod{n'}$, $\alpha_0 < n$, and if $\mathfrak{X}'$ is a matrix (analogous to $\mathfrak{X}$) which catalogues a selection of the united points of an involution of order n' in the neighbourhoods of an $(\alpha_0, 1)$ united point, the corresponding matrix $\mathfrak{X}$ for the original involution is given by (5.11). And conversely.

This result is of importance in Parts II and III, where it leads to a corresponding and, because of the greater arithmetical complexity of the problems involved, more welcome simplification.

A fact worth noting, which can easily be shown by examples, is that if a point represented by a column of $\mathfrak{X}$ does not lie in a neighbourhood of any perfect united point of I_n , nevertheless the point in the neighbourhood of O' represented by the corresponding column of $\mathfrak{X}'$ may lie in a neighbourhood of a perfect united point of $I_{n'}$. Our conventional introduction of certain sets of united points in the neighbourhood of a perfect united point is necessitated by this fact.

6. This paragraph contains two results concerning the exceptional curves representing the neighbourhoods of O which are not of great intrinsic interest, but are of importance as lemmas for Part III of this paper. The proofs, which present no difficulties except those of complication, are merely sketched. The first is :

The components of the reducible exceptional curve representing the total neighbourhood of O on a model of F obtained by resolving O and a set of isolated united points of I_n in its neighbourhoods form a simply connected chain; that is, there are two components (end components of the chain) each of which meets only one other component and, apart from these two, every component meets two others; the end components are those representing the last upper and last lower free points which have been resolved.

The theorem is trivial if only the point O has been resolved and for the general case it is proved by induction. In fact, if after O and a certain number of united points in its neighbourhoods are resolved (transforming

F into a surface F') the components of the reducible exceptional curves representing the neighbourhoods of O do form a simply connected chain of which the end components represent the last upper and last lower free points resolved, then, by the results of §2, the actual isolated united points of the image on F' of I_n , representing united points in the neighbourhoods of O , must lie either (a) on the end components at points distinct from their intersections with other components, or (b) at the intersection of two components of the chain. And it is easily verified that if any one of these points is resolved, the components of the exceptional curve on the new surface obtained again form a simply connected chain of the same type.

The second theorem is:

If F is a model of F obtained by resolving the set of points (X_i, Y_i) , $i = 1, 2, \dots, s$, in the neighbourhoods of O , then the order in which the components representing the points (X_i, Y_i) occur in the reducible exceptional curve of F which represents the total neighbourhood of O is the order of the numerical magnitudes of the fractions $X_i/Y_i = x_i/y_i$.

That is, if the points (X_i, Y_i) are renumbered so that

$$\frac{X_i}{Y_i} < \frac{X_{i+1}}{Y_{i+1}} \quad (i = 1, 2, \dots, s-1),$$

then the component representing (X_i, Y_i) meets those representing (X_{i-1}, Y_{i-1}) and (X_{i+1}, Y_{i+1}) . The point O is, of course, $(1, 1)$.

Once again the theorem is trivial if O is the only point resolved and it can be proved in the general case by induction. In fact, if we assume that the theorem is true after a certain number of resolutions have been made, transforming F into a surface F' , then to prove it true after one more resolution has been made, transforming F' into F'' , it is necessary to show only

(i) that, if the curves of F' representing the neighbourhoods of (X_α, Y_α) and (X_β, Y_β) intersect and (X_γ, Y_γ) is the point resolved in transforming F' into F'' , and if on F'' the curve representing the neighbourhood of (X_γ, Y_γ) meets those representing (X_α, Y_α) and (X_β, Y_β) (which no longer intersect), then X_γ/Y_γ lies between X_α/Y_α and X_β/Y_β ; and

(ii) that, if (X_α, Y_α) is the last free point resolved in obtaining F' and if (X_γ, Y_γ) is the free point in its first neighbourhood whose resolution gives rise on F'' to the new end component of the chain, meeting the component representing (X_α, Y_α) , then X_γ/Y_γ is less than or greater than X_α/Y_α according as (X_α, Y_α) is an upper or a lower point.

The proof of (ii) is almost immediate and that of (i) is along the following lines. By the theory of exceptional curves[†], the curves of F' representing (X_α, Y_α) and (X_β, Y_β) can intersect only if one of these points is in some neighbourhood of the other. And if, say, (X_β, Y_β) is in a neighbourhood of (X_α, Y_α) , then the curves intersect if and only if (X_β, Y_β) is proximate to (X_α, Y_α) and no point in the first neighbourhood of (X_β, Y_β) which has been resolved is also proximate to (X_α, Y_α) . Again on F'' , since the curve representing (X_γ, Y_γ) is to meet those representing (X_α, Y_α) and (X_β, Y_β) , it follows that (X_γ, Y_γ) is proximate to (X_α, Y_α) and (X_β, Y_β) , and thus (X_γ, Y_γ) is the point in the first neighbourhood of (X_β, Y_β) which is proximate to (X_α, Y_α) . Thus the relations of (X_α, Y_α) , (X_β, Y_β) , (X_γ, Y_γ) are those of the points $O_{s_1+s_2+\dots+s_\mu-1-1}$, $O_{s_1+s_2+\dots+s_\mu-1}$, $O_{s_1+s_2+\dots+s_\mu}$ of the sequence (1.1). Now, applying the equations (3.1) and (3.2) for (X_γ, Y_γ) and analogous equations for (X_α, Y_α) and (X_β, Y_β) , it is easily verified that

$$X_\alpha Y_\gamma - X_\gamma Y_\alpha = X_\gamma Y_\beta - X_\beta Y_\gamma = \pm 1,$$

whence the required result, that X_γ/Y_γ lies between X_α/Y_α and X_β/Y_β , follows immediately.

PART II.

The behaviour in the neighbourhood of an isolated united point of systems of curves compounded of a cyclic involution.

In Part II we investigate the properties of linear systems of curves compounded of the involution I_n . We consider such a system of curves not having a base-point at an isolated united point O of I_n and discover the rules which determine the base-multiplicities in the neighbourhoods of O of sub-systems of this system. First, in §7, we consider the problem for systems of a fairly general nature, in which, for example, fundamental curves through O are permitted, but obtain only crude results, referring entirely to the first neighbourhood of O . These results, not very interesting in themselves, are required for the theorems of §8, where, restricting the nature of the systems considered, we obtain results concerning the base-multiplicities of the sub-systems at the united points of I_n in all neighbourhoods of O .

We find, in fact, necessary conditions which must be satisfied by the base-multiplicities of the sub-systems in the neighbourhoods of O ; and,

[†] Barber and Zariski, (I), Theorem 5.

in general, there are several sets of base-multiplicities satisfying these conditions. It will be clear, however, from the methods employed, that the conditions are not sufficient; in other words, we cannot assert that a sub-system having any particular set of base-multiplicities actually exists, but only that any actual system has one of the sets as its base-system in the neighbourhoods of O .

In §§ 10-12, where we are concerned with the existence of particular sub-systems, this fact is important.

7. Let $|D|$ be a linear system of curves on F compounded of the involution I_n , but not of any other involution, having no base-point at O and of sufficiently high freedom, grade, and genus to allow its curves to satisfy the base-conditions which we shall impose upon them†. Then we prove

THEOREM III. *If O is a δ -perfect united point, the multiplicity at O of any sub-system of $|D|$ is an integral multiple of n/δ and the sum of the multiplicities at O of those branches of the sub-system which do not contain a united point of the involution in the neighbourhood of O is an integral multiple of δ .*

We describe branches through O but not through a united point in the neighbourhood of O , as "free" branches, though it must be remembered that these branches are not necessarily free in the sense of having variable tangents, since their points in the neighbourhoods of O may be fixed by conditions other than those mentioned in the theorem.

To prove the theorem we use the local parameters u, v of § 4 in the neighbourhood of O . Then the curves of $|D|$ are given by an equation of the form

$$\lambda_0 + \lambda_{10}u + \lambda_{01}v + \sum_{i+j \geq 2}^m \lambda_{ij}u^i v^j = 0,$$

where $\lambda_0 (\neq 0)$ and the λ_{ij} are variable but may be connected by linear equations. Then, by (4.1), the transformation T transforms this curve into

$$\lambda_0 + \lambda_{10}\epsilon^\beta u + \lambda_{01}\epsilon^\alpha v + \dots = 0,$$

the terms not written down being of degree 2 at least. Since $|D|$ is compounded of I_n , these two curves are identical.

Let $|D^*|$ be a sub-system of $|D|$ having a base-point at O and let the (or a) term of least order in the equation of a curve of $|D^*|$ be $u^p v^q$, so that $\mu = p+q$ is the multiplicity of $|D^*|$ at O . Then the equation of the

† For the method of construction of such a system, see Godeaux (1), 8, 9.

transform by T of this curve contains this (or these) terms of least order, each multiplied by a factor $\epsilon^{\beta p + \alpha q}$, and, since this must be identical with the original curve, we have

$$\beta p + \alpha q \equiv 0 \pmod{n}. \quad (7.1)$$

If, now, O is δ -perfect, we have by (4.2)

$$\delta(\alpha - \beta) \equiv 0 \pmod{n}.$$

Thus, multiplying (7.1) by δ and using this last congruence, we obtain

$$\delta\alpha(p + q) \equiv 0 \pmod{n},$$

$$\begin{aligned} \text{i.e.,} \quad & \delta\alpha\mu \equiv 0 \pmod{n}, \\ \text{and similarly} \quad & \delta\beta\mu \equiv 0 \pmod{n}. \end{aligned} \quad (7.2)$$

If α or β is prime to n , then one of these congruences gives

$$\mu \equiv 0 \pmod{n/\delta}. \quad (7.3)$$

If not, since we have shown (§4) that n , α , β have no common factor, α and β must be mutually prime and therefore there exist two integers l , m , such that

$$l\alpha - m\beta = 1.$$

Now from (7.2) we deduce

$$\delta(l\alpha - m\beta)\mu \equiv 0 \pmod{n},$$

$$\text{i.e.,} \quad \delta\mu \equiv 0 \pmod{n},$$

whence (7.3) follows. This proves the first part of the theorem.

Now, μ being the multiplicity of $|D^*|$ at O , let $u^{p_i}v^{q_i}$, $p_i + q_i = \mu$, $i = 0, 1, 2, \dots, t$, be the terms of least order in the equation of a generic curve of $|D^*|$. Then we can write

$$p_i = p_0 + r_i, \quad q_i = q_0 - r_i, \quad (7.4)$$

and, applying (7.1) to p_i , q_i and to p_0 , q_0 and using (7.4), we obtain

$$(\beta - \alpha)r_i \equiv 0 \pmod{n};$$

and therefore, by (4.2), r_i is a multiple of δ , say

$$r_i = k_i\delta. \quad (7.5)$$

Now let p_α and q_β be the least of the numbers p_i and q_i , respectively, $i = 0, 1, 2, \dots, t$. Then $p_\alpha + q_\beta$ is the multiplicity at O of those branches

of $|D^*|$ which touch $u = 0$ or $v = 0$, that is, which pass through the united points of I_n in the neighbourhood of O . But

$$\begin{aligned} p_a + q_\beta &= p_0 + q_0 + (p_a - p_0) + (q_a - q_0) \\ &= \mu + (k_a - k_\beta)\delta, \quad \text{by (7.4) and (7.5),} \\ &= \mu - k\delta, \quad \text{say,} \end{aligned}$$

and therefore the multiplicity at O of the free branches of $|D^*|$ is equal to

$$\mu - (p_a + q_\beta) = k\delta,$$

which is the required result.

Now consider the surface F_1 , obtained from F by resolving O , on which the neighbourhood of O is represented by the curve $E_0^{(1)}$, of virtual grade -1 . Let the variable parts of the images on F_1 of $|D|$ and $|D^*|$ be denoted by $|D^{(1)}|$ and $|D^{*(1)}|$. We have, then,

$$|D^{(1)}| \equiv |D^{*(1)} + kE_0^{(1)}|,$$

where k is an integer. Hence, using $(A.B)$ to denote the virtual intersection number of the linear systems $|A|$ and $|B|$, we have

$$(D^{*(1)}.E_0^{(1)}) + k(E_0^{(1)}.E_0^{(1)}) = (D^{(1)}.E_0^{(1)}).$$

But $(D^{(1)}.E_0^{(1)}) = 0$, since $|D|$ does not pass through O , while, as we have seen, $E_0^{(1)}$ is of grade -1 . Thus

$$\mu - k = 0,$$

i.e.,

$$k = \mu = \lambda n/\delta,$$

where λ is an integer by Theorem III. Hence, by a simple step, we have the

COROLLARY. *On any model of F on which the (total or partial) first neighbourhood of the δ -perfect united point O is represented by a simple curve, this curve counts $\lambda n/\delta$ times as a fixed component of the difference between the variable parts of the images of $|D|$ and $|D^*|$, where λ is an integer depending on the system $|D^*|$ but independent of the particular model of F .*

8. Not much more than what is given above can be said about a system of curves so general as $|D|$. We therefore consider, in this and the following paragraphs, linear systems of curves subject to more stringent conditions. These systems are such that their projective models are images of the involution on which the image of O is a single (simple or isolated multiple) point.

Let $|C|$ be a linear system of curves on F satisfying the following conditions:

- (i) it is compounded of I_n but not of any other involution,
- (ii) its freedom, grade, and genus are sufficiently high to allow its curves to satisfy the base conditions which we shall impose upon them,
- (iii) it has no fundamental curves passing through O ,
- (iv) there is no point of F , other than O , such that all curves of $|C|$ through O also pass through this point[†].

We add the further restriction that we do not consider any sub-systems of $|C|$ having fixed components passing through O . (It is not difficult to show that, if the imposing of certain base conditions at O causes the curves of $|C|$ to have a fixed component, then a partial sub-system of a sufficiently high multiple of $|C|$ exists satisfying conditions (i)–(iv) and such that the curves of this new system on which are imposed the same base conditions do not have a fixed component through O .)

In this paragraph we investigate the nature of those sub-systems of $|C|$ whose multiplicity at O is less than n . We find, in fact, necessary but not sufficient conditions for the sets of base-multiplicities of the sub-systems in the neighbourhoods of O .

Let $|C_a|$ be a sub-system of $|C|$ having multiplicity less than n at O and let (X_i, Y_i) be a united point of I_n in some neighbourhood of O , which is a base-point of $|C_a|$ at which curves of $|C_a|$ have a branch not passing through any united point of I_n in the neighbourhood of (X_i, Y_i) . As in §7, we describe such a branch as a “free” branch through (X_i, Y_i) , and a similar remark to the one made there applies to the meaning of the term.

Let F' be a model of F , obtained by resolving all the ancestors of (X_i, Y_i) , on which the image of (X_i, Y_i) is an actual point, P' say. Let I_n' be the image on F' of I_n and let $|C'|$ and $|C_a'|$ be the image of $|C|$ and the variable part of the image of $|C_a|$, respectively. The sub-system of $|C'|$ corresponding to $|C_a|$ consists of $|C_a'|$ together with a certain linear combination of the curves of F' representing neighbourhoods of O . Since $|C_a|$ has a free branch at (X_i, Y_i) , $|C_a'|$ has a free branch at P' , and clearly the system $|C'|$ satisfies the conditions required of the system $|D|$ of Theorem III. Thus, putting $\delta = n$ in that theorem, we see that, if (X_i, Y_i) is an imperfect united point, the sum of the multiplicities of free

[†] The work of the present paragraph does not require this condition, but it will be necessary for Part III.

branches of $|C'_a|$ at P' is a multiple of n and so the multiplicity of $|C_a|$ at O is greater than n , which contradicts our initial conditions. Hence:

The sub-system $|C_a|$ of $|C|$ cannot have a free branch at any imperfect united point of I_n in a neighbourhood of O .

Next let us suppose that the set (X_i, Y_i) , $i = 1, 2, \dots, s$, of united points of I_n in the neighbourhoods of O includes all those at which $|C_a|$ has a free branch, and, in addition, any other not imperfect united points of I_n in the neighbourhoods of O (the reasoning which follows still holds if imperfect united points are included, but nothing is gained thereby). For brevity, we denote (X_i, Y_i) by P_i and we call all the ancestors of these points P_k , $k = s+1, s+2, \dots, q$, in any order. Let $\nu_k^{(i)}$ denote the multiplicity at P_k of a branch of least order through P_i . By Theorem III, the sum of the multiplicities of free branches of $|C_a|$ at P_i , $i = 1, 2, \dots, s$, is a multiple of δ_i , where $P_i \equiv (X_i, Y_i)$ is δ_i -perfect; let it be $\delta_i \mu_{ia}$. Let σ_{ka} , $k = 1, 2, \dots, q$, denote the multiplicity of $|C_a|$ at P_k . Then

$$\sigma_{ka} = \sum_{i=1}^s \nu_k^{(i)} \delta_i \mu_{ia}. \quad (8.1)$$

Let F' be a model of F on which a point, P_j , and all its ancestors are represented by simple curves. Let I_n' denote the image on F' of I_n and let $|C'|$, $|C'_a|$ denote the image of $|C|$ and the variable part of the image of $|C_a|$, respectively. The total image of $|C_a|$ on F' consists of $|C'_a|$, together with a fixed part which is a linear combination, with positive integral coefficients, of the curves of F' which represent neighbourhoods of O . And, by the corollary to Theorem III, the curve representing the neighbourhood of P_j must count $\lambda_{ja} n / \delta_j$ times in this fixed part, where λ_{ja} is a positive integer (for the systems under consideration on F' clearly satisfy the same conditions as those considered in § 7). Thus:

On a surface on which the neighbourhood of P_j is represented by a simple curve, this curve counts $\lambda_{ja} n / \delta_j$ times as a fixed component of the difference between the variable parts of the images of $|C|$ and $|C_a|$.

Now on any surface on which all the points P_j , $j = 1, 2, \dots, q$, are represented by simple curves, the exceptional curve, G_k say, representing the total neighbourhood of P_k , contains the curve representing the total (or partial) first neighbourhood of P_j as a component $\nu_k^{(j)}$ times†. Further, since the multiplicity of $|C_a|$ at P_k is σ_{ka} , the curve G_k counts σ_{ka} times as a part of the difference between the variable parts of the images of $|C|$ and

† Barber and Zariski (1), Theorem 3.

$|C_a|$. Hence the number of times which the curve representing the (total or partial) first neighbourhood of P_j occurs as a fixed component of the difference between the variable parts of the images of $|C|$ and $|C_a|$ is

$$\sum_{k=1}^q \nu_k^{(j)} \sigma_{ka};$$

and, by the result italicized above, we have therefore

$$\sum_{k=1}^q \nu_k^{(j)} \sigma_{ka} = \lambda_{ja} n / \delta_j.$$

Thus, by (8.1),

$$\sum_{k=1}^q \left\{ \nu_k^{(j)} \sum_{i=1}^s \nu_k^{(i)} \delta_i \mu_{ia} \right\} = \lambda_{ja} n / \delta_j \quad (j = 1, 2, \dots, s),$$

$$i.e., \quad \sum_{i=1}^s \left\{ \delta_i \mu_{ia} \sum_{k=1}^q \nu_k^{(i)} \nu_k^{(j)} \right\} = \lambda_{ja} n / \delta_j \quad (j = 1, 2, \dots, s),$$

$$i.e., \quad \sum_{i=1}^s \delta_i \delta_j \rho_{ij} \mu_{ia} = n \lambda_{ja} \quad (j = 1, 2, \dots, s), \quad (8.2)$$

where ρ_{ij} is the number of intersections of primitive branches through $(X_i, Y_i), (X_j, Y_j)$. Therefore, by Theorem I,

$$\delta_i \delta_j \rho_{ij} = \delta_i \delta_j \min (X_i Y_j, X_j Y_i) = \min (x_i y_j, x_j y_i),$$

where x_i, y_i are the elements of the matrix $\mathfrak{X}$ of § 5 (assuming that the appropriate columns have been included in $\mathfrak{X}$).

Clearly, we can now take any number of sub-systems of $|C|$ having multiplicities less than n at O , say $|C_a|$, $a = 1, 2, \dots, t$, and, taking the matrix $\mathfrak{X}$ to include columns representing all united points of I_n at which any of the sub-systems $|C_a|$ has a free branch, obtain for each $|C_a|$ equations of the form (8.2). If we write all these st equations together in matrix form, we obtain

THEOREM IV. *Let $|C_a|$, $a = 1, 2, \dots, t$, be t subsystems of $|C|$ whose multiplicities at O are less than n .*

Let the matrix $\mathfrak{X}$ of § 4 include columns, of elements x_i, y_i , representing all united points of I_n in the neighbourhoods of O at which any of the systems $|C_a|$, $a = 1, 2, \dots, t$, has a free branch and let it also include columns representing any other not imperfect united points of I_n in the neighbourhoods of O .

Let $\mathfrak{R}$ be the square matrix of s rows and columns constructed from $\mathfrak{X}$ by the rule that its element in the i -th row and j -th column is

$$\min (x_i y_j, x_j y_i).$$

Let $\mathfrak{M}$ be the matrix of s rows and t columns whose element, μ_{ia} , in the i -th row and a -th column is given by the fact that $\delta_i \mu_{ia}$ is equal to the sum of the multiplicities at (X_i, Y_i) of free branches of $|C_a|$ through (X_i, Y_i) .

Then
$$\mathfrak{M}\mathfrak{N} = n\mathfrak{L}, \quad (8.3)$$

where $\mathfrak{L}$ is a matrix of integers of s rows and t columns. The element λ_{ia} in the i -th row and a -th column of $\mathfrak{L}$ is given by the fact that $(n/\delta_i) \lambda_{ia}$ is the number of times which, on any model of F on which the (partial or total) first neighbourhood of (X_i, Y_i) is represented on a simple curve, this curve counts as a fixed component of the difference between the variable parts of the images of $|C|$ and $|C_a|$.

Clearly the numbers μ_{ia} determine completely the system $|C_a|$, the multiplicities of $|C_a|$ at the ancestors of the points (X_i, Y_i) being determined by the proximity relations, *i.e.*, by the equations (8.1).

The equation (8.3) is sufficient to determine the sub-systems of $|C|$ in any simple case and, as we shall see in §§ 10–12, it leads to general formulae for certain important sub-systems of $|C|$ in any case, that is, for all possible values of n, α, β . Clearly the equation (8.3) is independent of the order of numbering of the points (X_i, Y_i) .

As an example, suppose that $n = 6, \alpha = 2, \beta = 1$. Then, taking only three columns, we have

$$\mathfrak{X} = \begin{pmatrix} 1, & 4, & 5 \\ 2, & 2, & 4 \end{pmatrix}$$

and so

$$\mathfrak{Y} = \begin{pmatrix} 1, & 2, & 5 \\ 2, & 1, & 4 \\ 1, & 2, & 1 \end{pmatrix}.$$

Hence

$$\mathfrak{R} = \begin{pmatrix} 2, & 2, & 4 \\ 2, & 8, & 10 \\ 4, & 10, & 20 \end{pmatrix}$$

and so equation (8.3) gives

$$\begin{pmatrix} 2, & 2, & 4 \\ 2, & 8, & 10 \\ 4, & 10, & 20 \end{pmatrix} \begin{pmatrix} 3, & 2, & 1 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix} = 6 \begin{pmatrix} 1, & 1, & 1 \\ 1, & 2, & 2 \\ 2, & 3, & 4 \end{pmatrix}.$$

The base multiplicities of the systems represented in $\mathfrak{M}$ are shown, in Figure 3, on the scheme described in § 2, the numbers indicating the multiplicities of the sub-systems at the points at which they are marked. It

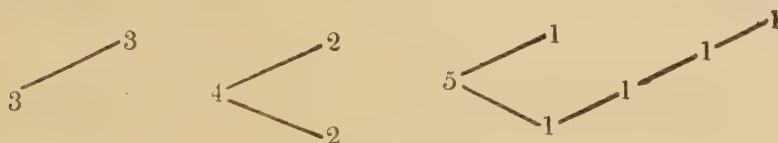


Fig. 3.

will be found that the solution, with integers, of the matrix equation (8.3) is quite simple even in much more complicated cases than this.

Though, for simplicity and because it is the most important case, we have confined our attention to the investigation of sub-systems of $|C|$ whose multiplicity at O is less than n , it is not difficult to remove this restriction. The only difference is that now the sub-systems may have free branches at imperfect united points (the sum of the multiplicities of such branches at such a point being a multiple of n) and so we must admit to $\mathfrak{X}$ columns representing imperfect united points.

9. In § 5, we showed that, if each pair of the numbers n, α, β had a common factor, so that

$$n = gh n', \quad \alpha = gk \alpha', \quad \beta = hk \beta',$$

and if $\mathfrak{X}'$ was a matrix (as in § 5) which catalogues a selection of the united points of a cyclic involution of order n' in the neighbourhoods of an (α', β') united point, then $\mathfrak{X}$ is a similar matrix for the neighbourhood of an (α, β) united point of an involution of order n , where

$$\mathfrak{X} = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix} \mathfrak{X}'. \quad (9.1)$$

We now show that a corresponding relation holds between the properties of systems of curves such as we have been discussing, in the two cases. In fact, let $\mathfrak{R}$ be the matrix of Theorem IV, § 8, obtained in the usual way from $\mathfrak{X}$, and let $\mathfrak{R}'$ be the corresponding matrix obtained from $\mathfrak{X}'$. Then, by (9.1), $\mathfrak{R} = gh \mathfrak{R}'$. Thus, if $\mathfrak{M}, \mathfrak{L}$ are matrices of integers such that

$$\mathfrak{R}' \mathfrak{M} = n' \mathfrak{L},$$

then also

$$\mathfrak{R} \mathfrak{M} = gh \mathfrak{R}' \mathfrak{M} = gh n' \mathfrak{L} = n \mathfrak{L},$$

and conversely. Thus, taking $\mathfrak{M}$ and $\mathfrak{L}$ to be the matrices of Theorem IV, we see that:

If $|C|$ and $|C'|$ are systems of curves satisfying the conditions of § 8 for the two involutions I_n and $I_{n'}$ in the neighbourhoods of isolated united points, O and O' , respectively, and if $|C'_j|$ is a sub-system of $|C'|$ given, in the notation of § 8, by the numbers, μ_{ij} and λ_{ij} (where $\delta_i' \mu_{ij}$ is the sum of the multiplicities of its free branches at the united point of $I_{n'}$, in the neighbourhood of O' , represented by the i -th column of $\mathfrak{X}'$), then there is a sub-system $|C_j|$ of $|C|$, having the same numbers μ_{ij} and λ_{ij} ($\delta_i \mu_{ij}$ now referring to the united point of I_n , in the neighbourhood of O , represented by the i -th column of $\mathfrak{X}$) and conversely.

The important thing about this result is that the matrices $\mathfrak{M}$ and $\mathfrak{L}$, that is, the numbers μ_{ij} , λ_{ij} are the same for both involutions, though the united points to which they refer are different, being related by (9.1).

It is clear that the further reduction, given in § 5, to the case in which $\beta = 1$ and $\alpha = \alpha_0$ is prime to n , applies here, since it depends on no more than the fact that the congruences (5.12) and (5.13) have the same solutions.

The difference between the virtual grades of $|C|$ and $|C_a|$ is conveniently expressed in terms of the matrices $\mathfrak{M}$ and $\mathfrak{L}$. Let F' be a model of F obtained by resolving the points $P_i, i = 1, 2, \dots, q$, that is the points

$$(X_i, Y_i) \equiv P_i \quad (i = 1, 2, \dots, s),$$

and all their ancestors (in the notation of § 8). Let $E'_i, i = 1, 2, \dots, q$, be the curve of F' representing the (total or partial) first neighbourhood of P_i . Let $|C'|$ and $|C'_a|$ be the variable parts of the images on F' of $|C|$ and $|C_a|$. Then, by definition of λ_{ia} , we have

$$|C'| \equiv |C'_a + \mathbf{E}_a'|, \quad (9.2)$$

$$\text{where} \quad \mathbf{E}_a' = n \sum_{i=1}^s \frac{\lambda_{ia}}{\delta_i} E'_i + \sum_{i=s+1}^q k_{ia} E'_i, \quad (9.3)$$

and the numbers k_{ia} are positive integers. Also, since $|C|$ has no base-point at O ,

$$(C' \cdot E'_i) = 0 \quad (i = 1, 2, \dots, q), \quad (C' \cdot \mathbf{E}_a') = 0, \quad (9.4)$$

and, since the sum of the multiplicities of free branches of $|C_a|$ at P_i is $\delta_i \mu_{ia}$, if $i = 1, 2, \dots, s$ and zero if $i = s+1, s+2, \dots, q$, we have

$$\left. \begin{aligned} (C'_a \cdot E'_i) &= \delta_i \mu_{ia} & (i = 1, 2, \dots, s), \\ &= 0 & (i = s+1, s+2, \dots, q). \end{aligned} \right\} \quad (9.5)$$

Now, by (9.2),

$$(C' \cdot E_a') = (C_a' \cdot E_a') + (E_a' \cdot E_a'),$$

whence, by (9.4)

$$\begin{aligned} (E_a' \cdot E_a') &= -(C_a' \cdot E_a') \\ &= -n \sum_{i=1}^s \frac{\lambda_{ia}}{\delta_i} \mu_{ia} \delta_i, \quad \text{by (9.3), (9.5)} \\ &= -n \sum_{i=1}^s \lambda_{ia} \mu_{ia}. \end{aligned}$$

But the difference, γ_a say, between the virtual grades of $|C|$ and $|C_a|$ is equal to the negative of the virtual grade of E_a' , that is,

$$(C^2) - (C_a^2) = n \sum_{i=1}^s \lambda_{ia} \mu_{ia}. \quad (9.6)$$

Thus γ_a is the diagonal element in the a -th row and column of the matrix, $n\mathfrak{M}'\mathfrak{X}$, where $\mathfrak{M}'$ indicates the transpose of $\mathfrak{M}$.

10. We next obtain details of one particular sub-system of $|C|$, which in all that follows we denote by $|C_1|$. It will be shown later† that, when $|C_1|$ *actually* exists, it is in fact the sub-system obtained by imposing on the curves of $|C|$ the single condition of passing through O . But since our method depends on the results of § 8, we can prove only the *virtual* existence of $|C|$.

We consider only the case in which $\beta = 1$ and $a = a_0$ is prime to n , since, by the results of §§ 5, 9, all other cases may be reduced to this. All the numbers which enter into our calculations are derived by an algorithmic process from the numbers a_0 and n . By the term “integer” we shall always understand positive, non-zero integer.

Let b_0 be the least integer such that $a_0 b_0 - 1 = k_0 n$, where k_0 is an integer. Let a_1, b_1 be the least integers such that $a_0 - a_1 = k_0 \theta_0$, $b_0 - b_1 = k_0 \phi_0$, where θ_0, ϕ_0 are integers, so that θ_0 is the integral part of $a_0/k_0, \dots$. Then it is easily verified that $a_1 b_1 - 1$ is divisible by k_0 ; let $a_1 b_1 - 1 = k_1 k_0$, and let a_2, b_2 be the least integers such that $a_1 - a_2 = k_1 \theta_1$, $b_1 - b_2 = k_1 \phi_1$, where θ_1 and ϕ_1 are integers. Then $a_2 b_2 - 1$ is divisible by k_1 ; let $a_2 b_2 - 1 = k_2 k_1, \dots$. And inductively, let $a_{i-1} b_{i-1} - 1 = k_{i-1} k_{i-2}$

† Part III, § 11.

and let a_i, b_i be the least integers such that

$$a_{i-1} - a_i = k_{i-1} \theta_{i-1}, \quad b_{i-1} b_i = k_{i-1} \phi_{i-1},$$

where θ_{i-1}, ϕ_{i-1} are integers; then $a_i b_i - 1$ is divisible by k_{i-1} ; let

$$a_i b_i - 1 = k_i k_{i-1}.$$

By definition, $a_{i+1} < a_i$ and $b_{i+1} < b_i$; therefore, for some value of i , $a_i = 1$ and, for some value of i , $b_i = 1$. It is easily shown that these two values of i are the same. Thus, taking the least value of i for which $a_i = b_i = 1$ to be $i = r - 1$, we have the equations

$$\left. \begin{aligned} a_i b_i - 1 &= k_i k_{i-1}, \\ a_i - a_{i+1} &= k_i \theta_i, \quad b_i - b_{i+1} = k_i \phi_i, \\ i &= 0, 1, 2, \dots, r-2, \\ a_{r-1} &= b_{r-1} = 1, \end{aligned} \right\} \quad (10.1)$$

where $a_i, b_i, \theta_i, \phi_i, k_i$ are integers, a_i and b_i being the least integers which satisfy the successive equations.

We first show how to obtain the matrix

$$\mathfrak{X} = \begin{pmatrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_r \end{pmatrix}$$

for the neighbourhoods of O including in it only columns representing united points of I_n which will be later shown to be base-points of $|C_1|$ at which $|C_1|$ has free branches. In fact, we show that (X_i, Y_i) is a δ_i -perfect united point of I_n , $\delta_i < n$ ($i \leq r$), (that is a not imperfect united point), where δ_i is the highest common factor of x_i, y_i , and n , and $X_i = x_i/\delta_i$, $Y_i = y_i/\delta_i$, and where x_i, y_i are defined inductively as follows:

If r is even, say $r = 2m$,

$$\left. \begin{aligned} x_1 &= 1, \quad x_2 = b_0, \\ x_{2i+1} &= \frac{1}{k_{2i-1} k_{2i-2}} \{ (a_{2i} b_{2i-1} - 1) x_{2i-1} + (a_{2i-1} - a_{2i}) x_{2i} \}, \\ x_{2i+2} &= \frac{1}{k_{2i-1} k_{2i-2}} \{ (b_{2i-1} - b_{2i}) x_{2i-1} + (a_{2i-1} b_{2i} - 1) x_{2i} \}, \\ i &= 1, 2, \dots, m-1. \end{aligned} \right\} \quad (10.2)$$

If r is odd, $r = 2m + 1$, the same equations apply for $i = 1, 2, \dots, m - 1$, while x_{2m+1} has the form which would be obtained from either of these

equations by putting $i = m$, ($a_{2m} = b_{2m} = 1$), that is

$$x_{2m+1} = \frac{1}{k_{2m-1} k_{2m-2}} \{ (b_{2m-1} - 1) x_{2m-1} + (a_{2m-1} - 1) x_{2m} \}.$$

And

$$y_1 = a_0, \quad y_2 = 1,$$

and y_i , $i = 3, 4, \dots, r$, is defined by a set of equations exactly similar to those defining x_i .

The proofs, by induction, of the following formulae present no difficulties except those of complicated arithmetic:

$$x_{2i+1} = x_{2i-1} + \theta_{2i-1} \{ \phi_{2i-2} x_{2i-1} + \phi_{2i-4} x_{2i-3} + \dots + \phi_0 x_1 \}, \quad (10.3)$$

and similar results for x_{2i+2} , y_{2i+1} , y_{2i+2} .

$$x_{2i} y_{2i-1} - x_{2i-1} y_{2i} = k_{2i-2} n, \quad (10.4)$$

$$x_{2i+1} y_{2i-1} - x_{2i-1} y_{2i+1} = \theta_{2i-1} n, \quad (10.5)$$

$$x_{2i} y_{2i+2} - x_{2i+2} y_{2i} = \phi_{2i-1} n, \quad (10.6)$$

and, in the case $r = 2m + 1$,

$$x_{2m+1} y_{2m-1} - x_{2m-1} y_{2m+1} = \theta_{2m-1} n, \quad (10.7)$$

$$x_{2m} y_{2m+1} - x_{2m+1} y_{2m} = \phi_{2m-1} n. \quad (10.8)$$

Since, by definition, x_1 and all the numbers θ_i , ϕ_i are integers, it follows by induction from (10.3) that x_{2i+1} is an integer; similarly x_{2i+2} , y_{2i+1} , y_{2i+2} are integers, that is, all the numbers of the matrix $\mathfrak{X}$ which we have defined are integers.

By definition, $a_0 x_1 \equiv y_1 \pmod{n}$, and using (10.4)–(10.8), we can show by induction that $a_0 x_i \equiv y_i \pmod{n}$, $i = 1, 2, \dots, r$. For example, if it is known that $a_0 x_{2i-1} \equiv y_{2i-1} \pmod{n}$, then, by substituting in (10.5), we obtain $a_0 x_{2i+1} \equiv y_{2i+1} \pmod{n}$. Thus, by the results of § 5, if δ_i is the highest common factor of x_i , y_i , and n , and if $X_i = x_i/\delta_i$, $Y_i = y_i/\delta_i$, then (X_i, Y_i) is a δ_i -perfect united point of I_n . Further it is not difficult to show from (10.1) and (10.2) that $x_i, y_i < n$ and therefore $\delta_i < n$, in other words, that (X_i, Y_i) , $i = 1, 2, \dots, r$, is not imperfect.

We next show that, if $\mathfrak{R}$ is derived as in § 8 from this matrix

$$\mathfrak{X} = \begin{pmatrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_r \end{pmatrix},$$

and if the first column of $\mathfrak{M}$ is the transpose of

$$(\phi_0, \theta_0, \phi_2, \theta_2, \dots, \phi_{2i}, \theta_{2i}, \dots, \phi_{2m-2}, \theta_{2m-2}) \quad (\text{if } r = 2m),$$

or of

$$(\phi_0, \theta_0, \phi_2, \theta_2, \dots, \phi_{2i}, \theta_{2i}, \dots, \phi_{2m-2}, \theta_{2m-2}, k_{2m-1}) \quad (\text{if } r = 2m+1),$$

then the equation (8.3), $\mathfrak{RM} = n\mathfrak{L}$, is satisfied (so far as the first columns of $\mathfrak{M}$ and $\mathfrak{L}$ are concerned) and that the elements of the first column of $\mathfrak{L}$ are all unity. That is, we show the virtual existence of a sub-system $|C_1|$ of $|C|$, having free branches only at the points (X_i, Y_i) , $i = 1, 2, \dots, r$, the sum of the multiplicities of free branches at (X_i, Y_i) being $\delta_i \mu_{i,1}$, where

$$\left. \begin{aligned} \mu_{2i+1,1} &= \phi_{2i} \\ \mu_{2i+2,1} &= \theta_{2i} \end{aligned} \right\} (i = 0, 1, 2, \dots, m-1), \quad \text{if } r = 2m, \quad (10.9)$$

$$\left. \begin{aligned} \mu_{2i+1,1} &= \phi_{2i} \\ \mu_{2i+2,1} &= \theta_{2i} \end{aligned} \right\} (i = 0, 1, 2, \dots, m-1), \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \text{if } r = 2m+1, \quad (10.10)$$

$$\mu_{2m+1,1} = k_{2m-1}$$

and that, in both cases,

$$\lambda_{i,1} = 1 \quad (i = 1, 2, \dots, r). \quad (10.11)$$

From the equations (10.4)–(10.8), we have

$$\left. \begin{aligned} \frac{x_1}{y_1} &< \frac{x_3}{y_3} < \dots < \frac{x_{2m-1}}{y_{2m-1}} < \frac{x_{2m}}{y_{2m}} < \frac{x_{2m-2}}{y_{2m-2}} < \dots < \frac{x_4}{y_4} < \frac{x_2}{y_2} \\ \frac{x_1}{y_1} &< \frac{x_3}{y_3} < \dots < \frac{x_{2m-1}}{y_{2m-1}} < \frac{x_{2m+1}}{y_{2m+1}} < \frac{x_{2m}}{y_{2m}} < \dots < \frac{x_4}{y_4} < \frac{x_2}{y_2} \end{aligned} \right\} \quad (10.12)$$

if $r = 2m$, and

if $r = 2m+1$.

Using these we can construct the matrix $\mathfrak{R}$, and, in the case $r = 2m$ (to which we confine ourselves for the present), the equation resulting from the $(2\alpha-1)$ -th row of $\mathfrak{R}$ and the first column of $\mathfrak{M}$ in the equation $\mathfrak{RM} = n\mathfrak{L}$ which we now have to prove is

$$\begin{aligned} y_{2\alpha-1} \sum_{i=1}^{\alpha} x_{2i-1} \mu_{2i-1,1} + x_{2,1-1} \sum_{i=\alpha+1}^m y_{2i-1} \mu_{2i-1,1} \\ + x_{2\alpha-1} \sum_{i=1}^{\alpha} y_{2i} \mu_{2i,1} + x_{2\alpha-1} \sum_{i=\alpha+1}^m y_{2i} \mu_{2i,1} = n. \end{aligned} \quad (10.13)$$

The proof is by induction from row to row, that is, with respect to a . If $a = 1$, the equation to be proved is

$$\sum_{i=1}^{2m} y_i \mu_{i,1} = n \quad (10.14)$$

and this is proved, using (10.1), (10.2), (10.9), by showing successively that

$$\begin{aligned} \sum_{i=2m-1}^{2m} y_i \mu_{i,1} &= \frac{1}{k_{2m-4}} \{ (b_{2m-3} - 1) y_{2m-3} + (a_{2m-3} - 1) y_{2m-2} \}, \\ \sum_{i=2m-3}^{2m} y_i \mu_{i,1} &= \frac{1}{k_{2m-6}} \{ (b_{2m-5} - 1) y_{2m-5} + (a_{2m-5} - 1) y_{2m-4} \}, \end{aligned}$$

and so on, each equation being obtained by adding two terms to both sides of the last. Hence, by induction, we reach the required result.

Next (10.13) is proved by induction with respect to a . The equation obtained from it by writing $a+1$ for a may be put in the form

$$\begin{aligned} y_{2a+1} \sum_{i=1}^a x_{2i-1} \mu_{2i-1,1} + x_{2a+1} \sum_{i=a+1}^m y_{2i-1} \mu_{2i-1,1} \\ + x_{2a+1} \sum_{i=1}^a y_{2i} \mu_{2i,1} + x_{2a+1} \sum_{i=a+1}^m y_{2i,1} \mu_{2i,1} = n. \end{aligned} \quad (10.15)$$

Multiplying (10.15) by x_{2i-1} and (10.13) by x_{2a+1} , and subtracting, we see that (10.15) is true if (10.13) is true and

$$(x_{2a+1} y_{2a-1} - x_{2a-1} y_{2a+1}) \sum_{i=1}^a x_{2i-1} \mu_{2i-1,1} = (x_{2a+1} - x_{2a-1}) n,$$

which, by (10.1), (10.2), (10.5), (10.9), reduces to

$$\frac{1}{k_{2a-2}} (x_{2a} - b_{2a-2} x_{2a-1}) - \sum_{i=1}^{a-1} x_{2i-1} \mu_{2i-1,1} = 0.$$

But, by (10.1) and (10.2), the first member of this equation can be proved to be equal to

$$\frac{1}{k_{2a-4}} (x_{2a-2} - b_{2a-4} x_{2a-3}) - \sum_{i=1}^{a-2} x_{2i-1} \mu_{2i-1,1},$$

and so, by induction, to

$$\frac{1}{k_0} (x_2 - b_0 x_1),$$

which is zero, by (10.1) and (10.2).

Thus the required result is proved. Similar methods serve to prove the equations arising from the rows of even number and also to prove the case $r = 2m + 1$.

Thus the work of this paragraph amounts to the following:

Given n, a_0 , we obtain the numbers $a_i, b_i, \theta_i, \phi_i, k_i, r$ by the algorithm (10.1). In terms of these we define the set of united points (X_i, Y_i) , $i = 1, 2, \dots, r$, by means of the equations (10.2). We then show the *virtual* existence of a sub-system, $|C_1|$, of $|C|$ which has free branches only at these united points, the sum of the multiplicities of its free branches at (X_i, Y_i) being $\delta_i \mu_{i,1}$, where $\mu_{i,1}$ is given by (10.9) and (10.10).

By (9.6), the difference between the virtual grades of $|C|$ and $|C_1|$ is

$$\begin{aligned} (C^2) - (C_1^2) &= n \sum_{i=1}^r \lambda_{i,1} \mu_{i,1} \\ &= n \sum_{i=1}^r \mu_{i,1}, \end{aligned} \quad (10.15)$$

since $\lambda_{i,1} = 1$, $i = 1, 2, \dots, r$, by (10.11).

In the sequel we always denote the sub-system of $|C|$ discussed in this paragraph by $|C_1|$ and the first columns of $\mathfrak{M}$ and $\mathfrak{L}$ are therefore those referring to this sub-system.

PART III.

The "natural" image of the involution and the image thereon of an isolated united point.

11. The projective model of any linear system of curves of F compounded of the involution I_n may be taken as an image of the involution. But we may regard as the "natural" image of the involution, in so far as the representation of the neighbourhood of O is concerned, a projective model Φ of a system $|C|$ satisfying the conditions of § 8. On Φ the image of the point O is a point, S say, which may be a simple or an isolated singular point of Φ . It is the object of Part III of this paper to discover the nature of the point S in terms of the numbers θ_i, ϕ_i, k_i , defined in § 10. First a few preliminaries are necessary.

In order to discuss conveniently the singularity of Φ at S , we assume Φ to be in a space of a sufficiently high number of dimensions to allow a prime section of Φ to pass through any of the singular points of Φ in the neighbourhoods of S , in other words, we assume $|C|$ to be of sufficiently high freedom. This can always be assured by taking as $|C|$ a sufficiently high multiple of a system of curves satisfying the conditions of § 8, or, if

the curves of this system are not compounded of I_n , extracting from it a partial system whose curves are compounded of I_n , which is then taken as the system $|C|$.

All the work that follows depends on the *actual* existence of the sub-system $|C_1|$ of $|C|$ whose *virtual* existence was proved in § 10. It will be clear in what follows that, since the system $|C_1|$ is used to define a transformation of F into another surface, the theorems we prove are not true unless $|C_1|$ actually exists.

We must further assume that the actual base system of $|C_1|$ coincides exactly with the virtual base system described in § 10. In particular, we must assume that $|C_1|$ has no base-points in the neighbourhoods of O , other than the united points of I_n and similar assumptions must be made for the other sub-systems of $|C|$ which are required for the proofs of theorems in Part III. This implies that what we have hitherto called a "free" branch through a united point (X_i, Y_i) , that is, a branch not passing through any united point in the neighbourhood of (X_i, Y_i) , is now actually free in the sense that it has no base-point at all in the neighbourhood of (X_i, Y_i) .

That cases do exist in which these conditions are satisfied is easily verified; for example, the case in which I_n is a cyclic homography of the plane.

If $|\Gamma|, |\Gamma_a|$ are the images on Φ of the systems of curves $|C|, |C_a|$ of § 8, then $|\Gamma|$ is the system of prime sections of Φ and $|\Gamma_a|$ is some sub-system of this system. Clearly the grade of any linear system of curves on F compounded of I_n but not of any other involution is n times that of the corresponding system on Φ (or on any other model of I_n), and so, by (9.6),

$$(\Gamma^2) - (\Gamma_a^2) = \sum_{i=1}^s \lambda_{ia} \mu_{ia}. \quad (11.1)$$

Thus the difference between the virtual grades of the two systems is the diagonal element in the a -th row and column of the matrix $\mathfrak{M}' \mathfrak{L}$, where $\mathfrak{M}'$ is the transpose of $\mathfrak{M}$.

Let (X_i, Y_i) be a δ_i -perfect united point of I_n in some neighbourhood of O ; then its image on Φ is a point, S_i say (singular or not) in some neighbourhood of S . Let Φ' be a model of Φ on which the total or partial neighbourhood of this point, and therefore also of (X_i, Y_i) , is represented by an irreducible curve, L'_i say. Let $|\Gamma'|, |\Gamma'_a|$ be the variable parts of the images on Φ' of $|\Gamma|, |\Gamma_a|$. Since the transformation defined in the neighbourhood of (X_i, Y_i) by T is of period δ_i , there is a $(\delta_i, 1)$ correspondence between the neighbourhoods of (X_i, Y_i) and S_i , that is, between the neighbourhood of (X_i, Y_i) and the curve L'_i . But, by definition of

$\mu_{i\alpha}$ (§ 8), the sum of the multiplicities of free branches of $|C_\alpha|$ at (X_i, Y_i) is $\delta_i \mu_{i\alpha}$. Thus, in view of the remark made above about free branches, we have:

(a) *The number of variable intersections of $|\Gamma'_\alpha|$ with L'_i is $\mu_{i\alpha}$; and $\mu_{i\alpha}$ is also the sum of the multiplicities at S_i of branches of $|\Gamma_\alpha|$ which pass through S_i , but not through any fixed point in the neighbourhood of S_i .*

Since every point of Φ' represents a group of n (possibly coincident) points of F and since the correspondence between the neighbourhood of (X_i, Y_i) and the curve L'_i is $(\delta_i, 1)$, each point of L'_i represents n/δ_i points of the first neighbourhood of (X_i, Y_i) . Thus if E_i^* is the curve representing the total or partial first neighbourhood of (X_i, Y_i) on some model of F , L'_i is the transform of $(n/\delta_i)E_i^*$. But, by definition of $\lambda_{i\alpha}$ (§ 8), the number of times which E_i^* counts as a fixed component of the difference between the variable parts of the images of $|C|$ and $|C_\alpha|$ is $(n/\delta_i)\lambda_{i\alpha}$. Hence, since $|\Gamma'|$ and $|\Gamma'_\alpha|$ are images of $|C|$ and $|C_\alpha|$, we have:

(b) *The number of times which L'_i counts as a fixed component of $|\Gamma' - \Gamma'_\alpha|$ is $\lambda_{i\alpha}$.*

Let Φ_1 be a projective model of the system $|C_1|$ of § 10. Then Φ_1 is a model of Φ on which the neighbourhoods of the points

$$(X_i, Y_i) \quad (i = 1, 2, \dots, r)$$

(the united points of I_n in the neighbourhoods of O at which $|C_1|$ has a free branch) are represented by curves, $L_i^{(1)}$ say, but on which every other point of F is represented by a point (singular or not). Let $|\Gamma^{(1)}|$ and $|\Gamma_1^{(1)}|$ be the variable parts of the images on Φ_1 of $|\Gamma|$ and $|\Gamma_1|$, that is, of $|C|$ and $|C_1|$.

Then, by (a) above, we have

$$(\Gamma_1^{(1)} \cdot L_i^{(1)}) = \mu_{i,1} \quad (i = 1, 2, \dots, r) \quad (11.2)$$

(where, as in all future intersection numbers, only variable intersections are counted). Again, by (10.11) and (b) above, $L_i^{(1)}$, $i = 1, 2, \dots, r$, counts *once* as a component of $|\Gamma^{(1)} - \Gamma_1^{(1)}|$. From this last it follows that the curves $L_i^{(1)}$, $i = 1, 2, \dots, r$, represent components of the first neighbourhood of S . On Φ_1 the total neighbourhood of S must be represented by a set of curves and points which are images of the neighbourhoods of O . Now, since $|C_1|$ has free branches only at the points (X_i, Y_i) , $i = 1, 2, \dots, r$, it follows that $|\Gamma_1^{(1)}|$ has intersections with no components of the neighbourhood of S , other than the curves $L_i^{(1)}$, which, as we have

seen, are all components of the first neighbourhood of S . Thus $|\Gamma_1^{(1)}|$ is the image on Φ_1 of the system of prime sections of Φ through S , which is therefore $|\Gamma_1|$. Thus the curves $L_i^{(1)}$ of Φ_1 represent the total first neighbourhood of S on Φ .

It follows that the tangent cone to Φ at S consists of r component cones (obviously rational), K_i say, $i = 1, 2, \dots, r$, corresponding to the curves $L_i^{(1)}$ and hence to the points (X_i, Y_i) . Since $|\Gamma_1^{(1)}|$ represents on Φ_1 the system of prime sections of Φ through S , it follows that the order of the cone K_i , $i = 1, 2, \dots, r$, is equal to the intersection number $(\Gamma_1^{(1)} \cdot L_i^{(1)})$, which, by (11.2) is $\mu_{i,1}$, where $\mu_{i,1}$ is given by (10.9), (10.10),

To complete our knowledge of the tangent cone at S it is necessary to know the intersection properties of the cones K_i . We consider a model F of F' on which the total first neighbourhood of the points (X_i, Y_i) , $i = 1, 2, \dots, r$, which correspond to the cones K_i are represented by irreducible curves. On such a model the partial first neighbourhoods of all the ancestors of (X_i, Y_i) are also represented by curves and, by the first theorem of §6, these two sets of curves together form a simply connected chain. Two of the cones K_i , say K_h and K_k , have a line in common if and only if the sub-chain on F connecting the components representing (X_h, Y_h) and (X_k, Y_k) does not contain a curve representing any of the other cones K_i , that is, any of the other points (X_i, Y_i) , $i = 1, 2, \dots, r$. And, taking the surface F of the second theorem of §6 to be our present F , we see that, if the cones K_i are arranged in order of magnitude of the fractions $X_i/Y_i = x_i/y_i$, $i = 1, 2, \dots, r$, then the cones placed adjacent in this order have a line in common. Thus, by (10.12), we see that K_h and K_k , $h = 1, 2, \dots, r-2$, $k = h+2$ or $h = r-1$, $k = r$, have a line in common, but otherwise K_h and K_k have only the point S in common. This is, of course, all on the assumption that Φ lies in a space of more than four dimensions; if not, there are in addition improper intersections of the component cones.

Two points may be noticed in passing. We have seen that $|\Gamma_1|$ is the system of prime sections of Φ through S . It follows that:

The system $|C_1|$ of curves of F , discussed in §10, is that sub-system of $|C|$ which is obtained by imposing on the curves of $|C|$ the single condition of passing through O (provided that $|C_1|$ is an actual system).

The surface Φ_1 is the transform by a birational transformation without exceptional elements of the surface obtained by resolving the point S of Φ . We may, therefore, without loss of generality, use Φ_1 to denote this last surface, for in considerations relating to neighbourhoods a transformation without exceptional elements makes no difference.

12. We have next to consider the fictitious singularities of Φ in the neighbourhoods of S , that is, in other words, the actual singularities of Φ_1 lying on the curves $L_i^{(1)}$, $i = 1, 2, \dots, r$, and any fictitious singularities of Φ_1 in their neighbourhoods.

We use here some theorems about fictitious singular points which would follow immediately from Du Val's work†, but which, since that work is not yet published, we prove briefly here from the classical point of view. Let Φ lie in a space R of σ dimensions. If the point S is resolved, R is transformed into a σ -fold, V say, on which the first neighbourhood of S as a point of R is represented by a linear space, Ω say, of dimension $\sigma - 1$. The image on V of Φ is a surface Φ_1 , which (apart from a possible birational transformation without exceptional elements, not affecting our considerations) is the Φ_1 of § 11. The first neighbourhood of S as a point of Φ is represented by the curve M of intersection of Φ_1 with Ω , so that in our case $M = \sum_{i=1}^r L_i^{(1)}$. Any point S_1 of Φ , in the first neighbourhood of S , is represented by a point, $S_1^{(1)}$, of M ; and, since M is the section of Φ_1 by Ω , it follows that the multiplicity of Φ_1 at $S_1^{(1)}$, which is also the multiplicity of Φ at S_1 , cannot exceed the multiplicity of M at $S_1^{(1)}$; and that, when these two multiplicities are equal, if M has distinct tangents at $S_1^{(1)}$, the generic prime section of Φ_1 through $S_1^{(1)}$ also has distinct tangents.

In our case $M = \sum_{i=1}^r L_i^{(1)}$ (being the image of the tangent cone $\sum_{i=1}^r K_i$ to Φ , the intersections of whose components were described in § 11) has as its only singularities ordinary double points at the intersections of the components $L_h^{(1)}$, $L_k^{(1)}$, $h = 1, 2, \dots, r-2$, $k = h+2$, or $h = r-1$, $k = r$. Thus the only singularities of Φ_1 on M lie at these points and are of multiplicity not greater than two for Φ_1 . Also their generic prime section has distinct tangents and so they must be simple points, conic nodes, or binodes.

Suppose, then, that S_1 is a binode in the first neighbourhood of S on Φ , so that $S_1^{(1)}$ is a binode of Φ_1 , lying at the intersection of $L_h^{(1)}$ and $L_k^{(1)}$. The composition of the binode is a succession of κ (say) double points $S_1^{(1)}, S_2^{(1)}, \dots, S_\kappa^{(1)}$, lying on a linear branch, of which $S_\kappa^{(1)}$ may be a conic node or a simple binode (of species 3). We seek to prove that none of the points $S_2, S_3, \dots, S_\kappa$, images on Φ of $S_2^{(1)}, S_3^{(1)}, \dots, S_\kappa^{(1)}$, is proximate to S ; and to do this it is necessary only to show that S_2 is not proximate to S .

Suppose, then, that S_2 is proximate to S . Then a primitive branch through S , S_1 , S_2 is an ordinary cusp and the corresponding branch on V ,

† Du Val (2).

which passes through $S_1^{(1)}$, $S_2^{(1)}$ is a linear branch touching Ω . Thus $S_2^{(1)}$, as well as $S_1^{(1)}$ lies on Ω , and, of course, on Φ_1 , and is therefore the point of M in the first neighbourhood of $S_1^{(1)}$. If then, as we are supposing, $S_2^{(1)}$ is a double point of Φ_1 , it must also be a double point of the section M of Φ_1 by Ω .

But, since the only double points of $M = \sum_{i=1}^r L_i^{(1)}$ are the *ordinary* double points at the intersections of its components, this is not the case; so we conclude that the double point $S_2^{(1)}$ in the first neighbourhood of $S_1^{(1)}$ does not lie on M , and therefore S_2 is not proximate to S .

Thus, to sum up, the only singular points of Φ in the neighbourhood of S are a finite number of binodes, whose images on Φ_1 are binodes at the intersection of two curves $L_h^{(1)}$, $L_k^{(1)}$, $h = 1, 2, \dots, r-2$, $k = h+2$, or $h = r-1$, $k = r$; and no singular point of Φ , other than those in the first neighbourhood of S , is proximate to S ; in other words, all the singular points of Φ in the neighbourhoods of S lie on linear branches through S . It follows that a prime section of Φ constrained to pass through any number of these singular points lying on one linear branch is not thereby also compelled to pass through another. This fact is used implicitly in the arguments which follow.

To preserve consistency in the arithmetic we shall use the terms "binode of species one" and "binode of species two" to mean simple point and conic node, respectively.

We consider now the case in which a binode, $S_1^{(1)}$ say, of Φ_1 at the intersection of $L_h^{(1)}$ and $L_k^{(1)}$ is of even species, 2κ say. Then the singularity at $S_1^{(1)}$ consists of κ double points $S_1^{(1)}$, $S_2^{(1)}$, ..., $S_\kappa^{(1)}$, each in the first neighbourhood of its predecessor, and $S_\kappa^{(1)}$ is a conic node.

Let $|\Gamma_2^{(1)}|$ be the sub-system of $|\Gamma_1^{(1)}|$ constrained to pass through the double points $S_1^{(1)}$, $S_2^{(1)}$, ..., $S_\kappa^{(1)}$. We discover the number κ by the fact that the difference between the grades of $|\Gamma_1^{(1)}|$ and $|\Gamma_2^{(1)}|$ is 2κ . Since $S_\kappa^{(1)}$ is a conic node, its first neighbourhood is irreducible and is therefore represented on F by the neighbourhood of a single united point of I_n in the neighbourhood of O , say the point (X_{r+1}, Y_{r+1}) . We now take the matrices $\mathfrak{X}$, $\mathfrak{Y}$ of § 5 to consist of the columns corresponding to the points (X_i, Y_i) , $i = 1, 2, \dots, r+1$, the first r of these being (as in § 11) the points of F representing the curves $L_i^{(1)}$ of Φ_1 and being given by (10.2). The matrix $\mathfrak{R}$ of § 8, Theorem IV, is then defined by the rule that its element in the i -th row and j -th column is

$$\min(x_i y_j, x_j y_i) \quad (i, j = 1, 2, \dots, r+1).$$

As usual, the first columns of $\mathfrak{M}$, $\mathfrak{X}$ refer to the system $|C_1|$, the image on F of $|\Gamma_1^{(1)}|$; so the elements of the first column of $\mathfrak{M}$ are $\mu_{i,1}$, $i = 1, 2, \dots, r$,

given by (10.9), (10.10), and

$$\mu_{r+1,1} = 0, \quad (12.1)$$

since $|\Gamma_1^{(1)}|$ does not pass through $S_k^{(1)}$. The elements of the first column of $\mathfrak{L}$ are

$$\lambda_{i,1} = 1 \quad (i = 1, 2, \dots, r+1). \quad (12.2)$$

This last equation follows, for $i = 1, 2, \dots, r$, from (10.11); and, for $i = r+1$, it follows by (b), § 11, from the facts that the total neighbourhood of $S_1^{(1)}$ counts once as a component of the neighbourhood of S , while the first neighbourhood of $S_k^{(1)}$ counts once as a component of the total neighbourhood of $S_1^{(1)}$ and therefore once as a component of the neighbourhood of S .

Now let $|C_2|$ be the sub-system of $|C_1|$ on F which corresponds to $|\Gamma_2^{(1)}|$. We take the second columns of $\mathfrak{M}$ and $\mathfrak{L}$ to correspond to $|C_2|$ and so we find the values for $\mu_{i,2}$ and $\lambda_{i,2}$, as follows:

Since $S_1^{(1)}$ lies on $L_h^{(1)}, L_k^{(1)}$, the number of intersections of $|\Gamma_2^{(1)}|$ with $L_i^{(1)}$, $i = 1, 2, \dots, r$, $i \neq h, k$, is the same as that of $|\Gamma_1^{(1)}|$ with the same curve, while the numbers of intersections of $|\Gamma_2^{(1)}|$ with $L_h^{(1)}, L_k^{(1)}$ are each one less than those of $|\Gamma_1^{(1)}|$ with the same curves. But, by remark (a), § 11, $\mu_{i,2} = (\Gamma_2^{(1)} \cdot L_i^{(1)})$, and so, by (11.2),

$$\left. \begin{aligned} \mu_{i,2} &= \mu_{i,1} \quad (i = 1, 2, \dots, r, \ i \neq h, k), \\ &= \mu_{i,1} - 1 \quad (i = h, k). \end{aligned} \right\} \quad (12.3)$$

Lastly, since $S_k^{(1)}$ is a double point and $|\Gamma_2^{(1)}|$ is cut by a system of primes through it, the multiplicity of $|\Gamma_2^{(1)}|$ at $S_k^{(1)}$ is two and therefore by (a), § 11,

$$\mu_{r+1,2} = 2. \quad (12.4)$$

Also, since $|\Gamma_2^{(1)}|$ does not contain any of the curves $L_i^{(1)}$, $i = 1, 2, \dots, r$, of Φ_1 as fixed components, we have, by (b), § 11, $\lambda_{i,2} = \lambda_{i,1}$, and therefore, by (12.2),

$$\lambda_{i,2} = 1 \quad (i = 1, 2, \dots, r). \quad (12.5)$$

The next step in the argument is to find $\lambda_{r+1,2}$, and this requires the use of the equation (8.3),

$$\mathfrak{R}\mathfrak{M} = n\mathfrak{L}.$$

From the equations (10.12), (10.2) we see that the elements of the first row of $\mathfrak{R}$ are y_i , $i = 1, 2, \dots, r$, and that therefore the equations arising from the first row of $\mathfrak{R}$ are

$$\sum_{i=1}^{r+1} y_i \mu_{ij} = \lambda_{i,j} n \quad (j = 1, 2).$$

Subtracting these equations and using (12.2)–(12.5), we have

$$y_h + y_k - 2y_{r+1} = 0,$$

$$\text{i.e.,} \quad y_{r+1} = \frac{1}{2}(y_h + y_k), \quad (12.6)$$

and similarly, using the second row of $\mathfrak{R}$, we obtain

$$x_{r+1} = \frac{1}{2}(x_h + x_k), \quad (12.7)$$

Let us now consider a typical case by putting $h = 2\gamma - 1$, $k = 2\gamma + 1$, $1 \leq 2\gamma - 1 \leq r - 2$. Then, from (10.12), (12.6), (12.7), we have

$$\frac{x_{2\gamma-1}}{y_{2\gamma-1}} < \frac{x_{r+1}}{y_{r+1}} < \frac{x_{2\gamma+1}}{y_{2\gamma+1}}. \quad (12.8)$$

Thus, using σ_{ij} to denote $\min(x_i y_j, x_j y_i)$, we have, by (12.6), (12.7), (12.8),

$$\left. \begin{aligned} \sigma_{r+1, 2\gamma-1} &= \frac{1}{2}x_{2\gamma-1}(y_{2\gamma-1} + y_{2\gamma+1}), \\ \sigma_{r+1, 2\gamma+1} &= \frac{1}{2}(x_{2\gamma-1} + x_{2\gamma+1})y_{2\gamma+1}, \\ \sigma_{r+1, r+1} &= \frac{1}{4}(x_{2\gamma-1} + x_{2\gamma+1})(y_{2\gamma-1} + y_{2\gamma+1}). \end{aligned} \right\} \quad (12.9)$$

Now, from the last row of $\mathfrak{R}$ in the equation $\mathfrak{R}\mathfrak{M} = n\mathfrak{L}$, we have the equations

$$\sum_{i=1}^{r+1} \sigma_{r+1, i} \mu_{i, j} = n\lambda_{r+1, j} \quad (j = 1, 2).$$

Subtracting these and using (12.1), (12.3), (12.4), we have

$$2\sigma_{r+1, r+1} - \sigma_{r+1, 2\gamma-1} - \sigma_{r+1, 2\gamma+1} = n(\lambda_{r+1, 2} - \lambda_{r+1, 1});$$

hence by (12.2),

$$\begin{aligned} n(\lambda_{r+1, 2} - 1) &= 2\sigma_{r+1, r+1} - \sigma_{r+1, 2\gamma-1} - \sigma_{r+1, 2\gamma+1} \\ &= \frac{1}{2}(x_{2\gamma+1}y_{2\gamma-1} - x_{2\gamma-1}y_{2\gamma+1}) \quad \text{by (12.9)} \\ &= \frac{1}{2}\theta_{2\gamma-1}n \quad \text{by (10.5)}. \end{aligned}$$

$$\text{Therefore} \quad \lambda_{r+1, 2} = 1 + \frac{1}{2}\theta_{2\gamma-1}. \quad (12.10)$$

Thus we have

$$\begin{aligned} (\Gamma_1^{(1)} \cdot \Gamma_1^{(1)}) - (\Gamma_2^{(1)} \cdot \Gamma_2^{(1)}) &= \{(\Gamma^{(1)} \cdot \Gamma^{(1)}) - (\Gamma_2^{(1)} \cdot \Gamma_2^{(1)})\} - \{(\Gamma^{(1)} \cdot \Gamma^{(1)}) - (\Gamma_1^{(1)} \cdot \Gamma_1^{(1)})\} \\ &= \sum_{i=1}^{r+1} (\lambda_{i, 2} \mu_{i, 2} - \lambda_{i, 1} \mu_{i, 1}) \quad \text{by (11.1)} \\ &= \theta_{2\gamma-1}, \end{aligned}$$

by (12.1)–(12.5) and (12.10). That is, the difference between the grade of a general prime section of Φ_1 and that of a general prime section of Φ_1 through the κ double points $S_1^{(1)}, S_2^{(1)}, \dots, S_\kappa^{(1)}$ is $\theta_{2\gamma-1}$; hence $2\kappa = \theta_{2\gamma-1}$.

Thus the species, when it is even, of the binode on Φ_1 at the intersection of the curves $L_{2\gamma-1}^{(1)}$ and $L_{2\gamma+1}^{(1)}$ is $\theta_{2\gamma-1}$.

It may also be proved that when $S_1^{(1)}$ is a binode of odd species, say $2\kappa+1$, its species is again $\theta_{2\gamma-1}$. The proof differs in some important respects from that just given, but space permits only a sketch of the differences. In this case the singularity at $S_1^{(1)}$ consists of a succession of n double points $S_1^{(1)}, S_2^{(1)}, \dots, S_\kappa^{(1)}$, of which the last is a binode of species three. The first neighbourhood of $S_\kappa^{(1)}$ consists of a pair of rational curves intersecting in one simple point. The neighbourhood of this last point we now take to be that represented on F by the united point (X_{r+1}, Y_{r+1}) ; and $|\Gamma_2^{(1)}|$ we now take to be the sub-system of $|\Gamma_1^{(1)}|$ through this point. Changing the meaning of the rest of our notation in an obvious way, we find the following alterations to the formulae already given (where, as before, $h = 2\gamma-1$, $k = 2\gamma+1$)

$$(12.2) \quad \lambda_{i,1} = 1 \quad (i = 1, 2, \dots, r),$$

$$\lambda_{r+1,1} = 2,$$

$$(12.4) \quad \mu_{r+1,2} = 1,$$

$$(12.6) \quad y_{r+1} = y_h + y_k,$$

$$(12.7) \quad x_{r+1} = x_h + x_k,$$

(12.9) the numerical fraction in the second member of each equation is omitted,

$$(12.10) \quad \lambda_{r+1,2} = 2 + \theta_{2\gamma-1},$$

and the formulae lead in the same way as before to the result

$$(\Gamma_1^{(1)} \Gamma_1^{(1)}) - (\Gamma_2^{(1)} \Gamma_2^{(1)}) = \theta_{2\gamma-1},$$

from which it follows easily that $S_1^{(1)}$ is of species $\theta_{2\gamma-1}$.

There are also to be considered separately the cases $h = 2\gamma$, $k = 2\gamma+2$, $2 \leq 2\gamma \leq r-2$, and $h = r-1$, $k = r$, but these differ only arithmetically from those already dealt with. We merely include the results in the theorem, stated below, which summarises all our results concerning the singularity at S .

All these considerations have been for a point S corresponding to an $(a_0, 1)$ united point O of I_n , where a_0 is prime to n . Now we showed in

§§ 5, 9 how the investigation of the neighbourhoods of an (α, β) united point and of the behaviour in its neighbourhood of the system $|C|$ could in all cases be reduced to the investigation of an $(a_0, 1)$ united point, a_0 being prime to n . A corresponding reduction takes place in the work of Part III of this paper.

The descriptive part of the work clearly applies to the general case, while the only numbers which enter into our final results are the following :

(i) the numbers λ_{ia}, μ_{ia} which characterize certain sub-systems of $|C|$; and these, as we have seen in § 9, are unaltered when the investigation of the general case is reduced to that of an $(a_0, 1)$ united point,

(ii) the order of the numerical magnitudes, but not the actual values, of the fractions x_i/y_i (which determine the intersection properties of the cones K_i); but, by (5.11), this order is unaltered in the reduction of the general case to the simple case.

Thus we conclude that, in investigating the nature of the singularity at S in the general case, we may first perform the reduction of §§ 5, 9 to the case of an $(a_0, 1)$ united point (a_0 prime to n) and then state the results which we have obtained in Part III. Hence we have

THEOREM V. *Let O be an isolated (α, β) united point of a cyclic involution, I_n , of order n on an algebraic surface F . Let $|C|$ be a linear system of curves compounded of I_n and satisfying the conditions of § 8. Let Φ be a projective model of $|C|$.*

If the sub-system $|C_1|$ of $|C|$, whose virtual existence was proved in § 10, is an actual system and if also certain sub-systems† of $|C_1|$ are actual, then the nature of the image point S on Φ of O is given in terms of n, α, β , as follows :

First, by the process described in § 5, we reduce the general case to one in which $\beta = 1$ and $\alpha = a_0$ is prime to n . Then if $r, \theta_i, \phi_i, k_i, i = 0, 1, 2, \dots, r-2$, are the integers derived from n, a_0 , by the equations (10.1), the multiplicity of Φ at S is

$$\sum_{i=0}^{\frac{1}{2}r-1} (\theta_{2i} + \phi_{2i}), \quad \text{when } r \text{ is even,}$$

$$\sum_{i=0}^{\frac{1}{2}(r-3)} (\theta_{2i} + \phi_{2i}) + k_{r-2}, \quad \text{when } r \text{ is odd.}$$

† *Viz.*, the sub-systems of which the $|C_2|$ of the present paragraph is a typical one.

The tangent cone to Φ at S consists of r rational components $K_1, K_2, \dots, K_r$, the order $\mu_{i,1}$ of K_i being given, when r is even, by

$$\left. \begin{aligned} \mu_{2i+1,1} &= \phi_{2i} \\ \mu_{2i+2,1} &= \theta_{2i} \end{aligned} \right\} \quad (i = 0, 1, 2, \dots, \tfrac{1}{2}r-1),$$

and, when r is odd, by

$$\left. \begin{aligned} \mu_{2i+1,1} &= \phi_{2i} \\ \mu_{2i+2,1} &= \theta_{2i} \end{aligned} \right\} \quad \{i = 0, 1, 2, \dots, \tfrac{1}{2}(r-3)\},$$

$$\mu_{r,1} = k_{r-2}.$$

The cones K_i and K_j ($i < j$) have a line in common if $i = 1, 2, \dots, r-2$, $j = i+2$ or $i = r-1$, $j = r$, but otherwise they have only the point S in common (provided that the ambient space of Φ is of more than four dimensions).

If the point S is resolved, then, on the surface Φ_1 obtained, the first neighbourhood of S is represented by the set of curves $L_i^{(1)}$, $i = 1, 2, \dots, r$; and the curves $L_i^{(1)}$ and $L_j^{(1)}$ intersect if and only if $i = 1, 2, \dots, r-2$, $j = i+2$, or $i = r-1$, $j = r$.

At the intersection of two intersecting curves $L_h^{(1)}$, $L_k^{(1)}$, $h < k$ ($k = h+2$, if $h = 1, 2, \dots, r-2$, or $k = r$, if $h = r-1$) Φ_1 has a binode of species u_h , where

$$(a) \text{ if } \quad h = 2\gamma-1, \quad u_h = \theta_{2\gamma-1},$$

$$(b) \text{ if } \quad h = 2\gamma, \quad u_h = \phi_{2\gamma-1},$$

$$(\gamma = 1, 2, \dots, \tfrac{1}{2}r-1, \quad \text{when } r \text{ is even};$$

$$\gamma = 1, 2, \dots, \tfrac{1}{2}(r-1), \quad \text{when } r \text{ is odd}),$$

$$(c) \text{ if } h = r-1, \text{ when } r \text{ is even, } u_h = k_{r-2}.$$

In this, the terms binode of species two and binode of species one are to be understood to mean conic node and simple point, respectively.

The images of these points constitute the only singularities of Φ in the neighbourhoods of S . Each singular point of Φ in the neighbourhood of S is proximate only to its immediate ancestor.

We have seen that, if any two of α, β, n have a common factor, we may, before calculating the nature of the point S , divide these two by their common factor and repeat the process until the integers thus derived from α, β, n are all mutually prime. This leads to some interesting results.

First, let $\alpha = k\alpha'$, $\beta = k\beta'$, where $\alpha'\beta'$ is a multiple of n ; then, by the method indicated we reduce the problem to one in which $n = 1$, that is, to the case in which S is the image of a simple point O of F on a model Φ of

a system of curves of F not compounded of any involution. So in this case S is a simple point.

For example, consider the case $n = 6$, $\alpha = 3$, $\beta = 4$. A cyclic homography of order 6 of the plane, having the origin as a (3, 4) united point, is

$$\left. \begin{aligned} x_1 &= \epsilon^4 x, \\ y_1 &= \epsilon^3 y, \end{aligned} \right\} T,$$

where ϵ is a primitive 6-th root of unity. A system of curves compounded of the involution is

$$\lambda_0 + \lambda_1 y^2 + \lambda_2 x^3 + \lambda_3 x^3 y^2 = 0,$$

and it is not difficult to verify that this can be taken as $|C|$, the conditions of § 8 being satisfied, and also that the system $|C_1|$ of § 10 is the *actual* sub-system of $|C|$ given by putting $\lambda_0 = 0$. A projective model of $|C|$ is given parametrically by

$$\frac{X_0}{1} = \frac{X_1}{y^2} = \frac{X_2}{x^3} = \frac{X_3}{x^3 y^2},$$

where $X_0, \dots, X_3$ are homogeneous coordinates in [3]. Eliminating x, y , we see that this surface is the quadric whose equation is $X_0 X_3 = X_1 X_2$; and so the image of O the point (1, 0, 0, 0) is non-singular, as our general theory indicated.

It may be noted that in this case the lines $x = 0$, $y = 0$ are united curves of T^2 and T^3 , respectively. It appears to be true that in all cases of the type we have just been considering there are curves through O which are united curves, not of T , but of some power of T .

The case $\alpha = \beta$, which can be reduced to $\alpha (= \alpha_0) = \beta = 1$, that is, the case in which O is a perfect united point, has not been considered explicitly in our work. The result in this case is given by Godeaux†; S is then an n -ple point of Φ whose tangent cone is rational and irreducible. Again, if $\alpha = k\alpha'$, $\beta = k\beta'$, where $\alpha'\beta'$ is a factor of n , say $n = \alpha'\beta'n'$ (k, n' being mutually prime), then, by the usual reduction, we can take S as being given by $\alpha = \beta = 1$ for an involution of order n' , and therefore the tangent cone at S is again irreducible.

Two cases already worked out in great detail by Godeaux‡ are those given by $\alpha_0 = n - 1$ and by $\alpha_0 = 2$ (n being odd in the latter, since otherwise the case would be included in that which we have just considered). From our

† Godeaux (1), 12-13.

‡ Godeaux (1).

general results it follows immediately that in the former case S is a binode of species $n-1$ ($a_0=b_0=n-1$, $k_0=n-2$, $r=2$, $\theta_0=\phi_0=1$) and in the latter S is a $\frac{1}{2}(n+1)$ -ple point, the tangent cone consisting of a plane and a rational $\frac{1}{2}(n-1)$ -ic cone, there being no fictitious singularities in the neighbourhoods of S ($a_0=2$, $b_0=\frac{1}{2}(n+1)$, $k_0=1$, $r=2$, $\theta_0=1$, $\phi_0=\frac{1}{2}(n-1)$).

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SOME TWO-DIMENSIONAL PROBLEMS IN CONDUCTION OF HEAT WITH CIRCULAR SYMMETRY

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[Received 28 December, 1938.—Read 19 January, 1939.]

1. This paper is related to that by Goldstein in these *Proceedings*, (2), 34 (1932), 51–88. The problems treated are confined to the equation of conduction of heat; similar methods apply to the hydrodynamical problems discussed by Goldstein. Goldstein uses the Heaviside operational calculus as developed by Bromwich*. We employ the Laplace transformation†, since it seems to us that this method has distinct advantages over the operational methods in that: (1) one sees clearly how the subsidiary equation is obtained: (2) the passage from the solution of the subsidiary equation and its boundary conditions to that of the original differential equation depends on the well-known inversion theorem for the Laplace transformation: (3) the solution in the form of a line integral which this theorem gives can be rigorously expressed by Cauchy's theorem as an infinite series (or integral, according to the problem), while one of the disadvantages of the operational calculus is that the expansion theorem (or partial fraction expansion) is applied to continuous systems without any discussion whether this is permissible.

In obtaining the subsidiary equation and its boundary conditions, assumptions (*e.g.* as to the commutability of certain limiting operations) have to be made, so that the solution obtained by the programme above is not rigorous, though it may perhaps be regarded as adequate for most investi-

* See Bromwich, *Proc. London Math. Soc.* (2), 15 (1916), 401. Jeffreys, *Operational methods in mathematical physics* (1927), Cambridge Tracts in Mathematics and Mathematical Physics, No. 23.

† Titchmarsh in his *Fourier integrals*, Ch. X, uses an essentially similar method in the solution of partial differential equations. His work depends on general theorems requiring the Lebesgue integral and more advanced mathematics than we employ.

gations in applied mathematics. Strictly speaking, the solution obtained as a line integral by the application of the inversion theorem is to be regarded as purely formal, and it must be verified that it does satisfy the original differential equation and boundary conditions. This is the point of view adopted in this paper and in all the questions treated here such verification is possible (see, in particular, § 4).

The process of verification may be performed directly on the line integral solution, and in problems arising from differential equations of hyperbolic type this is often the only possible method; but in problems on conduction of heat it is simplest first to transform the line integral to a new path L' (§ 4). This is, in fact (apart from a trivial change of variable), the "Path P " of the method developed long ago by one of us for the solution of problems of this type*.

The principles of the method will be seen from §§ 2, 3, where two familiar problems are solved in this way. In § 4 the question of verification is examined. The sections which follow cover all the heat problems treated by Goldstein and include many now solved for the first time.

The Riemann integral is used throughout.

2. Flow of heat in a circular cylinder $r=a$. Initial temperature of the cylinder zero: the surface kept at a constant temperature v_0 :

Let v be the temperature at r at time t . Then we have to solve the equation

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (0 \leq r < a, \quad t > 0). \quad (1)$$

$$\text{with}^\dagger \quad v = v_0, \quad \text{when} \quad r = a, \quad t > 0, \quad (2)$$

$$v = 0, \quad \text{when} \quad t = 0, \quad 0 \leq r < a. \quad (3)$$

Multiply (1) by e^{-pt} , where $p > 0$, and integrate with regard to t from 0 to ∞ .

* See Carslaw, *Conduction of heat* (1921), 2 ed., Chapters X, XI; this book will be cited in future as C.H.; also papers by Carslaw, *Phil. Mag.* (6), 4 (1902), 162, and *Proc. Edinburgh Math. Soc.* (1), 21 (1903), 40, and Dougall, *ibid.*, 18 (1900), 50.

† Conditions (2) and (3) put more precisely are:

$$\text{for fixed time } t > 0, \quad \lim_{r \rightarrow a} v = v_0, \quad (2')$$

$$\text{for fixed } r \text{ in } 0 \leq r < a, \quad \lim_{t \rightarrow 0} v = 0. \quad (3')$$

Boundary conditions stated shortly as in (2) and (3) are always to be understood in this sense.

Then
$$\int_0^\infty e^{-pt} \frac{\partial v}{\partial t} dt = \kappa \int_0^\infty e^{-pt} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) dt. \quad (4)$$

But
$$\begin{aligned} \int_0^\infty e^{-pt} \frac{\partial v}{\partial t} dt &= \left[e^{-pt} v \right]_0^\infty + p \int_0^\infty e^{-pt} v dt \\ &= p \int_0^\infty e^{-pt} v dt, \end{aligned}$$

if we assume that $\lim_{t \rightarrow \infty} (e^{-pt} v) = 0$.

Also, assuming that the order of differentiation and integration on the right-hand of (4) can be altered, and writing*

$$\bar{v} = \int_0^\infty e^{-pt} v dt, \quad (5)$$

we have the "subsidiary equation"

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r} \frac{d\bar{v}}{dr} = q^2 \bar{v} \quad (0 \leq r < a), \quad (6)$$

where

$$q^2 = p/\kappa. \quad (7)$$

Also, from (2),

$$\bar{v} = v_0/p, \quad \text{when } r = a, \quad (8)$$

if we assume that

$$\lim_{r \rightarrow a} \int_0^\infty e^{-pt} v dt = \int_0^\infty e^{-pt} \lim_{r \rightarrow a} v dt.$$

From (6) and (8),

$$\bar{v} = \frac{v_0}{p} \frac{I_0(qr)}{I_0(qa)}, \quad (9)$$

and we have to determine v from the equation

$$\frac{v_0}{p} \frac{I_0(qr)}{I_0(qa)} = \int_0^\infty e^{-pt} v dt. \quad (10)$$

* The "bar" notation for the Laplace transform will be used throughout this paper.

The "inversion theorem"* gives

$$v = \frac{v_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}, \quad (11)$$

where

$$\mu = \sqrt{(\lambda/\kappa)}.$$

Now $I_0(\mu r)/(I_0(\mu a))$ is a single-valued function of λ . The zeros of $I_0(\mu a)$ are at $\lambda = -\kappa a_s^2$, where $\pm a_1, \pm a_2, \dots$ are the roots† of $J_0(aa) = 0$.

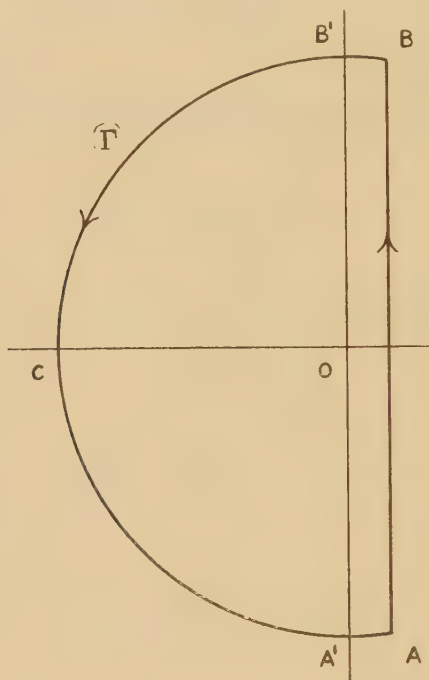


Fig. 1.

Let Γ be a circle, centre 0, radius‡ $R = \kappa(n + \frac{1}{2})^2 (\pi^2/a^2)$, cutting the line $R(\lambda) = \gamma$ at A and B and the imaginary axis at A' and B' .

When $n \rightarrow \infty$, the integral (11) taken over BCA of Fig. 1 tends to zero. This will be proved at the end of this section. Thus we can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ of (11) by the limit when $n \rightarrow \infty$ of the integral over the closed circuit $ABCA$.

* This asserts that $F(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} f(\lambda) d\lambda$ satisfies $f(p) = \int_0^\infty e^{-pt} F(t) dt$, provided that $f(\lambda)$ is analytic for $R(\lambda) \geq \gamma > 0$, and satisfies certain other conditions. See, e.g., Churchill, *Math. Zeitschrift*, 42 (1937), 567. As remarked in § 1, our solution as far as (11) is regarded as formal, so there is no need to show that $\bar{v}$ does satisfy these conditions.

† These are all real and simple. Cf. Watson, *Theory of Bessel functions* (1922), §§ 15.25, 15.21. This work will be cited in future as W.B.F.

‡ With this choice of R , no zero of $I_0(\mu a)$ lies on the circumference of Γ . W.B.F., § 15.32,

It follows, from the theory of residues*, that

$$\begin{aligned}\frac{v}{v_0} &= 1 + \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{J_0(\alpha_s r)}{[\lambda(d/d\lambda) I_0(\mu a)]_{\lambda=-\kappa \alpha_s^2}} \\ &= 1 + \frac{2}{a} \sum_1^{\infty} e^{-\kappa \alpha_s^2 t} \frac{J_0(\alpha_s r)}{\alpha_s J_0'(\alpha_s a)},\end{aligned}\quad (12)$$

the summation being taken over the positive roots of $J_0(\alpha a) = 0$ †.

We now return to the proof that

$$I = \int e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}, \quad (13)$$

taken over the arc $BB'C$ of the circle‡ Γ of radius $R = \kappa(n + \frac{1}{2})^2 (\pi^2/a^2)$, vanishes when $n \rightarrow \infty$.

LEMMA 1. On§ $BB'C$.

$$|\cosh(\mu a - \frac{1}{4}\pi i)| > C_2 \exp[(n + \frac{1}{2})\pi \cos \frac{1}{2}\theta].$$

On the circle Γ we have

$$\begin{aligned}& |\cosh(\mu a - \frac{1}{4}\pi i)|^2 \\ &= |\cosh[(n + \frac{1}{2})\pi e^{\frac{1}{2}i\theta} - \frac{1}{4}\pi i]|^2 \\ &= \frac{1}{2} \cosh[(2n+1)\pi \cos \frac{1}{2}\theta] \{1 + \sin[(2n+1)\pi \sin \frac{1}{2}\theta] \operatorname{sech}[(2n+1)\pi \cos \frac{1}{2}\theta]\}.\end{aligned}$$

$$\text{Let} \quad \beta = 2 \sin^{-1} 2n/(2n+1).$$

Then

$$\sin[(2n+1)\pi \sin \frac{1}{2}\theta] \operatorname{sech}[(2n+1)\pi \cos \frac{1}{2}\theta] \geq 0, \quad \text{when} \quad \pi \geq \theta \geq \beta,$$

* This is the justification for using the Heaviside expansion theorem in this example. Churchill, *Math. Zeitschrift*, 42 (1937), 567, and 43 (1938), 743, has given expansion theorems generalizing Heaviside's by means of which a series expansion of $v(t)$ is obtained provided that $\bar{v}(p)$ satisfies certain conditions. We use the procedure above as being completely self-contained and in practice almost equivalent.

† Cf. C.H., § 110 (1).

‡ We are indebted to a Referee for the remark that Lemma I could have been avoided by the use of a parabolic contour in place of the circle Γ [Cf. Churchill, *Math. Zeitschrift*, 43 (1938), 743], but it seems worth while showing that the obvious circular contour may be used and gives a relatively elementary exposition which can be made rigorous.

§ Here and in the sequel $C, C_1, \dots$ will be used for positive absolute constants, $n_1, n_2, \dots$ for fixed values of n , etc.

and

$$|\sin [(2n+1)\pi \sin \tfrac{1}{2}\theta] \operatorname{sech} [(2n+1)\pi \cos \tfrac{1}{2}\theta]| < \operatorname{sech} [(2n+1)\pi \cos \tfrac{1}{2}\theta] \\ < C_1, \quad \text{when } \beta \geq \theta \geq 0.$$

Thus

$$|\cosh[(n+\tfrac{1}{2})\pi e^{i\theta} - \tfrac{1}{4}\pi i]| > [\tfrac{1}{2}(1-C_1)]^{\frac{1}{2}} \cosh^{\frac{1}{2}}[(2n+1)\pi \cos \tfrac{1}{2}\theta] \\ > C_2 \exp[(n+\tfrac{1}{2})\pi \cos \tfrac{1}{2}\theta].$$

LEMMA II. On $BB'C$,

$$|I_0(\mu r)/I_0(\mu a)| < Cn^{\frac{1}{2}} \exp[-\pi(n+\tfrac{1}{2})(1-r/a) \cos \tfrac{1}{2}\theta]$$

for $0 \leq r \leq a$, if $n > n_1$.

By using the asymptotic expansion of the Bessel functions together with Lemma I, we find that, when $\lambda = \kappa(n+\tfrac{1}{2})^2(\pi^2/a^2)e^{i\theta}$,

$$|I_0(\mu a)| > C_3 n^{-\frac{1}{2}} \exp[(n+\tfrac{1}{2})\pi \cos \tfrac{1}{2}\theta], \quad \text{if } n > n_1,$$

and the Lemma follows when we use the result*

$$|I_0(\mu r)| \leq \exp\left[(n+\tfrac{1}{2}) \frac{\pi r}{a} \cos \tfrac{1}{2}\theta\right].$$

Returning to the integral (13) and using Lemma II, we obtain:

$$\text{Along } BB' \quad |I| < Cn^{\frac{1}{2}} e^{\gamma t} \sin^{-1}(\gamma/R).$$

Along $B'C$, if $t > 0$, $0 \leq r \leq a$,

$$|I| < Cn^{\frac{1}{2}} \int_{\frac{1}{2}\pi}^{\pi} e^{Rt \cos \theta} d\theta < \frac{\pi Cn^{\frac{1}{2}}}{2Rt},$$

and, if $t \geq 0$, $0 \leq r < a$,

$$|I| < Cn^{\frac{1}{2}} \int_{\frac{1}{2}\pi}^{\pi} \exp\left[-\pi(n+\tfrac{1}{2})\left(1-\frac{r}{a}\right) \cos \tfrac{1}{2}\theta\right] d\theta < \frac{aC}{n^{\frac{1}{2}}(a-r)}.$$

Thus finally the integral over the arc $BB'C$, and similarly that over $AA'C$, tends to zero as $n \rightarrow \infty$, provided that either

$$t > 0, \quad 0 \leq r \leq a, \quad \text{or} \quad t \geq 0, \quad 0 \leq r < a.$$

In the other problems of this paper it may be shown in the same way that the integrals along Γ tend to zero as $R \rightarrow \infty$. In all cases a result similar to that of Lemma II is obtained, and the remainder of the proof is as above.

* W.B.F., § 3.31.

3. *The solid bounded internally by $r=a$. Initial temperature zero: the surface $r=a$ at a constant temperature* v_0 .*

Here we have to solve the equation

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (r > a, \quad t > 0), \quad (1)$$

$$v = v_0, \quad \text{when} \quad r = a, \quad t > 0, \quad (2)$$

$$v = 0, \quad \text{when} \quad t = 0, \quad r > a. \quad (3)$$

The subsidiary equation is

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r} \frac{d\bar{v}}{dr} = q^2 \bar{v} \quad (r > a), \quad (4)$$

with
$$\bar{v} = \frac{v_0}{p}, \quad \text{when} \quad r = a. \quad (5)$$

From (4) and (5),
$$\bar{v} = \frac{v_0}{p} \frac{K_0(qr)}{K_0(qa)}, \quad (6)$$

and, when we use the inversion theorem, we get

$$\frac{v}{v_0} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{K_0(\mu r)}{K_0(\mu a)} \frac{d\lambda}{\lambda}, \quad (7)$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

In this case we take Fig. 2 and suppose that $-\pi < \arg \lambda < \pi$, so that the integrand is single-valued inside the closed circuit of the diagram. The circle Γ has radius R which tends to infinity, and the radius of the small circle tends to zero. It is known that there are no zeros of $K_0(\mu a)$ in this region or on its boundary†.

The integral round Γ tends to zero‡ as $R \rightarrow \infty$. Thus, when $R \rightarrow \infty$, we can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ by the sum of the integrals over CD , the small circle at the origin, and EF .

* For a treatment along these lines of the problems of §§ 2 and 3 with the surface temperature $v_0 \cos \omega t$, see Carslaw and Jaeger, *Phil. Mag.* (7), 26 (1938), 473.

† W.B.F., § 15.7.

‡ From the asymptotic expansion of $K_0(z)$ it follows that, for $\lambda = Re^{i\theta}$,

$$\left| \frac{K_0(\mu r)}{K_0(\mu a)} \right| < C \exp \left[-\frac{(a-r)}{a} \left(\frac{R}{\kappa} \right)^{\frac{1}{2}} \cos \frac{1}{2}\theta \right],$$

if R is greater than some fixed R_0 . The remainder of the proof follows as in § 2, but is simpler.

The integral along the small circle gives 1 in the limit as its radius tends to 0. (8)

On CD we put $\lambda = \kappa u^2 e^{-i\pi}$ and obtain for the integral

$$-\frac{1}{\pi i} \int_0^\infty e^{-\kappa u^2 t} \frac{K_0(rue^{-\frac{1}{2}i\pi})}{K_0(au e^{-\frac{1}{2}i\pi})} \frac{du}{u} = -\frac{1}{\pi i} \int_0^\infty e^{-\kappa u^2 t} \frac{J_0(ru) + iY_0(ru)}{J_0(au) + iY_0(au)} \frac{du}{u}, \quad (9)$$

since $K_0(ze^{-\frac{1}{2}i\pi}) = \frac{1}{2}i\pi H_0^{(1)}(z) = \frac{1}{2}i\pi[J_0(z) + iY_0(z)]$.

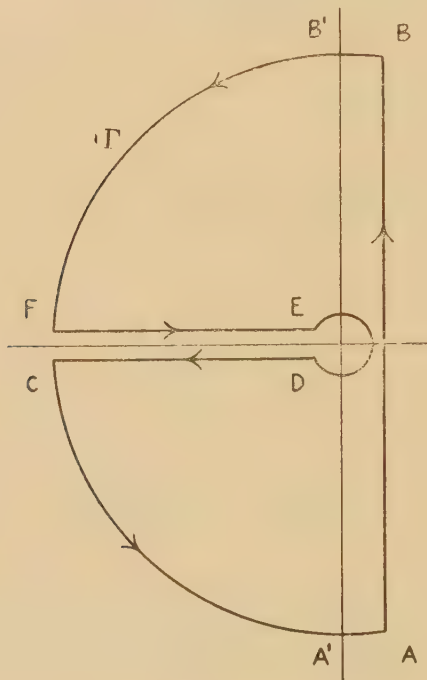


Fig. 2.

On EF we put $\lambda = \kappa u^2 e^{i\pi}$ and obtain the conjugate of (9). Adding these results, we have finally*

$$v = v_0 + \frac{2v_0}{\pi} \int_0^\infty e^{-\kappa u^2 t} \frac{J_0(ru) Y_0(au) - Y_0(ru) J_0(au)}{J_0^2(au) + Y_0^2(au)} \frac{du}{u}. \quad (10)$$

4. In obtaining the results of §§ 2 and 3 various assumptions have been made. Thus to complete the proof we should verify† that the values of

* Goldstein, *loc. cit.*, 53. Nicholson, *Proc. Royal Soc. (A)*, 100 (1921), 226, obtains the correct result, but his reasoning seems to be wrong.

† It is possible to state fairly general conditions on $\bar{v}(p)$ under which most of the verification process below may be performed. Cf. Carslaw and Jaeger, "A problem in conduction of heat", Theorem II, *Proc. Cambridge Phil. Soc.*, 35 (1939), 394-404.

v found in § 2 (11) and § 3 (7) satisfy the given differential equations and the initial and boundary conditions. It is usually easier to perform the verification on line integrals of this type than on the final expressions § 2 (12) and § 3 (10); and, since these have been deduced rigorously from the line integrals, this is all that is necessary.

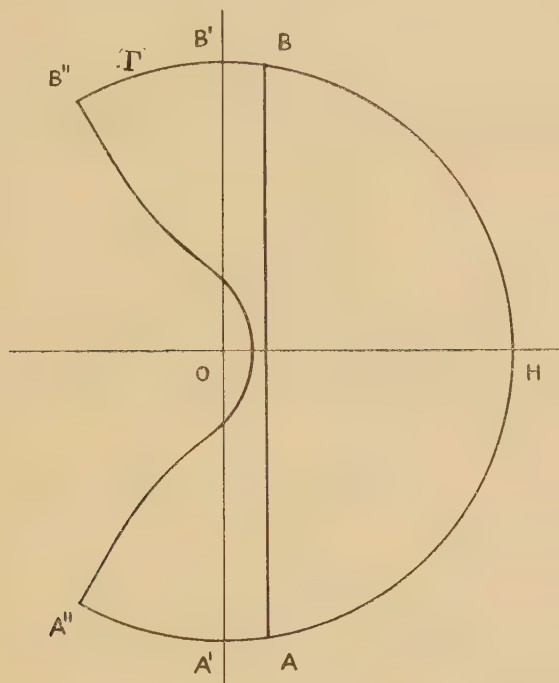


Fig. 3.

We have, from § 2 (11),

$$v = \frac{v_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}. \quad (1)$$

We show first that this can be transformed into the integral along the path L' of Fig. 3, which begins at infinity in the direction $\arg \lambda = -\frac{1}{2}\pi - \beta$, where $0 < \beta < \frac{1}{2}\pi$, passes to the right of the origin keeping all poles of the integrand to the left, and ends in the direction $\arg \lambda = \frac{1}{2}\pi + \beta$. We then show that the integral along L' converges uniformly and can be differentiated under the sign of integration, with regard to both r and t .

Let $\lambda = \kappa \rho e^{i\theta}$, $\mu = \sqrt{\rho} e^{\frac{1}{2}i\theta}$, where $\pi > \theta_0 \geq \theta \geq 0$, θ_0 being fixed. Then, from the asymptotic expansion for $I_0(z)$, it follows that

$$|2\pi\mu a|^{\frac{1}{2}} |I_0(\mu a)| > C_0 \exp[-a\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta], \quad \text{if } \rho > \rho_0, \quad (2)$$

where ρ_0 and C_0 are constants independent of θ .

Thus, using the result* $|I_0(z)| \leq \exp|R(z)|$, we have, when $0 \leq r \leq a$,

$$\left| \frac{I_0(\mu r)}{I_0(\mu a)} \right| < C\rho^{\frac{1}{2}} \exp[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta] \quad \text{if } \rho > \rho_0. \quad (3)$$

Similarly we find that

$$\left| \frac{\partial}{\partial r} \frac{I_0(\mu r)}{I_0(\mu a)} \right| < \tfrac{1}{2} r C \rho^{\frac{5}{2}} \exp[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta] \quad \text{if } \rho > \rho_0. \quad (4)$$

$$\left| \frac{\partial^2}{\partial r^2} \frac{I_0(\mu r)}{I_0(\mu a)} \right| < \tfrac{1}{2} C \rho^{\frac{5}{2}} (1 + \tfrac{1}{2} \rho r^2) \exp[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta] \quad \text{if } \rho > \rho_0. \quad (5)$$

I. To prove that the path L may be deformed into L' , we have to show that the integrals

$$I = \int e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda},$$

taken over the arcs $BB'B''$ and $A''A'A$ of the circle of radius κR , tend to zero as $R \rightarrow \infty$.

Along BB' , by (3),

$$|I| < C R^{\frac{1}{2}} e^{\gamma t} \sin^{-1} \gamma / (\kappa R).$$

Along $B'B''$, by (3),

$$\begin{aligned} |I| &< C R^{\frac{1}{2}} \exp[-(a-r) R^{\frac{1}{2}} \cos(\tfrac{1}{2}\beta + \tfrac{1}{4}\pi)] \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi + \beta} e^{\kappa R t \cos \theta} d\theta \\ &< \begin{cases} \frac{\pi C}{2\kappa t} R^{-\frac{1}{2}}, & \text{if } t > 0, \quad a \geq r \geq 0 \\ \beta C R^{\frac{1}{2}} \exp[-(a-r) R^{\frac{1}{2}} \cos(\tfrac{1}{2}\beta + \tfrac{1}{4}\pi)], & \text{if } t \geq 0, \quad a > r \geq 0. \end{cases} \end{aligned}$$

Thus $\lim_{R \rightarrow \infty} I = 0$ if $t > 0, a \geq r \geq 0$, or if $t \geq 0, a > r \geq 0$, and in these ranges we have

$$v = \frac{v_0}{2\pi i} \int_{L'} e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}. \quad (6)$$

II. Uniform convergence of the integral (6).

By (3), with $\lambda = \kappa \rho e^{i(\beta + \frac{1}{2}\pi)}$, the modulus of the integrand of (6) is less than that of

$$C \int_0^\infty \rho^{\frac{1}{2}} \exp[-\kappa \rho t \sin \beta - (a-r)\rho^{\frac{1}{2}} \cos(\tfrac{1}{2}\beta + \tfrac{1}{4}\pi)] d\rho.$$

* W.B.F., § 3.31.

Thus (6) is uniformly convergent

$$\left. \begin{array}{l} \text{with respect to } t \text{ in } t \geq 0, \text{ for fixed } r \text{ in } 0 \leq r < a, \\ \text{and} \\ \text{with respect to } r \text{ in } 0 \leq r \leq a, \text{ for fixed } t > 0. \end{array} \right\} \quad (7)$$

III. *The value (6) of v satisfies the boundary condition § 2 (2').*

From II, (6) is uniformly convergent with respect to r in $0 \leq r \leq a$ for fixed $t > 0$, and consequently is a continuous function of r in this range.

Therefore
$$\lim_{r \rightarrow a} v = \frac{v_0}{2\pi i} \int_{L'} e^{\lambda t} \frac{d\lambda}{\lambda} = v_0.$$

IV. *The value (6) of v satisfies the boundary condition § 2 (3').*

From II, (6) is uniformly convergent with respect to t in $t \geq 0$, for fixed r in $0 \leq r < a$, and consequently is a continuous function of t .

Therefore
$$\lim_{t \rightarrow 0} v = \frac{v_0}{2\pi i} \int_{L'} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}.$$

To evaluate this integral, consider

$$I = \int \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\lambda}{\lambda}$$

taken round the arc $B'' BHAA''$ of the circle of radius κR in Fig. 3. Here, by (3),

$$|I| < 2CR^{\frac{1}{2}} \exp[-(a-r) R^{\frac{1}{2}} \cos(\frac{1}{4}\pi + \frac{1}{2}\beta)] \int_0^{\frac{1}{2}\pi + \beta} d\theta.$$

Thus, since $r < a$, $\lim_{R \rightarrow \infty} |I| = 0$; and, since there are no poles of

$$I_0(\mu r)/[\lambda I_0(\mu a)]$$

in the closed path formed by L' and this arc, we have finally $\lim_{t \rightarrow 0} v = 0$.

V. *The value (6) of v satisfies the differential equation § 2 (1).*

As in II it follows from (3), (4), and (5) respectively that the integrals

$$\int_{L'} e^{\lambda t} \frac{I_0(\mu r)}{I_0(\mu a)} d\lambda, \quad \int_{L'} e^{\lambda t} \frac{\partial}{\partial r} \left[\frac{I_0(\mu r)}{I_0(\mu a)} \right] \frac{d\lambda}{\lambda}, \quad \int_{L'} e^{\lambda t} \frac{\partial^2}{\partial r^2} \left[\frac{I_0(\mu r)}{I_0(\mu a)} \right] \frac{d\lambda}{\lambda}$$

are uniformly convergent in each of the ranges (7). Thus the necessary differentiations under the integral sign may be performed.

The verification that (6) and thus § 2, (11) and (12) satisfy the conditions of the problem of § 2 is now complete. A similar argument applies to all the problems of this paper.

5. *Circular cylinder of radius a . Radiation at $r=a$ into a medium at zero. The initial temperature of the cylinder constant v_0 .*

The equations for v are

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (0 \leq r < a, \quad t > 0),$$

with $v = v_0$, when $t = 0$, $0 \leq r < a$,

and $\frac{\partial v}{\partial r} + hv = 0$, when $t > 0$, $r = a$.

Thus the subsidiary equation obtained as in § 2 is

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r} \frac{d\bar{v}}{dr} - q^2 \bar{v} = -\frac{v_0}{\kappa} \quad (0 \leq r < a),$$

with $\frac{d\bar{v}}{dr} + h\bar{v} = 0$, when $r = a$.

Hence $\bar{v} = \frac{1}{p} \left\{ 1 - \frac{hI_0(qr)}{qI_0'(qa) + hI_0(qa)} \right\}$,

and so* $\frac{v}{v_0} = 1 - \frac{h}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu r)}{\mu I_0'(\mu a) + hI_0(\mu a)} \frac{d\lambda}{\lambda}$.

The integrand is a single-valued function of λ with poles at $\lambda = 0$, and at $\lambda = -\kappa a_s^2$, where $\pm a_s$, $s = 1, 2, \dots$, are the roots† of

$$aJ_0'(aa) + hJ_0(aa) = 0.$$

Then, as in § 2, the integral may be replaced by $2\pi i$ times the sum of the residues at these poles.

The residue at $\lambda = 0$ is $1/h$. Also

$$\left[\lambda \frac{d}{d\lambda} \{ \mu I_0'(\mu a) + hI_0(\mu a) \} \right]_{\lambda = -\kappa a_s^2} = -\frac{1}{2} a (h^2 + a_s^2) J_0(aa_s).$$

* Cf. C.H., § 110 (IV).

† All real and simple. W.B.F., §§ 15.25, 15.23.

Thus finally

$$v = \frac{2hv_0}{a} \sum_{s=1}^{\infty} e^{-\kappa_s^2 t} \frac{J_0(r\alpha_s)}{(h^2 + \alpha_s^2) J_0(a\alpha_s)}.$$

6. *Solid bounded internally by $r=a$. Radiation at the surface into a medium at zero. Initial temperature v_0 .*

The subsidiary equation obtained as in §5 is

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r} \frac{d\bar{v}}{dr} - q^2 \bar{v} = -\frac{v_0}{\kappa} \quad (r > a),$$

with
$$\frac{d\bar{v}}{dr} = h\bar{v} \quad \text{at} \quad r = a.$$

Thus
$$\bar{v} = \frac{v_0}{p} \left\{ 1 + \frac{hK_0(qr)}{qK_0'(qa) - hK_0(qa)} \right\}$$

and
$$v = v_0 + \frac{hv_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{K_0(\mu r)}{\mu K_0'(\mu a) - hK_0(\mu a)} \frac{d\lambda}{\lambda}.$$

The integrand has a branch point at $\lambda = 0$, so we use the contour of Fig. 2. There are no poles inside or on this path since $zK_0'(z) - hK_0(z)$ has no zeros* for $R(z) \geq 0$ provided that $h \geq 0$. Thus, proceeding as in §3, we may replace the line integral by integrals along CD , the small circle, and EF , and we obtain

$$v = -\frac{2hv_0}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \times \frac{J_0(ur)[uY_1(ua) + hY_0(ua)] - Y_0(ur)[uJ_1(ua) + hJ_0(ua)]}{[uJ_1(ua) + hJ_0(ua)]^2 + [uY_1(ua) + hY_0(ua)]^2} \frac{du}{u}.$$

7. *Hollow cylinder $a < r < b$. The surface $r=a$ kept at v_0 and $r=b$ at zero. Initial temperature zero.*

Here the subsidiary equation is

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r} \frac{d\bar{v}}{dr} - q^2 \bar{v} = 0 \quad (a < r < b),$$

with
$$\bar{v} = 0, \quad \text{when} \quad r = b,$$

$$\bar{v} = \frac{v_0}{p}, \quad \text{when} \quad r = a.$$

* See § 16.

$$\text{Thus} \quad \bar{v} = \frac{v_0}{p} \frac{I_0(qr) K_0(qb) - K_0(qr) I_0(qb)}{I_0(qa) K_0(qb) - K_0(qa) I_0(qb)},$$

$$\text{and} \quad \frac{v}{v_0} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu r) K_0(\mu b) - K_0(\mu r) I_0(\mu b)}{I_0(\mu a) K_0(\mu b) - K_0(\mu a) I_0(\mu b)} \frac{d\lambda}{\lambda}, \quad (1)$$

where, as usual, $\mu = \sqrt{\lambda/\kappa}$.

The integrand of (1) is a single-valued function of λ with poles at $\lambda = 0$ and $\lambda = -\kappa a_s^2$, where $\pm a_s$, $s = 1, 2, \dots$, are the roots* of

$$J_0(aa) Y_0(ab) - Y_0(aa) J_0(ab) = 0. \quad (2)$$

The integral over Γ of Fig. 1 vanishes† in the limit $R \rightarrow \infty$, so the integral in (1) may be replaced by $2\pi i$ times the sum of the residues at its poles.

$$\text{The pole at } \lambda = 0 \text{ gives} \quad \frac{\log(b/r)}{\log(b/a)}.$$

To find the residue at $\lambda = -\kappa a_s^2$, we require the formula

$$\begin{aligned} & \left[\lambda \frac{d}{d\lambda} \{I_0(\mu a) K_0(\mu b) - K_0(\mu a) I_0(\mu b)\} \right]_{\lambda = -\kappa a_s^2} \\ &= [\tfrac{1}{2} a \mu \{I_0'(\mu a) K_0(\mu b) - K_0'(\mu a) I_0(\mu b)\} \\ & \quad + \tfrac{1}{2} \mu b \{I_0(\mu a) K_0'(\mu b) - K_0(\mu a) I_0'(\mu b)\}]_{\lambda = -\kappa a_s^2}. \quad (3) \end{aligned}$$

Now, for this value of λ ,

$$\frac{I_0(\mu a)}{I_0(\mu b)} = \frac{K_0(\mu a)}{K_0(\mu b)} = \frac{J_0(a a_s)}{J_0(b a_s)} = \sigma, \quad \text{say,}$$

and, when we use this result, (3) becomes

$$\begin{aligned} & [\tfrac{1}{2} \mu a \sigma^{-1} \{I_0'(\mu a) K_0(\mu a) - I_0(\mu a) K_0'(\mu a)\} \\ & \quad + \tfrac{1}{2} \mu b \sigma \{I_0(\mu b) K_0'(\mu b) - K_0(\mu b) I_0'(\mu b)\}]_{\lambda = -\kappa a_s^2} \\ &= \tfrac{1}{2} (\sigma^{-1} - \sigma) \\ &= \tfrac{1}{2} \frac{J_0^2(b a_s) - J_0^2(a a_s)}{J_0(b a_s) J_0(a a_s)}. \end{aligned}$$

Also

$$[I_0(\mu r) K_0(\mu b) - K_0(\mu r) I_0(\mu b)]_{\lambda = -\kappa a_s^2} = \tfrac{1}{2} \pi \{J_0(b a_s) Y_0(r a_s) - J_0(r a_s) Y_0(b a_s)\}.$$

* All real and simple. Gray and Mathews, *Treatise on Bessel functions* (1922), 2 ed., 82.

† R is to tend to infinity through a discrete series of values chosen to lie between the roots of (2). The proof then follows as in § 2. The same remark applies to §§ 10 and 11.

Thus finally

$$\frac{v}{v_0} = \frac{\log(b/r)}{\log(b/a)} + \pi \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{J_0(a a_s) J_0(b a_s)}{J_0^2(b a_s) - J_0^2(a a_s)} \times [J_0(b a_s) Y_0(r a_s) - J_0(r a_s) Y_0(b a_s)]. \quad (4)$$

If $r = a$ is maintained at zero and $r = b$ at v_1 , we have merely to interchange a and b and to replace v_0 by v_1 in (4). Adding this result to (4), we obtain for the cylinder $a < r < b$ with zero initial temperature, $r = a$ kept at v_0 and $r = b$ at v_1 for $t > 0$,

$$v = \pi \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{J_0(a a_s) J_0(b a_s)}{J_0^2(b a_s) - J_0^2(a a_s)} \{v_0 [J_0(b a_s) Y_0(r a_s) - J_0(r a_s) Y_0(b a_s)] - v_1 [J_0(a a_s) Y_0(r a_s) - J_0(r a_s) Y_0(a a_s)]\} + \frac{v_0 \log(b/r) - v_1 \log(a/r)}{\log(b/a)}. \quad (5)$$

8. The following results for the hollow cylinder with zero initial temperature and other boundary conditions may be obtained as in § 7.

(i) *Hollow cylinder* $a < r < b$. *Zero initial temperature. For* $t > 0$ *the surface* $r = b$ *kept at* v_1 , *and at* $r = a$ *radiation into a medium at* v_0 .

$$v = \pi \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{\{h v_0 J_0(b a_s) + v_1 [a_s J_0'(a a_s) - h J_0(a a_s)]\} \times [a_s J_0'(a a_s) - h J_0(a a_s)] [J_0(r a_s) Y_0(b a_s) - Y_0(r a_s) J_0(b a_s)]}{(h^2 + a_s^2) J_0^2(b a_s) - [a_s J_0'(a a_s) - h J_0(a a_s)]^2} + \frac{v_1 + a h v_1 \log(r/a) + a h v_0 \log(b/r)}{1 + a h \log(b/a)}. \quad (1)$$

where $\pm a_s$ are the roots* (all real) of

$$J_0(b a) [a Y_0'(a a) - h Y_0(a a)] - Y_0(b a) [a J_0'(a a) - h J_0(a a)] = 0. \quad (2)$$

(ii) *Hollow cylinder* $a < r < b$. *Zero initial temperature. For* $t > 0$ *the surface* $r = a$ *kept at* v_0 *and at* $r = b$ *radiation into a medium at* v_1 .

$$v = \pi \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{\{v_0 [a_s J_0'(b a_s) + h J_0(b a_s)] - h v_1 J_0(a a_s)\} \times [a_s J_0'(b a_s) + h J_0(b a_s)] [J_0(r a_s) Y_0(a a_s) - Y_0(r a_s) J_0(a a_s)]}{(h^2 + a_s^2) J_0^2(a a_s) - [a_s J_0'(b a_s) + h J_0(b a_s)]^2} + \frac{v_0 - b h v_0 \log(r/b) - b h v_1 \log(a/r)}{1 - b h \log(a/b)}, \quad (3)$$

* See § 16.

where $\pm \alpha_s$ are the roots* (all real) of

$$J_0(a\alpha)[\alpha Y_0'(b\alpha) + hY_0(b\alpha)] - Y_0(a\alpha)[\alpha J_0'(b\alpha) + hJ_0(b\alpha)] = 0. \quad (4)$$

(iii) *Hollow cylinder* $a < r < b$. *Initial temperature zero. Radiation at* $r = a$ *into a medium at* v_0 , *and at* $r = b$ *into a medium at zero.*

$$\begin{aligned} = \pi h v_0 \sum_{s=1}^{\infty} \frac{e^{-\kappa \alpha_s^2 t}}{\alpha_s^2 + h^2} \frac{[\alpha_s J_0'(a\alpha_s) - hJ_0(a\alpha_s)][\alpha_s J_0'(b\alpha_s) + hJ_0(b\alpha_s)]}{[\alpha_s J_0'(b\alpha_s) + hJ_0(b\alpha_s)]^2 - [\alpha_s J_0'(a\alpha_s) - hJ_0(a\alpha_s)]^2} \\ \times \{J_0(r\alpha_s)[\alpha_s Y_0'(b\alpha_s) + hY_0(b\alpha_s)] - Y_0(r\alpha_s)[\alpha_s J_0'(b\alpha_s) + hJ_0(b\alpha_s)]\} \\ + v_0 \frac{a + abh \log(b/r)}{a + b + abh \log(b/a)}. \end{aligned} \quad (5)$$

where $\pm \alpha_s$ are the roots* (all real) of

$$\begin{aligned} [\alpha J_0'(a\alpha) - hJ_0(a\alpha)][\alpha Y_0'(b\alpha) + hY_0(b\alpha)] \\ - [\alpha J_0'(b\alpha) + hJ_0(b\alpha)][\alpha Y_0'(a\alpha) - hY_0(a\alpha)] = 0. \end{aligned} \quad (6)$$

9. *Unit instantaneous source at* $t = 0$ *along the axis† of the cylinder* $r = a$. *The surface of the cylinder kept at temperature zero.*

Here the equations for v are

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (0 \leq r < a, t > 0) \quad (1)$$

$$v = 0 \quad \text{when} \quad r = a \quad (t > 0), \quad (2)$$

$$v = \frac{1}{4\pi\kappa t} e^{-r^2/(4\kappa t)} + w = u + w,$$

where

$$\frac{\partial w}{\partial t} = \kappa \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \quad (0 \leq r < a, t > 0)$$

and

$$\lim_{t \rightarrow 0} w = 0 \quad (0 \leq r < a).$$

As usual, let

$$\bar{u} = \int_0^\infty e^{-pt} u dt, \text{ etc.}$$

We know‡ that

$$\bar{u} = \frac{1}{2\pi\kappa} K_0(qr).$$

* See § 16.

† Cf. C.H., 247, Ex. 39. This problem was set in the Mathematical Tripos, Part II, 1895. For a source not at the centre see Carslaw and Jaeger, "On Green's functions in the theory of heat conduction", *Bull. American Math. Soc.*, 45 (1939), 407-413.

‡ W.B.F., 6.22 (15).

Also
$$\frac{d^2 \bar{w}}{dr^2} + \frac{1}{r} \frac{d\bar{w}}{dr} = q^2 \bar{w} \quad (0 \leq r < a),$$

and
$$\bar{v} = \bar{u} + \bar{w}.$$

Thus
$$\bar{w} = -\frac{1}{2\pi\kappa} K_0(qa), \quad \text{when } r = a.$$

Therefore
$$\bar{w} = -\frac{1}{2\pi\kappa} \frac{K_0(qa) I_0(qr)}{I_0(qa)} \quad (3)$$

and
$$\bar{v} = \frac{1}{2\pi\kappa} \frac{I_0(qa) K_0(qr) - I_0(qr) K_0(qa)}{I_0(qa)}.$$

Thus, from the inversion theorem,

$$v = \frac{1}{4\pi^2 i\kappa} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu a) K_0(\mu r) - I_0(\mu r) K_0(\mu a)}{I_0(\mu a)} d\lambda, \quad (4)$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

The integrand is a single-valued function of λ , and we take Fig. 1, where the radius of the circle Γ is $\kappa(n + \frac{1}{2})^2(\pi^2/a^2)$, so that it does not pass through any zero of $I_0(\mu a)$.

We can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ in (4) by the integral over the closed circuit of this diagram, since it will be found that when $n \rightarrow \infty$ the integrals over AA' , $A'C$, BB' , and $B'C$ vanish.

Thus we have

$$v = -\frac{1}{2\pi\kappa} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{I_0(i a_s r) K_0(i a_s a)}{[(d/d\lambda) I_0(\mu a)]_{\lambda = -\kappa a_s^2}}.$$

But
$$\frac{d}{d\lambda} I_0(\mu a) = \frac{a}{2\sqrt{(\lambda\kappa)}} I_1(\mu a),$$

and
$$I_0(z) K_1(z) + I_1(z) K_0(z) = \frac{1}{z}.$$

It follows that

$$v = \frac{1}{\pi a^2} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{J_0(a_s r)}{J_1^2(a_s a)}. \quad (5)$$

To verify the solution (4), we have to show that it satisfies the differential equation (1), that $\lim_{r \rightarrow a} v = 0$ for $t > 0$, and that $\lim_{t \rightarrow 0} w = 0$ for $0 \leq r < a$, where w is obtained from (3). These results follow along the lines of §4.

For the same problem with no escape of heat at $r = a$, the solution is

$$v = \frac{1}{\pi a^2} \left\{ 1 + \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{J_0(r \alpha_s)}{J_0^2(a \alpha_s)} \right\}, \quad (6)$$

where $\pm \alpha_s$, $s = 1, 2, 3, \dots$, are the roots of $J_1(a \alpha) = 0$ other than $\alpha = 0$.

For the same problem with radiation at $r = a$ into a medium at temperature zero, the solution is

$$v = \frac{1}{\pi a^2} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{\alpha_s^2 J_0(r \alpha_s)}{(\kappa^2 + \alpha_s^2) J_0^2(a \alpha_s)}, \quad (7)$$

where $\pm \alpha_s$, $s = 1, 2, \dots$, are the roots* of

$$a J_0'(a \alpha) + h J_0(a \alpha) = 0.$$

10. *Instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$. The surface $r = a$ kept at temperature zero.*

We know that the temperature due to an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$ in the infinite region $r \geq 0$ is

$$u = \frac{Q}{4\pi\kappa t} e^{-(r^2+r'^2)/(4\kappa t)} I_0\left(\frac{rr'}{2\kappa t}\right). \quad (1)$$

Thus for this source in the cylinder $r = a$, with the surface kept at temperature zero, we have to solve the equations

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (0 \leq r < a, t > 0) \quad (2)$$

$$v = 0 \quad \text{when } r = a \quad (t > 0) \quad (3)$$

and

$$v = u + w,$$

where

$$\frac{\partial w}{\partial t} = \kappa \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \quad (0 \leq r < a, t > 0) \quad (4)$$

and

$$\lim_{t \rightarrow 0} w = 0 \quad (0 \leq r < a).$$

We know† that

$$\left. \begin{aligned} \bar{u} &= \frac{Q}{2\pi\kappa} I_0(qr') K_0(qr) \quad (r > r') \\ &= \frac{Q}{2\pi\kappa} I_0(qr) K_0(qr') \quad (r < r') \end{aligned} \right\}. \quad (5)$$

* All real and simple. W.B.F., §§ 15.25, 15.23.

† This result may be deduced from W.B.F., § 13.7 (2), but it is not difficult to prove it directly.

But $\bar{v} = u + \bar{w}$, so that

$$\begin{aligned}\bar{w} &= -\bar{u}, \text{ when } r = a \\ &= -\frac{Q}{2\pi\kappa} I_0(qr') K_0(qa), \text{ when } r = a.\end{aligned}$$

Thus

$$\bar{w} = -\frac{Q}{2\pi\kappa} \frac{I_0(qr') K_0(qa)}{I_0(qa)} I_0(qr),$$

and

$$\begin{aligned}\bar{v} &= \frac{Q}{2\pi\kappa} \frac{I_0(qr') (I_0(qa) K_0(qr) - I_0(qr) K_0(qa))}{I_0(qa)} \quad (r > r'), \\ &= \frac{Q}{2\pi\kappa} \frac{I_0(qr)}{I_0(qa)} (I_0(qa) K_0(qr') - I_0(qr') K_0(qa)) \quad (r < r').\end{aligned}$$

Therefore

$$v = \frac{Q}{4\pi^2 i\kappa} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_0(\mu r') [I_0(\mu a) K_0(\mu r) - I_0(\mu r) K_0(\mu a)]}{I_0(\mu a)} d\lambda,$$

when $r > r'$, and we interchange r and r' when $r < r'$.

The integrand is a single-valued function of λ and we again use Fig. 1, taking the radius of the circle Γ as $\kappa(n + \frac{1}{2})^2(\pi^2/a^2)$, so that it does not pass through a zero of $I_0(\mu a)$.

We can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ by the limit of the integral over the closed circuit of this diagram, since the integral over the circle Γ in the diagram tends to zero when $n \rightarrow \infty$.

Then, from the theory of residues, we have

$$v = \frac{Q}{2\pi\kappa} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{I_0(i a_s r') [-I_0(i a_s r) K_0(i a_s a)]}{[(d/d\lambda) I_0(\mu a)]_{\lambda = -\kappa a_s^2}},$$

the summation being taken over the positive roots of $J_0(aa) = 0$. Thus, as in § 9, this result reduces to

$$v = \frac{Q}{\pi a^2} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{J_0(\alpha_s r) J_0(\alpha_s r')}{J_1^2(\alpha_s a)} \quad (6)$$

and it holds for $r \geq r'$.

For the same problem but with no flow of heat* over the surface $r = a$ the result is

$$v = \frac{Q}{\pi a^2} \left\{ 1 + \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{J_0(r' a_s) J_0(r a_s)}{J_0^2(a a_s)} \right\} \quad \text{for } (r \geq r'), \quad (7)$$

the summation being taken over the positive roots of $J_0'(aa) = 0$.

* Cf. Goldstein, *loc. cit.*, 83 (199).

For the same problem with radiation at $r = a$ into a medium at temperature zero, the result is

$$v = \frac{Q}{\pi a^2} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{\alpha_s^2 J_0(r \alpha_s) J_0(r' \alpha_s)}{(h^2 + \alpha_s^2) J_0^2(a \alpha_s)} \quad \text{for } r \geq r', \quad (8)$$

the summation being taken over the positive roots of $\alpha J_0'(a \alpha) + h J_0(a \alpha) = 0$.

11. In the results of § 10, equations (6), (7), and (8) put

$$Q = 2\pi r' f(r') dr'$$

and integrate from 0 to a with respect to r' .

Then we obtain the solution of the problems in which the initial temperature of the cylinder is $f(r)$ and (i) the surface $r = a$ is kept at zero temperature, (ii) there is no flow of heat across it, and (iii) radiation takes place into a medium at temperature zero.

In these cases, if $f(r)$ is such that the orders of integration and summation may be interchanged, we have, respectively*

$$v = \frac{2}{a^2} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{J_0(\alpha_s r)}{J_1^2(\alpha_s a)} \int_0^a r' f(r') J_0(\alpha_s r') dr', \quad (1)$$

$$v = \frac{2}{a^2} \int_0^a r' f(r') dr' + \frac{2}{a^2} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{J_0(\alpha_s r)}{J_0^2(\alpha_s a)} \int_0^a r' f(r') J_0(\alpha_s r') dr', \quad (2)$$

$$v = \frac{2}{a^2} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{\alpha_s^2 J_0(\alpha_s r)}{(h^2 + \alpha_s^2) J_0^2(\alpha_s a)} \int_0^a r' f(r') J_0(\alpha_s r') dr', \quad (3)$$

where in (1) the summation is taken over the positive roots of $J_0(a \alpha) = 0$; in (2), over the positive roots of $J_0'(a \alpha) = 0$; and in (3), over the positive roots of $\alpha J_0'(a \alpha) + h J_0(a \alpha) = 0$.

Solid bounded internally by the cylinder $r = a$.

12. *Instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$. The solid extends from $r = a$ to ∞ and the boundary $r = a$ is kept at temperature zero.*

Here we have to solve

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right), \quad (r > a, \quad t > 0), \quad (1)$$

$$v = 0 \quad \text{when} \quad r = a \quad (t > 0), \quad (2)$$

$$v = u + w,$$

* Cf. C.H., 57 (I), (II), and (III); but an initial term has to be added to the expansion given in (II). This was pointed out by Goldstein, *loc. cit.*, 83 (200), where he obtained the expansion (2).

where

$$u = \frac{Q}{4\pi\kappa t} e^{-(r^2+r'^2)/(4\kappa t)} I_0\left(\frac{rr'}{2\kappa t}\right) \quad (3)$$

and

$$\frac{\partial w}{\partial t} = \kappa \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \quad (r > a, \quad t > 0),$$

$$\lim_{t \rightarrow 0} w = 0 \quad (r > a).$$

Then proceeding as in § 10 we have, using § 10 (5),

$$\frac{d^2 \bar{w}}{dr^2} + \frac{1}{r} \frac{d\bar{w}}{dr} = q^2 \bar{w} \quad (r > a),$$

with

$$\bar{w} = -\bar{u}, \quad \text{when } r = a,$$

$$= -\frac{Q}{2\pi\kappa} I_0(qa) K_0(qr'), \quad \text{when } r = a.$$

Thus

$$\bar{w} = -\frac{Q}{2\pi\kappa} \frac{I_0(qa) K_0(qr')}{K_0(qa)} K_0(qr).$$

Also

$$\bar{v} = \frac{Q}{2\pi\kappa} \left(I_0(qr) K_0(qa) - I_0(qa) K_0(qr) \right) \frac{K_0(qr')}{K_0(qa)} \quad (r < r'),$$

$$= \frac{Q}{2\pi\kappa} \left(I_0(qr') K_0(qa) - I_0(qa) K_0(qr') \right) \frac{K_0(qr)}{K_0(qa)} \quad (r > r').$$

Then

$$v = \frac{Q}{4\pi^2 i\kappa} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \left(I_0(\mu r) K_0(\mu a) - I_0(\mu a) K_0(\mu r) \right) \frac{K_0(\mu r')}{K_0(\mu a)} d\lambda, \quad \text{when } r < r',$$

and we interchange r and r' , when $r > r'$.

We take $|\arg \lambda| \leq \pi$, and we know that $K_0(\mu a)$ has no zeros in this range.

Also the integrand is single-valued within the closed circuit of Fig. 2.

Along CD , we take $\arg \lambda = -\pi$ and along EF , π . The asymptotic expansions show that the limit of the integral over the circular arcs of Γ in this Figure tends to zero when the radius of Γ tends to infinity. Thus in the limit we can replace $\int_{\gamma-i\infty}^{\gamma+i\infty}$ above by the integrals over CD , the small circle, and EF , when C and F tend to $-\infty$ and the radius of the small circle tends to zero.

The integral over the small circle gives zero.

On CD , put $\lambda = \kappa u^2 e^{-i\pi}$, and we have

$$\frac{Q}{2\pi^2 i} \int_0^\infty e^{-\kappa u^2 t} \frac{K_0(ur' e^{-\frac{1}{2}i\pi})}{K_0(au e^{-\frac{1}{2}i\pi})} \{I_0(rue^{-\frac{1}{2}i\pi}) K_0(au e^{-\frac{1}{2}i\pi}) - I_0(au e^{-\frac{1}{2}i\pi}) K_0(rue^{-\frac{1}{2}i\pi})\} u du.$$

But $K_0(ze^{-\frac{1}{2}i\pi}) = \frac{1}{2}i\pi H_0^{(1)}(z) = \frac{1}{2}i\pi [J_0(z) + iY_0(z)]$

and $I_0(ze^{-\frac{1}{2}i\pi}) = J_0(z),$

so this becomes

$$\frac{iQ}{4\pi} \int_0^\infty e^{-\kappa u^2 t} \frac{J_0(ur') + iY_0(ur')}{J_0(ua) + iY_0(ua)} \{J_0(ur) Y_0(ua) - J_0(ua) Y_0(ur)\} u du.$$

Similarly from EF , putting $\lambda = \kappa u^2 e^{i\pi}$, we get the conjugate expression. Adding these we have finally*

$$v = \frac{Q}{2\pi} \int_0^\infty e^{-\kappa u^2 t} \times \frac{\{J_0(ur) Y_0(ua) - J_0(ua) Y_0(ur)\} \{J_0(ur') Y_0(ua) - J_0(ua) Y_0(ur')\}}{J_0^2(ua) + Y_0^2(ua)} u du, \quad (4)$$

and this holds for both $r > r'$ and $r < r'$.

For the same problem but with no flow of heat over $r = a$, the result is

$$v = \frac{Q}{2\pi} \int_0^\infty e^{-\kappa u^2 t} \times \frac{\{J_1(ua) Y_0(ur') - J_0(ur') Y_1(ua)\} \{J_1(ua) Y_0(ur) - J_0(ur) Y_1(ua)\}}{J_1^2(ua) + Y_1^2(ua)} u du, \quad (5)$$

for $r \geq r'$.

For the same problem with radiation at $r = a$ into a medium at temperature zero†, the result is

$$v = \frac{Q}{2\pi} \int_0^\infty e^{-\kappa u^2 t} \frac{C_0(ur) C_0(ur')}{[uJ_1(au) + hJ_0(au)]^2 + [uY_1(au) + hY_0(au)]^2} u du, \text{ for } r \geq r', \quad (6)$$

where

$$C_0(z) = J_0(z)[uY_1(au) + hY_0(au)] - Y_0(z)[uJ_1(au) + hJ_0(au)]. \quad (7)$$

* Cf. Goldstein, *loc. cit.*, 85 (215).

† Cf. Goldstein, *loc. cit.*, 87 (223).

13. In the results of § 12, equations (4), (5), and (6), put

$$Q = 2\pi r' f(r') dr'$$

and integrate from a to ∞ with respect to r' .

Then we obtain the solutions of the problems in which the initial temperature of the cylinder is $f(r)$ and (i) the surface $r=a$ is kept at zero temperature, (ii) there is no flow of heat across $r=a$, and (iii) there is radiation at $r=a$ into a medium at temperature zero.

In these cases we have respectively*

$$v = \int_a^\infty f(r') r' dr' \int_0^\infty e^{-\kappa u^2 t} \times \frac{[J_0(ur) Y_0(ua) - J_0(ua) Y_0(ur)][J_0(ur') Y_0(ua) - J_0(ua) Y_0(ur')]}{J_0^2(ua) + Y_0^2(ua)} u du, \quad (1)$$

$$v = \int_a^\infty f(r') r' dr' \int_0^\infty e^{-\kappa u^2 t} \times \frac{[J_0(ur) Y_1(ua) - J_1(ua) Y_0(ur)][J_0(ur') Y_1(ua) - J_1(ua) Y_0(ur')]}{J_1^2(ua) + Y_1^2(ua)} u du, \quad (2)$$

$$v = \int_a^\infty f(r') r' dr' \int_0^\infty e^{-\kappa u^2 t} \frac{C_0(ur) C_0(ur')}{[u J_1(ua) + h J_0(ua)]^2 + [u Y_1(ua) + h Y_0(ua)]^2} u du, \quad (3)$$

where $C_0(z)$ is defined in § 12 (7).

14. *Hollow cylinder* $a < r < b$. *Instantaneous cylindrical surface source of strength* Q *at* $t = 0$ *over* $r = r'$. *The surfaces* $r = a$ *and* $r = b$ *kept at temperature zero.*

We have to solve

$$\frac{\partial v}{\partial t} = \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) \quad (a < r < b, \quad t > 0), \quad (1)$$

$$\text{with} \quad \left. \begin{array}{l} v = 0, \quad \text{when } r = a \\ v = 0, \quad \text{when } r = b, \end{array} \right\} \quad (t > 0) \quad (2)$$

$$\text{and} \quad v = u + w, \quad (3)$$

$$\text{where} \quad u = \frac{Q}{4\pi\kappa t} e^{-(r^2+r'^2)/(4\kappa t)} I_0\left(\frac{rr'}{2\kappa t}\right), \quad (4)$$

$$\text{and} \quad \lim_{t \rightarrow 0} w = 0 \quad (a < r < b), \quad (5)$$

* Cf. Goldstein, *loc. cit.*, 86-7 (216) and (224).

while w satisfies

$$\frac{\partial w}{\partial t} = \kappa \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \quad (a < r < b, \quad t > 0). \quad (6)$$

The subsidiary equation for w is

$$\frac{d^2 \bar{w}}{dr^2} + \frac{1}{r} \frac{d\bar{w}}{dr} - q^2 \bar{w} = 0 \quad (a < r < b), \quad (7)$$

$$\text{with} \quad \bar{w} = -\bar{u}, \quad \text{when} \quad r = a, \quad \text{and when} \quad r = b. \quad (8)$$

Now, by § 10 (5),

$$\left. \begin{aligned} \bar{u} &= \frac{Q}{2\pi\kappa} I_0(qr) K_0(qr') \quad (r < r') \\ &= \frac{Q}{2\pi\kappa} I_0(qr') K_0(qr) \quad (r > r'). \end{aligned} \right\}$$

$$\text{Thus} \quad \bar{w} = AI_0(qr) + BK_0(qr),$$

$$\text{where} \quad \left. \begin{aligned} AI_0(qa) + BK_0(qa) &= -\frac{Q}{2\pi\kappa} I_0(qa) K_0(qr'), \\ AI_0(qb) + BK_0(qb) &= -\frac{Q}{2\pi\kappa} K_0(qb) I_0(qr'). \end{aligned} \right\}$$

Solving, we find that

$$\begin{aligned} \bar{v} &= \bar{u} + \bar{w} \\ &= \frac{Q}{2\pi\kappa} \frac{[I_0(qa) K_0(qr) - K_0(qa) I_0(qr)][I_0(qb) K_0(qr') - K_0(qb) I_0(qr')]}{I_0(qa) K_0(qb) - K_0(qa) I_0(qb)}, \end{aligned}$$

when $r < r'$. Thus

$$\begin{aligned} v &= \frac{Q}{4\pi^2 i \kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \\ &\quad \times \frac{[I_0(\mu a) K_0(\mu r) - K_0(\mu a) I_0(\mu r)][I_0(\mu b) K_0(\mu r') - K_0(\mu b) I_0(\mu r')]}{I_0(\mu a) K_0(\mu b) - K_0(\mu a) I_0(\mu b)} d\lambda, \end{aligned}$$

when $r < r'$, and we interchange r and r' when $r > r'$. The denominator of the integrand is the same as that of § 7 (1) (except for the factor λ) so we use the results of that section and get

$$\begin{aligned} v &= \frac{\pi Q}{4} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{a_s^2 J_0(a a_s) J_0(b a_s)}{J_0^2(a a_s) - J_0^2(b a_s)} C(a a_s, r a_s) C(b a_s, r' a_s) \\ &= \frac{\pi Q}{4} \sum_{s=1}^{\infty} e^{-\kappa a_s^2 t} \frac{a_s^2 J_0^2(a a_s)}{J_0^2(a a_s) - J_0^2(b a_s)} C(b a_s, r a_s) C(b a_s, r' a_s), \end{aligned} \quad (9)$$

where $C(x, y) = J_0(x) Y_0(y) - Y_0(x) J_0(y)$, (10)

and $\pm \alpha_s$, $s = 1, 2, \dots$, are the roots of

$$J_0(a\alpha) Y_0(ab) - J_0(ab) Y_0(a\alpha) = 0. \quad (11)$$

The result (9) being symmetrical in r and r' holds for both $r > r'$ and $r < r'$.

The results for the hollow cylinder $a < r < b$ with an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$ and radiation at one or both surfaces are as follows:

(i) Radiation at $r = a$ into a medium at temperature zero. The surface $r = b$ kept at temperature zero.

$$v = \frac{\pi Q}{4} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{\alpha_s^2 [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 C(r\alpha_s, b\alpha_s) C(r'\alpha_s, b\alpha_s)}{[a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 - (\alpha_s^2 + h^2) J_0^2(b\alpha_s)}, \quad (12)$$

for $r \geq r'$, where $C(x, y)$ is defined by (10), and $\pm \alpha_s$, $s = 1, 2, \dots$, are the roots of

$$J_0(b\alpha)[a Y_0'(a\alpha) - h Y_0(a\alpha)] - Y_0(b\alpha)[a J_0'(a\alpha) - h J_0(a\alpha)] = 0. \quad (13)$$

(ii) The solution for the case in which there is radiation at $r = b$ into a medium at temperature zero, while $r = a$ is kept at temperature zero is obtained by interchanging a and b and changing the sign of h in (12) and (13).

(iii) Radiation at $r = a$ and at $r = b$ into a medium at temperature zero.

$$v = \frac{\pi Q}{4} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} \frac{\alpha_s^2 [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 F(r\alpha_s) F(r'\alpha_s)}{(\alpha_s^2 + h^2) \{ [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 - [a_s J_0'(b\alpha_s) + h J_0(b\alpha_s)]^2 \}}, \quad (14)$$

where

$$F(r\alpha_s) = J_0(r\alpha_s)[a_s Y_0'(b\alpha_s) + h Y_0(b\alpha_s)] - Y_0(r\alpha_s)[a_s J_0'(b\alpha_s) + h J_0(b\alpha_s)], \quad (15)$$

and $\pm \alpha_s$, $s = 1, 2, \dots$, are the roots of

$$[a J_0'(a\alpha) - h J_0(a\alpha)][a Y_0'(b\alpha) + h Y_0(b\alpha)] - [a J_0'(b\alpha) + h J_0(b\alpha)][a Y_0'(a\alpha) - h Y_0(a\alpha)] = 0. \quad (16)$$

15. In the results of § 14, equations (9), (12), and (14) put

$$Q = 2\pi r' f(r') dr'$$

and integrate from a to b with respect to r' . Then we obtain the solutions of the problems in which the initial temperature of the cylinder is $f(r)$ and

(i) the surfaces $r = a$ and $r = b$ are kept at zero temperature, (ii) $r = b$ is

kept at temperature zero and at $r = a$ there is radiation into a medium at temperature zero, (iii) $r = a$ is kept at temperature zero and at $r = b$ there is radiation into a medium at temperature zero, and (iv) there is radiation into a medium at temperature zero at both $r = a$ and $r = b$.

Then, using the notation

$$C(r\alpha_s, b\alpha_s) = J_0(r\alpha_s) Y_0(b\alpha_s) - Y_0(r\alpha_s) J_0(b\alpha_s), \quad (1)$$

$$F(r\alpha_s) = J_0(r\alpha_s)[\alpha_s Y_0'(b\alpha_s) + h Y_0(b\alpha_s)] - Y_0(r\alpha_s)[\alpha_s J_0'(b\alpha_s) + h J_0(b\alpha_s)], \quad (2)$$

and assuming that $f(r)$ is such that the order of integration and summation may be interchanged, we have in these cases respectively

$$v = \frac{\pi^2}{2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{\alpha_s^2 J_0^2(a\alpha_s) C(r\alpha_s, b\alpha_s)}{J_0^2(a\alpha_s) - J_0^2(b\alpha_s)} \int_a^b r' f(r') C(r' \alpha_s, b\alpha_s) dr', \quad (3)$$

$$v = \frac{\pi^2}{2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{\alpha_s^2 [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 C(r\alpha_s, b\alpha_s)}{[a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 - (\alpha_s^2 + h^2) J_0^2(b\alpha_s)} \times \int_a^b r' f(r') C(r' \alpha_s, b\alpha_s) dr', \quad (4)$$

$$v = \frac{\pi^2}{2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{\alpha_s^2 [a_s J_0'(b\alpha_s) + h J_0(b\alpha_s)]^2 C(r\alpha_s, a\alpha_s)}{[a_s J_0'(b\alpha_s) + h J_0(b\alpha_s)]^2 - (\alpha_s^2 + h^2) J_0^2(a\alpha_s)} \times \int_a^b r' f(r') C(r' \alpha_s, a\alpha_s) dr', \quad (5)$$

$$v = \frac{\pi^2}{2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{\alpha_s^2 [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 F(r\alpha_s)}{(\alpha_s^2 + h^2) \{ [a_s J_0'(a\alpha_s) - h J_0(a\alpha_s)]^2 - [a_s J_0'(b\alpha_s) + h J_0(b\alpha_s)]^2 \}} \times \int_a^b r' f(r') F(r' \alpha_s) dr', \quad (6)$$

where the α_s in (3) to (6) respectively are the roots of

$$J_0(a\alpha) Y_0(b\alpha) - J_0(b\alpha) Y_0(a\alpha) = 0, \quad (7)$$

$$J_0(b\alpha) [\alpha Y_0'(a\alpha) - h Y_0(a\alpha)] - Y_0(b\alpha) [\alpha J_0'(a\alpha) - h J_0(a\alpha)] = 0, \quad (8)$$

$$J_0(a\alpha) [\alpha Y_0'(b\alpha) + h Y_0(b\alpha)] - Y_0(a\alpha) [\alpha J_0'(b\alpha) + h J_0(b\alpha)] = 0, \quad (9)$$

$$[\alpha J_0'(a\alpha) - h J_0(a\alpha)] [\alpha Y_0'(b\alpha) + h Y_0(b\alpha)] - [\alpha J_0'(b\alpha) + h J_0(b\alpha)] [\alpha Y_0'(a\alpha) - h Y_0(a\alpha)] = 0. \quad (10)$$

16. It remains to prove certain results which have been used in the foregoing: (i) that the roots of the equations § 15 (7), (8), (9), (10) are all real and simple: (ii) that, when $k > 0$, $zK_0'(z) - kK_0(z)$ has no zeros for $R(z) \geq 0$.

It is known* that, if a and b are real and positive, the roots of

$$J_n(aa)Y_n(ba) - J_n(ba)Y_n(aa) = 0 \quad (1)$$

are all real and simple. The equation § 15 (7) is the case $n = 0$ of this. The same method† may be used to show that § 15 (8), (9), (10) have no complex roots, and that their roots are not repeated. It remains to prove that they have no purely imaginary roots. When we put $a = iy$ the left-hand sides of these equations become (except for a change of sign)

$$I_0(by)[yK_1(ay) + hK_0(ay)] + K_0(by)[yI_1(ay) - hI_0(ay)], \quad (2)$$

$$I_0(ay)[yK_1(by) - hK_0(by)] + K_0(ay)[yI_1(by) + hI_0(by)], \quad (3)$$

$$[yI_1(ay) - hI_0(ay)][yK_1(by) - hK_0(by)] \\ - [yI_1(by) + hI_0(by)][yK_1(ay) + hK_0(ay)]. \quad (4)$$

These are even functions of y , so we consider only $y > 0$.

The expression (2) may be written

$$y[I_0(by)K_1(ay) + K_0(by)I_1(ay)] + h[I_0(by)K_0(ay) - K_0(by)I_0(ay)],$$

and all terms of this are positive, since, from the infinite series, $I_n(ay) > 0$ for $y > 0$; also‡ $K_n(ay) > 0$ for $y > 0$ and $n = 0$ or 1 ; finally

$$I_0(by)K_0(ay) - K_0(by)I_0(ay) > 0;$$

for this expression has no real zeros, since such zeros would be purely imaginary roots of (1), and, from the asymptotic expansions, the expression is positive for large y . Thus, since $h \geq 0$, the expression (2) has no real zeros.

Similarly (3) has no real zeros.

Also (4) may be written

$$y^2[I_1(ay)K_1(by) - K_1(ay)I_1(by)] + h^2[I_0(ay)K_0(by) - I_0(by)K_0(ay)] \\ - hy[I_1(ay)K_0(by) + I_0(ay)K_1(by) + I_1(by)K_0(ay) + I_0(by)K_1(ay)],$$

and all three terms of this are negative. Thus (4) has no real zeros.

Finally we show that, when $k > 0$, $zK_n'(z) - kK_n(z)$ has no zeros for $\text{Re}(z) \geq 0$. Since $K_{-n}(z) = K_n(z)$, we may take $n \geq 0$. That there are no

* Gray and Mathews, *Treatise on Bessel functions* (1922), 2 ed., 82, Th. 10.

† See also Weyrich, *Die Zylinderfunktionen und ihre Anwendungen* (Teubner, 1937), § 11. Here the more general equation arising from boundary conditions of the third kind at a and b is discussed.

‡ W.B.F., § 6.22 (5).

complex zeros follows from the formula*

$$(\lambda^2 - \mu^2) \int_a^\infty x K_n(\lambda x) K_n(\mu x) dx \\ = a \{ \mu K_n(\lambda a) K_n'(\mu a) - \lambda K_n(\mu a) K_n'(\lambda a) \}, \quad \mathbf{R}(\lambda + \mu) > 0.$$

It follows from the recurrence formula and the fact that $K_n(x) > 0$ for real positive x that there are no purely real zeros.

To show that there are no purely imaginary zeros $z = iy$, we have

$$iy K_n'(iy) - k K_n(iy) = \frac{1}{2} \pi i e^{3ni\pi} \{ -y J_n'(y) + k J_n(y) + iy Y_n'(y) - ik Y_n(y) \}$$

and the real and imaginary parts of this must vanish.

This requires that

$$J_n(y) Y_n'(y) - Y_n(y) J_n'(y) = 0;$$

but the expression on the left is equal to $2/(\pi y)$, and so the result follows.

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* Gray and Mathews, *loc. cit.*, 70 (30).

THE ASYMPTOTIC EXPANSION OF THE GENERALIZED HYPERGEOMETRIC FUNCTION

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[Received 3 May, 1939.—Read 18 May, 1939.]

1.1. In a paper* with the same title published in the *Journal* of this Society a few years ago I gave without proof the complete asymptotic expansion of the function

$${}_pF_q(z) = \sum_{n=0}^{\infty} \frac{f(n)}{\Gamma(n+1)} z^n$$

for all large values of z , where

$$f(t) = \prod_{r=1}^p \Gamma(\beta_r + \alpha_r t) \left\{ \prod_{r=1}^q \Gamma(\mu_r + \rho_r t) \right\}^{-1}$$

and is regular at $t = n$ ($n = 0, 1, 2, \dots$), the numbers $\alpha_1, \dots, \alpha_p, \rho_1, \dots, \rho_q$ are all real and positive and

$$\kappa = 1 + \rho_1 + \dots + \rho_q - \alpha_1 - \dots - \alpha_p > 0.$$

The results were given in the form of six theorems. The first five of these are special cases of results which I have since proved† for a much wider class of functions. The sixth theorem, however, is not included in these more general results. Since this theorem is perhaps the most striking of the six, I prove it here by a new method, especially suitable to these particular functions. The other five theorems could be proved by this method, but this is unnecessary in view of the preceding remarks.

As in Wright, 2, we write

$$h = \left(\prod_{r=1}^p \alpha_r^{\alpha_r} \right) \left(\prod_{r=1}^q \rho_r^{-\rho_r} \right), \quad \vartheta = \sum_{r=1}^p \beta_r - \sum_{r=1}^q \mu_r + \frac{1}{2}(q-p),$$

* Wright, 2.

† Wright, 3.

and we use ϵ and δ to denote arbitrary positive numbers and M a positive integer*. K is a positive number (not always the same at each occurrence) depending at most on ϵ , δ , τ , ν , M , N , and

$$\alpha_1, \dots, \alpha_p, \rho_1, \dots, \rho_q, \beta_1, \dots, \beta_p, \mu_1, \dots, \mu_q, \quad (1.11)$$

but not on z ; the number K_1 is a fixed number of the type of K . The constant implied in the $O(\)$ notation is a K .

LEMMA 1. *If $\arg t$, $\arg(\beta_r + \alpha_r t)$ and $\arg(\mu_r + \rho_r t)$ all lie between $-\pi + \epsilon$ and $\pi - \epsilon$, then we can calculate numbers $A_0, A_1, A_2, \dots$ such that*

$$\left| \frac{f(t)}{(h\kappa^*)^t \Gamma(t+1)} - \sum_{m=0}^{M-1} \frac{A_m}{\Gamma(\kappa t - \vartheta + m + 1)} \right| < \frac{K}{\Gamma(\kappa t - \vartheta + M + 1)}. \quad (1.12)$$

In particular,

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)} \kappa^{\frac{1}{2}-\vartheta} \prod_{r=1}^p \rho_r^{\frac{1}{2}-\mu_r} \prod_{r=1}^q \alpha_r^{\beta_r-\frac{1}{2}}.$$

This lemma is given in Wright, 2, and suffices to define A_m . We write $I(X)$ for the "exponential" asymptotic expansion

$$X^{\vartheta} e^X \left\{ \sum_{m=0}^{M-1} A_m X^{-m} + O(X^{-M}) \right\}$$

and $J(y)$ for the "algebraic" asymptotic expansion

$$\sum_{r=0}^p \sum_{l=0}^{l_r} P_{r,l} y^{-(l+\beta_r)/\alpha_r} + O(y^{-N+\delta})$$

defined by means of the residues of $\Gamma(-t)f(t)y^t$ at the poles of $f(t)$ as in Wright, 2†. If $f(t)$ has only a finite number of poles, then $P_{r,l} = 0$ when l is greater than some fixed l_r and we write

$$H(y) = \sum_{r=0}^p \sum_{l=0}^{l_r} P_{r,l} y^{-(l+\beta_r)/\alpha_r}.$$

In this case

$$J(y) = H(y) + O(y^{-N+\delta}),$$

where N may be taken as large as we please.

* In this paper M is any positive integer i.e., it is always at our choice. In Wright, 3, M played a similar role, but was supposed given by the data of the problem.

† If $f(t)$ has a pole of order s at the point $-(l+\beta_r)/\alpha_r$, $sP_{r,l}y^{-(l+\beta_r)/\alpha_r}$ is the residue of $\Gamma(-t)f(t)y^t$ at that point; if $s > 1$ $P_{r,l}$ is a polynomial in $\log y$ of degree $s-1$.

We choose $\zeta = \arg z$ and $\eta = \arg(-z)$ so that

$$-\pi < \zeta \leq \pi, \quad -\pi < \eta \leq \pi$$

and write

$$Z = \kappa(h|z|)^{1/\kappa} e^{i\zeta/\kappa},$$

$$Z_1 = |Z| e^{i(\eta+\pi)/\kappa}, \quad Z_2 = |Z| e^{i(\eta-\pi)/\kappa}.$$

We observe that

$$R(Z_2) - R(Z_1) = 2|Z| \sin(\eta/\kappa) \sin(\pi/\kappa),$$

and so, when $\kappa > 1$,

$$Z = Z_1, \quad R(Z_2) < R(Z_1) \quad \text{for} \quad -\pi < \eta < 0,$$

$$Z = Z_2, \quad R(Z_1) < R(Z_2) \quad \text{for} \quad 0 < \eta \leq \pi,$$

and $R(Z_1) = R(Z_2)$ when $\eta = 0$.

1.2. We restate here the six theorems of the former paper.

THEOREM 1. *If $\kappa > 0$ and $|\zeta| \leq \frac{1}{2}\pi \min(\kappa, 2) - \epsilon$, then*

$${}_pF_q(z) = I(Z).$$

THEOREM 2. *If $\kappa > 2$ and $|\eta| \leq \pi$, then*

$${}_pF_q(z) = I(Z_1) + I(Z_2).$$

THEOREM 3. *If $\kappa = 2$ and $|\eta| \leq \pi$, then*

$${}_pF_q(z) = I(Z_1) + I(Z_2) + J(-z).$$

THEOREM 4. *If $0 < \kappa < 2$ and $|\eta| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon$, then*

$${}_pF_q(z) = J(-z).$$

THEOREM 5. *If $0 < \kappa < 2$, $|\zeta| \leq \min(\pi, \frac{3}{2}\pi\kappa - \epsilon)$ and $|\eta| \leq \pi$, then*

$${}_pF_q(z) = I(Z) + J(-z).$$

The result of Theorem 5 holds good in the neighbourhood of the "critical lines" on which $\zeta = \pm \frac{1}{2}\pi\kappa$; on these the exponential expansion $I(Z)$ and the algebraic expansion $J(-z)$ are each of importance. In the regions covered by Theorems 1 and 2, $J(-z)$ is negligible compared with the error term of $I(Z)$.

If we write $x = h\kappa^t z$ and

$$\phi(t) = \frac{f(t) \Gamma(\kappa t - \vartheta + 1)}{(h\kappa^t)^t \Gamma(t + 1)},$$

then

$${}_pF_q(z) = \sum_{n=0}^{\infty} \frac{\phi(n) x^n}{\Gamma(\kappa n - \vartheta + 1)}.$$

For any fixed number N , $\phi(t)$ is regular for $\mathbf{R}(t) > N$ except for a finite number of poles and, by Lemma 1, satisfies condition A of Wright, 3, for $\mathbf{R}(t) > N$ and $|t| > K$. Hence the theorems of that paper apply and Theorems 1-5 above are an immediate corollary.

1.3. THEOREM 6. *If $f(t)$ has only a finite number of poles or none, then $\kappa \geq 1$ and the asymptotic expansion of ${}_pF_q(z)$ is given by*

$${}_pF_q(z) = I(Z_1) + I(Z_2) + H(-z) \quad (1 < \kappa < 2),$$

$${}_pF_q(z) = I(Z) + H(-z) \quad (\kappa = 1),$$

where $H(-z)$ is the finite expression defined above. If $f(t)$ has no poles, then $H(-z) = 0$.

For $|\zeta| \leq \frac{1}{2}\pi\kappa$, i.e. $\frac{1}{2}\pi(2-\kappa) \leq |\eta| \leq \pi$, this theorem is an immediate corollary of Theorem 5. We have

$$z^{-N+\delta} = O(Z^{3-M} e^Z)$$

for sufficiently large N , since $\mathbf{R}(Z) \geq 0$ for such values of ζ ; and so the difference between $H(-z)$ and $J(-z)$ can be absorbed in the error term of one of the I expansions.

For $|\eta| < \frac{1}{2}\pi(2-\kappa)$, however, there is a definite advance; the error term $O(z^{-N+\delta})$ in $J(-z)$ is now replaced by an exponential expansion with an exponentially small error term. If $f(t)$ has no poles (i.e. if $p = 0$), $H(-z)$ vanishes and we obtain an exponentially small expansion for ${}_pF_q(z)$ in this part of the plane.

2.1. We use $I_s(X)$ to denote an asymptotic expansion

$$X^{\vartheta_s} e^X \left\{ \sum_{m=0}^{M-1} A_{m,s} X^{-m} + O(X^{-M}) \right\}$$

and $H_s(y)$ to denote a finite sum

$$H_s(y) = \sum_{r=1}^h y^{r,s} \sum_{l=0}^{I_{r,s}} B_{r,l,s} (\log y)^l.$$

The numbers ϑ_s , $A_{m,s}$, $B_{r,l,s}$... are numbers independent of X , y , z , ..., but possibly depending on the parameters appearing in their suffixes or in (1.11).

Our calculations are shortened and simplified by the use of the following lemma.

LEMMA 2. If $1 \leq \kappa < 2$, if $0 \leq \alpha < \frac{1}{2}\pi\kappa \leq \beta < \gamma < \pi$, if $F(z)$ is regular and

$$F(z) = H_1(-z) + I_1(Z) + O(z^{-N}) \quad (A_{0,1} \neq 0)$$

for every N when $|z| > K$ and $\alpha \leq \zeta \leq \gamma$, and if

$$F(z) = H_2(-z) + I_2(Z) \quad (A_{0,2} \neq 0)$$

when $|z| > K$ and $\beta \leq \zeta \leq \gamma$, then $\vartheta_1 = \vartheta_2$, $A_{m,1} = A_{m,2}$ and

$$F(z) = H_1(-z) + I_1(Z)$$

when $|z| > K$ and $\alpha \leq \zeta \leq \gamma$.

In my original proof I used a lemma equivalent to this with $\alpha = \beta = \gamma = \frac{1}{2}\pi\kappa$; the regularity of $F(z)$ is then unnecessary. I am indebted to Mr. W. H. J. Fuchs for the above generalization which simplifies some of the subsequent calculations*; the following proof is his.

Without loss of generality we may take $\beta > \frac{1}{2}\pi\kappa$. If $\beta \leq \zeta \leq \gamma$, then $\cos \arg Z < -K$ and so

$$I_1(Z) = O(z^{-N}), \quad I_2(Z) = O(z^{-N})$$

for all N . Hence

$$H_1(-z) - H_2(-z) = I_2(Z) - I_1(Z) + O(z^{-N}) = O(z^{-N})$$

for all N and all z such that $|z| > K$ and $\beta \leq \zeta \leq \gamma$. It follows that $H_1(-z) = H_2(-z)$.

We may suppose that $H_1(-z) = 0$, since otherwise we replace $F(z)$ by $F(z) - H_1(-z)$. We have then

$$F(z) = I_1(Z) + O(z^{-N}) \quad (\alpha \leq \zeta \leq \gamma), \quad (2.11)$$

$$F(z) = I_2(Z) \quad (\beta \leq \zeta \leq \gamma). \quad (2.12)$$

We require the two following well-known lemmas. We use S to denote the region of the z -plane in which $\alpha \leq \zeta \leq \gamma$ and $|z| > K$.

* We have to prove (3.12) only under the condition (3.13) instead of under the wider condition $\frac{1}{2}\pi\kappa \leq \zeta \leq \pi$. In the latter case the proof is the same in principle but heavier in detail and the contour C of § 3.2 has to be replaced by a more complicated contour.

LEMMA 3 (*Phragmén and Lindelöf*, 1; *Titchmarsh*, 1, 177–178). If

- (i) $0 < \gamma - \alpha < \pi$,
- (ii) $G(z)$ is regular and $|G(z)| < Ke^{K|z|}$ in S , and
- (iii) $|G(z)| < K$ on the boundary of S ,

then $|f(z)| < K$ when $|z| \geq K$ and $\alpha \leq \zeta \leq \gamma$.

LEMMA 4 (*Montel*, 1, *Titchmarsh*, 1, 170–171). If

- (i) $\alpha < \delta < \gamma$, $\alpha < \alpha' < \gamma' < \gamma$,
- (ii) $g(z)$ is regular and $|g(z)| < K$ in S , and
- (iii) $\lim_{z \rightarrow \infty} g(re^{i\delta}) = A$,

then $g(z) \rightarrow A$ uniformly as $z \rightarrow \infty$, provided that $\alpha' \leq \zeta \leq \gamma'$.

We apply Lemma 3 to $G(z) = Z^{-\phi} e^{-Z} F(z)$, where

$$\phi = \vartheta_1 \quad (\mathbf{R}(\vartheta_1) \geq \mathbf{R}(\vartheta_2)), \quad \phi = \vartheta_2 \quad (\mathbf{R}(\vartheta_2) > \mathbf{R}(\vartheta_1)).$$

Since $\kappa \geq 1$, we have by (2.11)

$$|G(z)| < Ke^{K|Z|} \leq Ke^{K|z|}$$

when $\alpha \leq \zeta \leq \gamma$; this is condition (ii). By (2.11),

$$G(z) = O(1) + O(e^{-Z} z^K) = O(1)$$

when $\zeta = \alpha$, since $R(Z) > K|Z|$, and, by (2.12), $G(z) = O(1)$ when $\zeta = \gamma$; hence condition (iii) is satisfied. It follows that $|G(z)| < K$ when $\alpha \leq \zeta \leq \gamma$ and $|z| > K$.

Hence, if we put $g(z) = G(z)$ in Lemma 4, condition (ii) of that lemma is satisfied. We may suppose that $\alpha' < \frac{1}{2}\pi\kappa < \beta < \gamma'$.

If $\mathbf{R}(\vartheta_2) > \mathbf{R}(\vartheta_1)$, then $\phi = \vartheta_2$ and $g(z) \rightarrow A_{0,2}$ on $\zeta = \delta = \beta$. Hence, by Lemma 4, $g(z) \rightarrow A_{0,2} \neq 0$ when $\alpha' \leq \zeta \leq \gamma'$. But $g(z) \sim A_{0,1} Z^{\beta_1 - \beta_2} \rightarrow 0$ on $\zeta = \alpha'$ by (2.11). This is a contradiction.

Hence $\mathbf{R}(\vartheta_1) \geq \mathbf{R}(\vartheta_2)$, $\phi = \vartheta_1$ and $g(z) \rightarrow A_{0,1}$ on $\zeta = \alpha'$. It follows that $g(z) \rightarrow A_{0,1} \neq 0$ when $\alpha' \leq \zeta \leq \gamma'$. By (2.12) $g(z) \sim A_{0,2} Z^{\beta_2 - \beta_1}$ on $\zeta = \gamma'$ and so we have $A_{0,2} = A_{0,1}$ and $\vartheta_2 = \vartheta_1$.

Next we put $g(z) = Z\{G(z) - A_{0,1}\}$ and prove that

$$g(z) \rightarrow A_{1,1} = A_{1,2}$$

when $\alpha' \leq \zeta \leq \gamma'$. Continuing this process we prove that

$$\vartheta_1 = \vartheta_2, \quad A_{m,1} = A_{m,2} \quad (m = 0, 1, 2, \dots), \quad (2.13)$$

$$G(z) = \sum_{m=0}^{M-1} A_{m,1} Z^{-m} + O(Z^{-M}),$$

and so

$$F(z) = I_1(Z) = I_2(Z)$$

when $\alpha' \leq \zeta \leq \gamma'$.

When $\alpha \leq \zeta \leq \alpha'$, we have $\mathbf{R}(Z) > K|Z|$ and so, by (2.11),

$$F(z) = I_1(Z) + O(z^{-N}) = I_1(Z),$$

the $O(Z^{-N})$ term being absorbed in the error term of $I_1(Z)$. When $\gamma' \leq \zeta \leq \gamma$,

$$F(z) = I_2(Z) = I_1(Z)$$

by (2.13).

This completes the proof of Lemma 2.

3.1. We first consider the function ${}_0F_q(z)$ for which $H(-z) = 0$. Apart from the trivial case of ${}_0F_0(z) = e^z$, we have $\kappa > 1$ here and so, in Theorem 6, we are only concerned with $1 < \kappa < 2$.

Hence what we have to prove is

LEMMA 5. *If $1 < \kappa < 2$ and $|\eta| \leq \pi$, then*

$${}_0F_q(z) = I(Z_1) + I(Z_2).$$

We use a method of induction to prove this lemma. By Theorem 1 of Wright, 1,

$${}_0F_1(z) = I_1(Z_1) + I_1(Z_2). \quad (3.11)$$

When $0 \leq \zeta \leq \pi - \epsilon$ we have $Z_1 = Z$,

$$\cos \arg Z_2 < \cos \arg Z_1 - K$$

and so ${}_0F_1(z) = I_1(Z)$ by (3.11). By Theorem 5

$${}_0F_1(z) = I(Z) + O(z^{-N})$$

for all N . Putting $\alpha = 0$, $\beta = \frac{1}{2}\pi\kappa + \epsilon$, $\gamma = \pi - \epsilon$ in Lemma 2, we see that $\vartheta_1 = \vartheta$, $A_{m,1} = A_m$ and so $I(Z)$ and $I_1(Z)$ are identical. Hence by (3.11) Lemma 5 is true for $q = 1$. To complete the induction we have to prove

LEMMA 6. *If Lemma 5 is true for $q = Q - 1$, it is true for $q = Q$.*

In view of what we have just proved, we may take $Q \geq 2$. Let us assume for the moment that

$${}_0F_Q(z) = I_1(Z_1) + I_1(Z_2) \quad (3.12)$$

when

$$\frac{1}{2}\pi\kappa + \epsilon \leq \zeta \leq \pi. \quad (3.13)$$

As above, when $\frac{1}{2}\pi + \epsilon \leq \zeta \leq \pi - \epsilon$, (3.12) becomes ${}_0F_Q(z) = I_1(Z)$. By Theorem 5,

$${}_0F_Q(z) = I(Z) + O(z^{-N})$$

when $0 \leq \zeta \leq \pi - \epsilon$. Hence by Lemma 2 with $\alpha = 0$, $\beta = \frac{1}{2}\pi\kappa + \epsilon$, $\gamma = \pi - \epsilon$ we have $\vartheta_1 = \vartheta$, $A_{m,1} = A_m$, $I(Z) = I_1(Z)$ and

$${}_0F_Q(z) = I(Z) = I(Z_1) + I(Z_2)$$

when $0 \leq \zeta \leq \pi - \epsilon$. It follows from (3.12) that the last result is true for $0 \leq \zeta \leq \pi$. The proof for $-\pi < \zeta < 0$ is similar.

It remains only to deduce (3.12) from (3.13) and the hypothesis of Lemma 6. We take κ to be the κ of ${}_0F_Q(z)$, i.e.

$$\kappa = 1 + \rho_1 + \rho_2 + \dots + \rho_Q;$$

and we write

$$\rho = \rho_Q, \quad \mu = \mu_Q, \quad \sigma = \frac{\rho}{\kappa - \rho},$$

$${}_0F_{Q-1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho_1 n + \mu_1) \dots \Gamma(\rho_{Q-1} n + \mu_{Q-1}) \Gamma(n+1)}.$$

In the latter function $\kappa - \rho$ replaces κ . Since $Q \geq 2$,

$$\kappa - \rho = 1 + \rho_1 + \dots + \rho_{Q-1} > 1 \quad (3.14)$$

and, since $\kappa < 2$, $0 < \rho < 1$ and so $0 < \sigma < 1$. Also

$$\frac{\rho}{\kappa} = \frac{\sigma}{1 + \sigma}. \quad (3.15)$$

It follows from (3.13) that $\zeta = \eta + \pi$ and $Z = Z_1$. We write

$$Z_3 = Z_1 \exp\left(-\frac{2\pi i}{1 + \sigma}\right), \quad Z_4 = Z_2 \exp\left(\frac{2\pi i}{1 + \sigma}\right),$$

$$Z_5 = Z_1^{1+\sigma} Z_2^{-\sigma}, \quad Z_6 = Z_2^{1+\sigma} Z_1^{-\sigma},$$

so that

$$|Z_i| = |Z| \quad (1 \leq i \leq 6);$$

also

$$y_1 = (\kappa - \rho) \{h\rho^{\rho} |z|\}^{1/(\kappa - \rho)} \exp\left(\frac{i\zeta}{\kappa - \rho}\right) = \frac{1}{\sigma} \left(\frac{\rho Z_1}{\kappa}\right)^{1+\sigma}, \quad (3.16)$$

$$y_2 = y_1 \exp\left(-\frac{2\pi i}{\kappa - \rho}\right) = \frac{1}{\sigma} \left(\frac{\rho Z_2}{\kappa}\right)^{1+\sigma}. \quad (3.17)$$

3.2. We denote by C the contour in the u -plane consisting of the three parts:

C' , that part of the straight line $\arg u = (\zeta - 2\pi)/\kappa$ on which $|u| \geq 1$,

C'' , the arc of the circle $|u| = 1$ on which $(\zeta - 2\pi)/\kappa \leq \arg u \leq \zeta/\kappa$,

C''' , that part of the straight line $\arg u = \zeta/\kappa$ on which $|u| \geq 1$,

the whole described in a counter-clockwise direction round 0. Since $1 < \kappa < 2$ we have

$$-\frac{3}{2}\pi < \frac{\zeta - 2\pi}{\kappa} < -\frac{1}{2}\pi, \quad \frac{1}{2}\pi < \frac{\zeta}{\kappa} < \pi$$

by (3.13). Hence

$$\frac{1}{\Gamma(\rho n + \mu)} = \frac{1}{2\pi i} \int_C u^{-\rho n - \mu} e^u du$$

$$\text{and} \quad {}_0F_Q(z) = \frac{1}{2\pi i} \int_C u^{-\mu} e^u F_{Q-1}(zu^{-\rho}) du, \quad (3.21)$$

the change of order of summation and integration being easily justified.

If $\theta = \arg u$, we have

$$\frac{\zeta - 2\pi}{\kappa} \leq \theta \leq \frac{\zeta}{\kappa}, \quad -\frac{\rho\zeta}{\kappa} \leq -\rho\theta \leq -\frac{\rho(\zeta - 2\pi)}{\kappa}$$

on C and so

$$-\pi < \frac{(\kappa - \rho)\zeta}{\kappa} - \pi \leq \zeta - \pi - \rho\theta \leq \frac{(\zeta - 2\pi)(\kappa - \rho)}{\kappa} + \pi < \pi.$$

Hence $\arg(-zu^{-\rho})$ lies between $-\pi$ and π and so, by the hypothesis of Lemma 6,

$${}_0F_{Q-1}(zu^{-\rho}) = I_2(y_1 u^{-\sigma}) + I_2(y_2 u^{-\sigma}). \quad (3.22)$$

If we substitute from (3.22) in (3.21) and write

$$G(\tau, y) = \frac{1}{2\pi i} \int_C u^{-\tau} \exp(u + y u^{-\sigma}) du,$$

we obtain

$$\begin{aligned} {}_0F_Q(z) &= \sum_{m=0}^{M-1} A_{m,2} \{ y_1^{j_2-m} G(\mu + \sigma(j_2 - m), y_1) + y_2^{j_2-m} G(\mu + \sigma(j_2 - m), y_2) \} \\ &\quad + O(E_1 + E_2), \end{aligned} \quad (3.23)$$

where

$$E_1 = |y_1^{j_2-M}| \int_C |u^{-\mu - \sigma(j_2 - M)} \exp(u + y_1 u^{-\sigma})| |du|$$

and E_2 is the same except that y_2 replaces y_1 .

By Wright, 1, if $|\arg(-y)| < \pi$,

$$G(\mu + \sigma(\vartheta_2 - m), y) = I_{(m)}(Y_1) + I_{(m)}(Y_2), \quad (3.24)$$

where

$$Y_1 = (1 + \sigma) \{ \sigma^{-\sigma} |y|\}^{1/(1+\sigma)} \exp \left\{ \frac{i(\arg(-y) + \pi)}{1 + \sigma} \right\},$$

$$Y_2 = Y_1 \exp \left(-\frac{2\pi i}{1 + \sigma} \right),$$

$$I_{(m)}(Y) = Y^{\frac{1}{2} - \mu - \sigma(\vartheta_2 - m)} e^Y \left\{ \sum_{n=0}^{M-1} A_{n,(m)} Y^{-n} + O(Y^{-M}) \right\}.$$

If $y = y_1$, we take

$$\arg(-y_1) = \frac{\zeta}{\kappa - \rho} - \pi;$$

then

$$|\arg(-y_1)| < \pi, \quad Y_1 = Z_1, \quad Y_2 = Z_3.$$

If $y = y_2$, we take

$$\arg(-y_2) = \frac{\zeta - 2\pi}{\kappa - \rho} + \pi;$$

then

$$|\arg(-y_2)| < \pi, \quad Y_1 = Z_4, \quad Y_2 = Z_2.$$

It follows from (3.15), (3.16), (3.17), (3.23) and (3.24) that

$${}_0F_Q(z) = I_1(Z_1) + I_1(Z_2) + e^{2\pi i \vartheta_2} I_1(Z_3) + e^{-2\pi i \vartheta_2} I_1(Z_4) + O(E_1 + E_2), \quad (3.25)$$

where

$$\begin{aligned} I_1(Z) &= Z^{\frac{1}{2} - \mu + \vartheta_2} e^Z \left[\sum_{m=0}^{M-1} A_{m,2} \left(\sigma^\sigma (1 + \sigma)^{-(1+\sigma)} \right)^{\vartheta_2 - m} Z^{-m} \right. \\ &\quad \left. \times \left\{ \sum_{n=0}^{M-1} A_{n,(m)} Z^{-n} + O(Z^{-M}) \right\} \right] \\ &= Z^{\frac{1}{2} - \mu + \vartheta_2} e^Z \left[\sum_{m=0}^{M-1} A_{m,1} Z^{-m} + O(Z^{-M}) \right] \end{aligned}$$

and

$$\vartheta_1 = \frac{1}{2} - \mu + \vartheta_2. \quad (3.26)$$

3.3. LEMMA 7. *If (3.13) is satisfied, then*

$$R(Z_i) \leq R(Z_1) - K|Z| \quad (i = 3, 4, 5, 6), \quad (3.31)$$

$$R(Z_2) \leq R(Z_1) < -K|Z|. \quad (3.32)$$

also, on C ,

$$\cos \arg(y_1 u^{-\sigma}) < -K, \quad \cos \arg(y_2 u^{-\sigma}) < -K. \quad (3.34)$$

We have

$$\begin{aligned}\cos \arg Z_4 - \cos \arg Z_1 &= \cos \left(\frac{\zeta - 2\pi}{\kappa} + \frac{2\pi}{1+\sigma} \right) - \cos \frac{\zeta}{\kappa} \\ &= \cos \left(\frac{\zeta - 2\pi(1+\rho)}{\kappa} \right) - \cos \frac{\zeta}{\kappa} \\ &= 2 \sin \left(\frac{\zeta - \pi(1+\rho)}{\kappa} \right) \sin \frac{\pi(1+\rho)}{\kappa} < -K,\end{aligned}$$

since $0 < \pi(1+\rho) < \pi\kappa$, $-\pi\kappa + \epsilon < \zeta - \pi(1+\rho) \leq -\pi\rho$

by (3.13). This gives (3.31) when $i = 4$. The proofs for $i = 3, 5, 6$ are of a similar character. Again

$$\begin{aligned}\cos \arg Z_1 - \cos \arg Z_2 &= \cos \frac{\zeta}{\kappa} - \cos \frac{\zeta - 2\pi}{\kappa} \\ &= 2 \sin \frac{\pi - \zeta}{\kappa} \sin \frac{\pi}{\kappa} \geq 0\end{aligned}$$

and $\frac{1}{2}\pi + \frac{\epsilon}{\kappa} \leq \frac{\zeta}{\kappa} \leq \frac{\pi}{\kappa}$, $\cos \arg Z_1 < -K$,

and (3.32) follows.

$$\text{On } C \quad \frac{\zeta - 2\pi}{\kappa} \leq \theta \leq \frac{\zeta}{\kappa}, \quad -\frac{\sigma\zeta}{\kappa} \leq -\sigma\theta \leq \frac{\sigma(2\pi - \zeta)}{\kappa},$$

and so

$$\begin{aligned}\frac{\pi}{2} + \frac{\epsilon}{\kappa} < \frac{\zeta}{\kappa} \leq \arg \left(\frac{y_1}{w^\sigma} \right) &\leq \frac{\zeta}{\kappa} + \frac{2\pi\sigma}{\kappa} \leq \frac{\pi(1+2\sigma)}{\kappa} < \frac{3\pi}{2}, \\ -\frac{3}{2}\pi < \frac{\pi}{2} - \frac{2\pi}{\kappa - \rho} &\leq \frac{\zeta}{\kappa} - \frac{2\pi}{\kappa - \rho} \leq \arg \left(\frac{y_2}{w^\sigma} \right) \\ &\leq \frac{\zeta}{\kappa} + \frac{2\pi\sigma}{\kappa} - \frac{2\pi}{\kappa - \rho} = \frac{\zeta - 2\pi}{\kappa} \leq -\frac{\pi}{\kappa} < -\frac{\pi}{2}.\end{aligned}$$

From these (3.34) follows.

3.4. LEMMA 8. *If $\frac{1}{2}\pi\kappa + \epsilon \leq \zeta \leq \pi$, then*

$$E_1 + E_2 = O(Z_1^{3_1 + \frac{1}{2} - M} e^{Z_1}).$$

We write

$$E_1 = E_1' + E_1'' + E_1''', \quad E_2 = E_2' + E_2'' + E_2''', \quad (3.41)$$

where E_1' is the part contributed to E_1 by the integral along C' and so on.

By (3.34),

$$\mathbf{R}(y_1 u^{-\sigma}) < -K|y_1| = -K|Z|^{1+\sigma}, \quad \mathbf{R}(y_2 u^{-\sigma}) < -K|Z|^{1+\sigma}$$

on C'' . Hence

$$\begin{aligned} E_1'' + E_2'' &= O(|Z|^K \exp(-K|Z|^{1+\sigma})) \\ &= O(\exp(-K|Z|^{1+\sigma})) \\ &= O(Z_1^{3_1+\frac{1}{2}-M} e^{Z_1}), \end{aligned} \tag{3.42}$$

since $\sigma > 0$.

On C' we put
$$u = \frac{\sigma Z_2 t}{1+\sigma},$$

so that t is real and positive. Then

$$u + y_1 u^{-\sigma} = \frac{\sigma}{1+\sigma} \left(Z_2 t + \frac{Z_5}{\sigma t^\sigma} \right),$$

$$u + y_2 u^{-\sigma} = \frac{\sigma Z_2}{1+\sigma} \left(t + \frac{1}{\sigma t^\sigma} \right).$$

If we write

$$g(t) = \frac{\sigma}{1+\sigma} \left(t + \frac{1}{\sigma t^\sigma} - 1 - \frac{1}{\sigma} \right)$$

and use (3.31) and (3.32), we have

$$\mathbf{R}(u + y_1 u^{-\sigma}) \leq \mathbf{R}(Z_1) \left(1 + g(t) \right),$$

$$\mathbf{R}(u + y_2 u^{-\sigma}) \leq \mathbf{R}(Z_1) \left(1 + g(t) \right).$$

On C''' we put
$$u = \frac{\sigma Z_1 t}{1+\sigma}$$

and the same results hold good. Hence, by (3.26),

$$\begin{aligned} E_1' + E_1'' + E_2' + E_2'' &\leq K |y_1^{3_2-M} Z_1^{1-\mu-\sigma(3_2-M)} e^{Z_1}| \int_0^\infty t^\tau e^{\mathbf{R}(Z_1)g(t)} dt \\ &\leq K |Z_1^{3_1+\frac{1}{2}-M} e^{Z_1}| \int_0^\infty t^\tau e^{\mathbf{R}(Z_1)g(t)} dt, \end{aligned} \tag{3.43}$$

where

$$\tau = -\mathbf{R}\{\mu + \sigma(3_2 - M)\}.$$

Now

$$\frac{d}{dt} g(t) = \frac{\sigma}{1+\sigma} (1 - t^{-1-\sigma})$$

and so, when $t > 0$,

$$g(t) \geq g(1) = 0.$$

Also $R(Z_1) < -K$ by (3.32); hence

$$\int_0^\infty t^r e^{R(Z_1)g(t)} dt < \int_0^\infty t^r e^{-Kg(t)} dt = K, \tag{3.44}$$

since the last integral is convergent. Lemma 8 follows from (3.41)–(3.44).

3.5. Since

$$|e^{Z_3}| < |e^{Z_1-K|Z_1|}|, \quad |e^{Z_4}| < |e^{Z_1-K|Z|}|$$

by (3.31), the terms $I_1(Z_3)$ and $I_1(Z_4)$ in (3.25) may be absorbed in the error term of $I_1(Z_1)$. Using Lemma 8, we have then

$${}_0F_Q(z) = I_1(Z_1) + I_1(Z_2) + O(Z_1^{3_1+\frac{1}{2}-M} e^{Z_1}).$$

We may replace M by $M+1$ in this formula; then

$$\begin{aligned} {}_0F_Q(z) &= \sum_{m=0}^M A_{m,1} (Z_1^{3_1-m} e^{Z_1} + Z_2^{3_1-m} e^{Z_2}) + O(Z_1^{3_1-\frac{1}{2}-M} e^{Z_1}) \\ &= \sum_{m=0}^{M-1} A_{m,1} (Z_1^{3_1-m} e^{Z_1} + Z_2^{3_1-m} e^{Z_2}) + O(Z_1^{3_1-M} e^{Z_1}) \\ &= I_1(Z_1) + I_1(Z_2). \end{aligned}$$

This is (3.12).

4.1. We now consider ${}_pF_q(z)$ for general p , where $f(t)$ has only a finite number of poles. We use $P_1(t), P_2(t), \dots$ to denote polynomials in t . The numbers B_1, B_2, B_3, B_4 are independent of z, x, u, v , and t , but may be functions of any of the parameters.

LEMMA 9. *If $f(t)$ has only a finite number of poles, then*

$$f(t) = \frac{P_1(t) e^{dt}}{P_2(t) \prod_{r=1}^{q'} \Gamma(\rho_r' t + \mu_r')},$$

where $\rho_1', \dots, \rho_{q'}'$ are all real and positive,

$$1 + \sum_{r=1}^{q'} \rho_r' = \kappa \tag{4.11}$$

and

$$d = \sum_{r=1}^{q'} \rho_r' \log \rho_r' + \log h. \tag{4.12}$$

Since

$$f(t) = \frac{\prod_{r=1}^p \Gamma(\alpha_r t + \beta_r)}{\prod_{r=1}^q \Gamma(\rho_r t + \mu_r)}$$

has only a finite number of poles, all the other poles of the numerator must also be poles of the denominator and of at least the same degree of multiplicity.

If the functions $\Gamma(\alpha_r t + \beta_r)$ and $\Gamma(\rho_R t + \mu_R)$ have just one pole (say $t = \lambda$) in common, we write

$$\frac{\Gamma(\alpha_r t + \beta_r)}{\Gamma(\rho_R t + \mu_R)} = \frac{P_3(t)}{P_4(t)} \frac{\Gamma(\alpha_r t + \beta_r + l)}{\Gamma(\rho_R t + \mu_R + L)},$$

where l and L are integers large enough to ensure that $t = \lambda$ is not a pole of the new Γ -functions, which have no pole in common.

If the two functions have more than one pole in common, let the two common poles with the greatest real parts be at the points

$$t = -\frac{\beta_r + l_1}{\alpha_r} = -\frac{\mu_R + L_1}{\rho_R}, \quad t = -\frac{\beta_r + l_2}{\alpha_r} = -\frac{\mu_R + L_2}{\rho_R}.$$

Hence

$$\frac{l_2 - l_1}{\alpha_r} = \frac{L_2 - L_1}{\rho_R}$$

and so the common poles of the two functions are given by

$$t = -\frac{\beta_r + l_1 + j(l_2 - l_1)}{\alpha_r} = -\frac{\mu_R + L_1 + j(L_2 - L_1)}{\rho_R} \quad (4.13)$$

for $j = 0, 1, 2, \dots$. We observe that the two functions have no poles, one pole or an infinity of poles in common.

By a well-known property of the Γ -function,

$$\Gamma(\alpha_r t + \beta_r) = (l_2 - l_1)^{\alpha_r t + \beta_r - \frac{1}{2}} (2\pi)^{-\frac{1}{2}(l_2 - l_1 - 1)} \prod_{l=0}^{l_2 - l_1 - 1} \Gamma\left(\frac{\alpha_r t + \beta_r + l}{l_2 - l_1}\right) \quad (4.14)$$

and

$$\Gamma(\rho_R t + \mu_R) = (L_2 - L_1)^{\rho_R t + \mu_R - \frac{1}{2}} (2\pi)^{-\frac{1}{2}(L_2 - L_1 - 1)} \prod_{L=0}^{L_2 - L_1 - 1} \Gamma\left(\frac{\rho_R t + \mu_R + L}{L_2 - L_1}\right). \quad (4.15)$$

If

$$l_1 \equiv l' \pmod{(l_2 - l_1)} \quad (0 \leq l' \leq l_2 - l_1 - 1),$$

$$L_1 \equiv L' \pmod{(L_2 - L_1)} \quad (0 \leq L' \leq L_2 - L_1 - 1),$$

then each of the functions

$$\Gamma\left(\frac{\alpha_r t + \beta_r + l'}{l_2 - l_1}\right), \quad \Gamma\left(\frac{\rho_R t + \mu_R + L'}{L_2 - L_1}\right)$$

has all the poles (4.13) and none of the other Γ -functions on the right-hand side of (4.14) and (4.15) has any of these poles. Also

$$\frac{\alpha_r t + \beta_r + l'}{l_2 - l_1} - \frac{\rho_R t + \mu_R + L'}{L_2 - L_1}$$

is an integer and so

$$\frac{\Gamma\{(\alpha_r t + \beta_r + l')/(l_2 - l_1)\}}{\Gamma\{(\rho_R t + \mu_R + L')/(L_2 - L_1)\}} = \frac{P_5(t)}{P_6(t)},$$

where one of $P_5(t)$, $P_6(t)$ is unity. Hence

$$\frac{\Gamma(\alpha_r t + \beta_r)}{\Gamma(\rho_R t + \mu_R)} = \frac{B_1^t P_5(t) \prod_{l=0}^{l_2-l_1-2} \Gamma(\alpha_l' t + \beta_l)}{P_6(t) \prod_{L=0}^{L_2-L_1-2} \Gamma(\rho_L' t + \mu_L)},$$

where on the right-hand side the numerator and the denominator have no pole in common.

By a repetition of these processes we eventually find that

$$f(t) = \frac{B_2^t P_1(t) \prod_{r=1}^{p'} \Gamma(\alpha_r' t + \beta_r')}{P_2(t) \prod_{r=1}^{q'} \Gamma(\rho_r' t + \mu_r')}, \quad (4.16)$$

where none of the Γ -functions in the numerator has a pole in common with any Γ -function in the denominator. Hence $f(t)$ has all the poles of the Γ -functions in the numerator except for those which lie among the zeros (finite in number) of $P_1(t)$. Since $f(t)$ has only a finite number of poles there are no Γ -functions in the numerator, so that $p' = 0$. This proves Lemma 9 except for the equality of B_2 and e^d , and for the relation (4.11). These follow at once if we apply the asymptotic expansion of the Γ -function to the original form of $f(t)$ and to the form (4.16).

We may suppose that $P_1(t)$ and $P_2(t)$ have no zero in common, since the factors corresponding to such zeros may be removed.

4.2. We can deduce from (4.11) that, as stated in Theorem 6, $\kappa \geq 1$ when $f(t)$ has only a finite number of poles. The cases $1 < \kappa < 2$ and $\kappa = 1$ have to be dealt with separately. We shall prove that

$${}_p F_q(z) = I_1(Z_1) + I_1(Z_2) + H_1(-z) \quad (4.21)$$

when $1 < \kappa < 2$ and

$${}_p F_q(z) = I_1(Z) + H_1(-z) \quad (4.22)$$

when $\kappa = 1$, provided that

$$|\eta| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon, \quad \text{i.e.} \quad \frac{1}{2}\pi\kappa + \epsilon \leq |\zeta| \leq \pi$$

in each case. By Theorem 5,

$${}_p F_q(z) = I(Z) + H(-z) + O(z^{-N})$$

when $1 \leq \kappa < 2$ and $0 \leq \zeta \leq \pi - \epsilon$. Hence $I_1(Z)$ and $I(Z)$ are identical by Lemma 2, and the results of Theorem 6 follow when $0 \leq \zeta \leq \pi$; the proof for negative ζ is similar.

We expand $P_1(t)/P_2(t)$ in partial fractions, and we obtain

$$\frac{P_1(t)}{P_2(t)} = P_7(t) + \sum_{v=1}^{k_1} \frac{D_{v,1}}{t - a_{v,1}} + \sum_{v=1}^{k_2} \frac{D_{v,2}}{(t - a_{v,2})^2} + \dots;$$

also

$$P_7(t) = D_{k_0} t(t-1) \dots (t-k_0+1) + D_{k_0-1} t(t-1) \dots (t-k_0+2) + \dots + D_0.$$

Hence, if we write

$$\begin{aligned} \gamma(t) &= \prod_{r=1}^{q'} \{\Gamma(\rho_r' t + \mu_r')\}^{-1}, \\ F_{(v)}(x) &= \sum_{n=0}^{\infty} \frac{n(n-1) \dots (n-v+1) \gamma(n) x^n}{\Gamma(n+1)}, \\ F_a^{(r)}(x) &= \sum_{n=0}^{\infty} \frac{\gamma(n) x^n}{(n-a)^r \Gamma(n+1)}, \end{aligned}$$

we have

$${}_p F_q(z) = \sum_{v=0}^{k_0} D_v F_{(v)}(e^d z) + \sum_{v=1}^{k_1} D_{0,1} F_{a_{v,1}}^{(1)}(e^d z) + \sum_{v=1}^{k_2} D_{v,2} F_{a_{v,2}}^{(2)}(e^d z) + \dots, \quad (4.23)$$

the number of terms being finite. The D are all functions of the parameters. We have now only to determine the asymptotic expansion of $F_a(x)$ and $F_a^{(r)}(x)$ when $|\arg(-x)| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon$.

4.3. *The case* $1 < \kappa < 2$. We write

$$\psi = \frac{1}{2}q' - \sum_{r=1}^{q'} \mu_r', \quad h' = \prod_{r=1}^{q'} (\rho_r')^{-\rho_r}, \quad \xi = \arg(-x) \quad (|\xi| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon),$$

$$X_1 = \kappa (h'(-x))^{1/\kappa} e^{\pi i/\kappa}, \quad X_2 = X_1 e^{-2\pi i/\kappa}. \quad (4.31)$$

Since

$$\begin{aligned} F_{(v)}(x) &= x^v \sum_{n=0}^{\infty} \frac{\gamma(n+v)}{\Gamma(n+1)} x^n \\ &= x^v \sum_{n=0}^{\infty} \frac{x^n}{\prod_{r=1}^{q'} \Gamma(\rho_r' n + \mu_r' + \rho_r' v) \Gamma(n+1)} \\ &= x^v {}_0 F_{q'}(x), \end{aligned}$$

we have

$$F_{(v)}(x) = x^v \{I_5(X_1) + I_5(X_2)\}$$

by Lemma 5, where

$$\vartheta_5 = \frac{1}{2}q' - \sum_{r=1}^{q'} (\mu_r' + \rho_r' v) = \psi - (\kappa - 1)v.$$

Since v is an integer,

$$x^v = \kappa^{-\kappa v} h'^{-v} X_1^{\kappa v} = \kappa^{-\kappa v} h'^{-v} X_2^{\kappa v}$$

and so
$$F_{(v)}(x) = I_6(X_1) + I_6(X_2) \quad (\vartheta_6 = \psi + v). \quad (4.32)$$

Since $F_a^{(0)}(x) = {}_0F_{q'}(x),$

$$F_a^{(0)}(x) = I_7(X_1) + I_7(X_2) \quad (\vartheta_7 = \psi) \quad (4.33)$$

by Lemma 5.

We use an induction to find the asymptotic expansion of $F_a^{(r)}(x)$ when $r \geq 1$. We write $I^{(r)}(X)$ for an exponential expansion

$$X^{\psi-r} e^X \left\{ \sum_{m=0}^{M-1} A_m^{(r)} X^{-m} + O(X^{-M}) \right\}$$

and $P^{(r)}(u)$ for a polynomial in u . The numbers $A_m^{(r)}$ and the coefficients in $P^{(r)}(u)$ depend on r , a , and the parameters $\rho_1', \dots, \rho_{q'}', \mu_1', \dots, \mu_{q'}'$.

LEMMA 11. *If $1 < \kappa < 2$, $r \geq 1$ and*

$$F_a^{(r-1)}(x) = I^{(r-1)}(X_1) + I^{(r-1)}(X_2) + (-x)^a P^{(r-1)}(\log(-x))$$

when $|x| \geq 1$ and $|\xi| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon$, then

$$F_a^{(r)}(x) = I^{(r)}(X_1) + I^{(r)}(X_2) + (-x)^a P^{(r)}(\log(-x))$$

for the same set of values of x .

We suppose that $|\arg u| \leq \frac{1}{2}\pi(2-\kappa) - \epsilon$ and write

$$U = \kappa(h'u)^{1/\kappa}, \quad G(u) = F_a^{(r-1)}(-u) - u^a P^{(r-1)}(\log u).$$

By the hypothesis of Lemma 11,

$$G(u) = I^{(r-1)}(Ue^{\pi i/\kappa}) + I^{(r-1)}(Ue^{-\pi i/\kappa}). \quad (4.34)$$

Now

$$\begin{aligned} |\arg(-Ue^{\pm\pi i/\kappa})| &= |\arg(Ue^{\pm\pi i(1-\kappa)/\kappa})| \\ &\leq \frac{\frac{1}{2}\pi(2-\kappa) - \epsilon}{\kappa} + \frac{\pi(\kappa-1)}{\kappa} = \frac{1}{2}\pi - \frac{\epsilon}{\kappa}, \end{aligned}$$

and so $R(Ue^{\pm\pi i/\kappa}) < -K|U|$. Hence

$$G(u) < Ke^{-K|U|},$$

and so $\int u^{-a-1} G(u) du$, taken along that arc of the circle $|u| = R$ on which $\arg u$ lies between 0 and $\arg(-x)$, tends to zero as $R \rightarrow \infty$. It follows by Cauchy's theorem that

$$\int_1^{-x} u^{-a-1} G(u) du = \int_1^{\infty} u^{-a-1} G(u) du - \int_{-x}^{\infty e^{i\theta}} u^{-a-1} G(u) du,$$

where the two contours in the integrals on the right are parts of straight lines drawn radially from the origin in the u -plane. Hence

$$\int_1^{-x} u^{-a-1} G(u) du = B_3 - (-x)^{-a} \int_1^{\infty} t^{-a-1} G(-xt) dt.$$

Also

$$\begin{aligned} (-x)^{-a} F_a^{(r)}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n \gamma(n) \{(-x)^{n-a-1}\}}{(n-a)^r \Gamma(n+1)} + F_a^{(r)}(-1) \\ &= \int_1^{-x} u^{-a-1} F_a^{(r-1)}(u) du + B_4 \\ &= \int_1^{-x} u^{-a-1} G(u) du + \int_1^{-x} u^{-1} P^{(r-1)}(\log u) du + B_4 \end{aligned}$$

and
$$\int_1^{-x} u^{-1} P^{(r-1)}(\log u) du + B_3 + B_4 = P^{(r)}\{\log(-x)\}.$$

Hence

$$F_a^{(r)}(x) = (-x)^a P^{(r)}\{\log(-x)\} - \int_1^{\infty} t^{-a-1} G(-xt) dt. \quad (4.35)$$

To evaluate the last integral we require the following lemma.

LEMMA 12. If $R(X) < -K$ and

$$\varpi(\nu, X) = \int_1^{\infty} T^{-\nu} e^{XT} dT,$$

then
$$\varpi(\nu, X) = -e^X \left\{ \sum_{l=0}^{L-1} \frac{\Gamma(\nu+l)}{\Gamma(\nu)} X^{-l-1} + O(X^{-L-1}) \right\}.$$

Writing $T = w/X$, we have

$$\varpi(\nu, X) = X^{\nu-1} \int_X^{\infty e^{i \arg X}} w^{-\nu} e^w dw = X^{\nu-1} \int_X^{X-\infty} w^{-\nu} e^w dw$$

by Cauchy's theorem, since $\mathbf{R}(X) < 0$. If we put $v = X - u$, we get

$$\begin{aligned}\varpi(\nu, X) &= -X^{\nu-1} e^X \int_0^\infty (X-u)^{-\nu} e^{-u} du \\ &= -\frac{e^X}{X} \int_0^\infty \left(1 - \frac{u}{X}\right)^{-\nu} e^{-u} du.\end{aligned}\quad (4.36)$$

Whenever $|u - X| > K$ and, in particular, for all $u \geq 0$,

$$\left(1 - \frac{u}{X}\right)^{-\nu} = 1 + \sum_{l=1}^{L-1} \frac{\Gamma(\nu+l)}{\Gamma(\nu)\Gamma(l+1)} \frac{u^l}{X^l} + O\left(\frac{u^L}{X^L}\right).$$

Using this in (4.36), we have Lemma 12.

If $t = T^\kappa$,

$$G(-xt) = I^{(r-1)}(X_1 T) + I^{(r-1)}(X_2 T)$$

by (4.34) and so

$$\begin{aligned}\int_1^\infty t^{-a-1} G(-xt) dt &= \kappa \sum_{m=0}^{M-1} A_m^{(r-1)} \{X_1^{\psi+1-r-m} \varpi(a\kappa - \psi + r + m, X_1) \\ &\quad + X_1^{\psi+1-r-m} \varpi(a\kappa - \psi + r + m, X_2)\} \\ &\quad + O\{X_1^{\psi+1-r-M} \varpi(\mathbf{R}(a\kappa - \psi + r + M), \mathbf{R}(X_1)) \\ &\quad \times \varpi(\mathbf{R}(a\kappa + \psi + r + M), \mathbf{R}(X_2))\} \\ &= -I^{(r)}(X_1) - I^{(r)}(X_2)\end{aligned}\quad (4.37)$$

by Lemma 12, since $\mathbf{R}(X_1) < -K$, $\mathbf{R}(X_2) < -K$. Lemma 11 follows from (4.35) and (4.37).

If we put $x = e^d z$, we have $X_1 = Z_1$ and $X_2 = Z_2$ by (4.12) and (4.31). Hence (4.21) follows from (4.23), (4.32), (4.33) and Lemma 11.

4.4. *The case $\kappa = 1$.* Here $q' = 0$, $h = e^d$, $\gamma(t) = 1$ and $Z = e^d z$; also

$$F_{(v)}(e^d z) = F_{(v)}(Z) = Z^v e^Z, \quad F_a^{(0)}(e^d z) = e^Z.$$

We write

$$I^{(r)}(Z) = e^Z Z^{-r} \left\{ \sum_{m=0}^{M-1} A_m^{(r)} Z^{-m} + O(Z^{-M}) \right\}.$$

To complete the proof of (4.22) we have only to prove

LEMMA 13. *If $\kappa = 1$, $r \geq 1$ and*

$$F_a^{(r-1)}(Z) = I^{(r-1)}(Z) + (-Z)^a P^{(r-1)}(\log(-Z))$$

when $|Z| \geq 1$ and $|\arg(-Z)| \leq \frac{1}{2}\pi - \epsilon$, then

$$F_a^{(r)}(Z) = I^{(r)}(Z) + (-Z)^a P^{(r)}(\log(-Z))$$

for the same values of Z .

The proof of this follows exactly the same lines as that of Lemma 11.

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SINGULAR INTEGRAL EQUATIONS†

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[Received and read 15 June, 1939.]

Introduction.

1.1. In the present paper I propose to discuss certain classes of singular integral equations. All the equations considered are linear and homogeneous, and we shall distinguish three main classes of equations.

The general forms of the equations of the three classes are as follows:

(i) in the *Volterra case*,

$$f(x) = \int_{-\infty}^x K(x-y)f(y)dy; \quad (1.1.1)$$

(ii) in the *Fredholm case*,

$$f(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy; \quad (1.1.2)$$

(iii) in what I propose to call the *Wiener-Hopf case*,

$$f(x) = \int_0^{\infty} K(x-y)f(y)dy. \quad (1.1.3)$$

We divide each of these classes into two types, according to the behaviour of $K(x)$ when $|x| \rightarrow \infty$. In the Volterra equations of the first type (discussed in Chapter 2) $K(x)$ is, roughly speaking, bounded for large x ; in those of the second type (Chapter 5) $K(x)$ behaves like x^{-k} , for some $k > 0$. Here $K(x)$ vanishes for negative x . In the Fredholm and Wiener-Hopf equations of the first type (Chapters 3 and 4 respectively)

† My warmest thanks are due to Prof. G. H. Hardy, whose encouragement and advice have been of the utmost value to me in the course of this work.

$K(x)$ behaves like $e^{-|x|}$ when $|x|$ is large; in those of the second type (Chapters 6 and 7) it behaves like $|x|^{-k} e^{-|x|}$ for some $k > 0$. The conditions that we actually impose are a little more restrictive than those indicated above. Various possible generalizations of the conditions are pointed out at the ends of the relevant chapters. A number of examples are discussed as illustrations of the general theorems.

The Laplace integral plays a considerable part in our discussions; its appropriateness for our purpose is suggested by the fact that in each of the three cases listed above the kernel is a function of $x-y$.

1.2. We do not endeavour to make a complete survey of the immense literature on equations of the above forms; numerous references will be found in Titchmarsh, 41†, Chapter 11, and in Hellinger and Toeplitz, 18, 1465. Reference may also be made to Bochner, 5, and to Bochner, 6, Chapters 5-6.

The methods that we use are modifications of one introduced by Wiener and Hopf, 46‡, who considered equations of the form (1.1.3). We show that these methods can be extended to equations (1.1.1) and (1.1.2) and made sufficiently delicate to enable us to prove, in a large range of cases, the exhaustiveness of the known solutions under the sole hypothesis that the equation considered has a meaning when the integration involved is taken in the sense of Lebesgue.

One of our most valuable tools is the following principle of analytic continuation§:

Let D_1 and D_2 be two non-overlapping open regions in the complex s -plane, and let C be a simple rectifiable curve forming part of the frontier of both regions; let $\phi_i(s)$ be regular in D_i and continuous in $D_i + C$ ($i = 1, 2$) and let $\phi_1(s) = \phi_2(s)$ on C ; then the functions $\phi_1(s)$ and $\phi_2(s)$ are analytic continuations of one another, and together define a function which is regular in $D_1 + D_2 + C$.

In Chapters 5-7 we make considerable use of an adaptation of Bochner's theory of generalized Fourier integrals||. Here again we have made the methods delicate enough to give results that are "best possible" in the sense already described. We do not use Bochner's special notation, which is of no particular advantage for our purposes.

† Numbers in clarendon type refer to the bibliography at the end of the paper.

‡ See also Hopf, 28, Ch. 4, and Paley and Wiener, 35, Ch. 4.

§ For a proof, see Titchmarsh, 40, 155-157.

|| Bochner, 4 and 6, Ch. 6.

1.3. Chapter 8 is devoted to a discussion of the integral equations of E. A. Milne. There are essentially two of these, viz.,

$$f(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy, \quad (1.3.1)$$

$$\text{and} \quad f(x) = \int_0^{\infty} K(x-y)f(y)dy, \quad (1.3.2)$$

$$\text{where} \quad K(x) = \frac{1}{2} \int_{|x|}^{\infty} \frac{e^{-y}}{y} dy.$$

Equation (1.3.2) arises in the theory of radiative equilibrium in the interior of a star†. If $f(x)$ is the total intensity of radiation at a point whose "optical depth" is x , then $f(x)$ satisfies (1.3.2). (1.3.1) is an approximation to (1.3.2) for great depths in the star.

The above equations also appear in the forms

$$F'(x) = \int_0^{\infty} \frac{F(x+y) - F(x-y)}{2y} e^{-y} dy, \quad (1.3.3)$$

$$F'(x) = \int_0^x \frac{F(x+y) - F(x-y)}{2y} e^{-y} dy + \int_x^{\infty} \frac{F(x+y)}{2y} e^{-y} dy, \quad (1.3.4)$$

respectively. The exact connection between the various forms is discussed in Chapter 8.

The existing literature on these equations is mostly devoted to (1.3.3) and (1.3.2). We give here an account of the results obtained by previous writers.

It is known that (1.3.3) possesses the solution

$$F(x) = Ax^2 + Bx + C,$$

where A , B , and C are arbitrary constants. Littlewood, 33, proved the exhaustiveness of this solution on the hypothesis that $F''(x)$ is continuous and bounded in $(-\infty, \infty)$. Hardy and Titchmarsh, 12‡, showed that (i) if

$$\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty,$$

then $F(x) \equiv 0$, (ii) if $|F(x)| \leq Ae^{\alpha|x|}$ for some A and some $\alpha < 1$, then $F(x)$ is a quadratic, and (iii) if

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!},$$

† Milne, 34.

‡ See also Hardy and Titchmarsh, 13, 14.

where $\sum_{n=0}^{\infty} a_{2n}$ and $\sum_{n=0}^{\infty} a_{2n+1}$ are convergent, then $F(x)$ is a quadratic. The condition in (ii) was improved by Hopf, 23, to

$$\int_{-\infty}^{\infty} |F(x)| e^{a|x|} dx < \infty.$$

More recently Hardy and Titchmarsh, 17, have proved the exhaustiveness of the quadratic solution under the condition that

$$\int_{-\infty}^{\infty} |g(x)|^2 [1 + \log^2 |g(x)|] dx < \infty,$$

where

$$g(x) = \frac{f(x)}{\cosh x}.$$

Equation (1.3.2) was at one time the object of considerable controversy. Several writers claimed to have shown that it does not possess a solution†. Their proofs were, however, valid only under conditions which they did not state explicitly, and which are not, in fact, satisfied by the actual solution. The limits of validity of these proofs were indicated in a paper by Freundlich, Hopf, and Wegner, 10‡. The existence of a solution was first proved by Hopf, 19, who also showed (20) that, apart from a multiplicative constant, it is the only positive solution§. He later applied his method to a more general class of equations with a positive kernel (21, 27). Various properties of the solution have been obtained by Hopf, 24, 26, Bronstein, 7, 8, 9, and Ambarzumian, 1.

Equations of type (1.3.2), with a general kernel $K(x-y)$, were discussed by Wiener and Hopf, 46, whose method was entirely different from Hopf's original one. They assumed that

$$\int_{-\infty}^{\infty} |K(x) e^{\sigma|x|}|^2 dx < \infty$$

for all $\sigma < 1$, and obtained an explicit formula for the solutions of the equation; this formula is given in Chapter 4 of the present paper. They proved the exhaustiveness of the solution so obtained under the condition that

$$f(x) = O(e^{ax}) \quad (x \rightarrow \infty)$$

† Ambarzumian and Kosirev, 2, 3, 29; Kostitzin, 30, 31; P. Parchomenko, 36, 37. See also Gerasimovitch, 11.

‡ See also Wegner, 42, 43, 44.

§ See also Hopf, 22, 25.

for some $\alpha < 1$. The Wiener-Hopf discussion is also given in Hopf, 28, Chapter 4, Paley and Wiener, 35, Chapter 4, and Titchmarsh, 41, 339-349.

In Chapters 3 and 6 we discuss equation (1.3.1) for a general kernel, and in Chapters 4 and 7 equation (1.3.2). In each case it is shown that the known solution of the equation is exhaustive under the condition that the equation has a meaning. In Chapter 8 we apply our general theorems to the equations of E. A. Milne, including the forms (1.3.3) and (1.3.4); again we obtain "best possible" results in the above sense.

1.4. We use the Lebesgue and Lebesgue-Stieltjes integrals throughout, and all functions are assumed measurable whenever measurability is not implied by the context. Equalities and inequalities between functions of real variables are in general to be understood as holding almost everywhere, *i.e.*, at all points of the range considered except those of some set of Lebesgue measure zero. We consider complex-valued functions of real variables throughout.

We denote by $L^p(a, b)$, where (a, b) is a finite or infinite interval, the class of measurable functions $f(x)$ such that

$$\int_a^b |f(x)|^p dx < \infty.$$

If $f(x)$ belongs to the class, we write $f(x) \in L^p(a, b)$. The class $L^1(a, b)$ will be denoted by $L(a, b)$. If $f(x)$ is of bounded variation in (a, b) , we say that $f(x)$ belongs to the class $V(a, b)$, and write $f(x) \in V(a, b)$.

As is now customary in the theory of Dirichlet series and Laplace integrals, we denote the complex variable by $s = \sigma + it$, where $\sigma = \Re s$, $t = \Im s$.

Except in Chapter 8, we use the letter C to denote a positive constant, not necessarily the same at each occurrence.

2. Volterra equations of Type 1.

2.1. In this chapter we consider equations of the form

$$f(x) = \int_{-\infty}^x K(x-y)f(y) dy. \quad (2.1.1)$$

We suppose that $K(x)$ satisfies the following conditions:

Hypothesis 1. $K(x)$ can be expressed in the form

$$K(x) = K_1(x) + K_2(x) \quad (x \geq 0),$$

where $K_1(x) \in V(0, \infty)$ and $K_2(x) \in L(0, \infty)$.

For convenience, we also suppose that

$$K(x) = K_1(x) = K_2(x) = 0 \quad (x \leq 0).$$

2.2. We now introduce the Laplace transform

$$\kappa(s) = \int_0^\infty K(x) e^{-sx} dx \quad (\Re s > 0)$$

of $K(x)$. We call $\kappa(s)$ the *characteristic function* corresponding to the kernel $K(x-y)$, and we call the equation

$$\kappa(s) = 1 \tag{2.2.1}$$

the *characteristic equation*.

LEMMA 1. $\kappa(s)$ is a regular function of the complex variable s in the half-plane $\Re s > 0$.

The integral
$$\int_0^\infty e^{-sx} K(x) dx$$

is uniformly convergent in any half-plane $\Re s \geq \delta > 0$. The result follows at once.

We now consider the typical solutions of (2.1.1).

THEOREM 1. Let $\Re s > 0$. A necessary and sufficient condition that e^{sx} , xe^{sx} , ..., $x^{m-1}e^{sx}$ should all be solutions of (2.1.1) is that s is an m -fold root of (2.2.1), i.e.,

$$\kappa(s) = 1, \quad \kappa'(s) = \dots = \kappa^{(m-1)}(s) = 0.$$

Suppose that s is an m -fold root of (2.2.1). Then

$$\begin{aligned} \int_{-\infty}^x K(x-y) y^{\mu-1} e^{sy} dy &= \int_0^\infty K(z) (x-z)^{\mu-1} e^{s(x-z)} dz \\ &= e^{sx} \left\{ x^{\mu-1} \kappa(s) + \binom{\mu-1}{1} x^{\mu-2} \kappa'(s) + \dots + \kappa^{(\mu-1)}(s) \right\} \\ &= x^{\mu-1} e^{sx} \quad (\mu = 1, 2, \dots, m), \end{aligned}$$

i.e., $x^{\mu-1} e^{sx}$ is a solution of (2.1.1) ($\mu = 1, 2, \dots, m$).

Conversely, if $x^{m-1} e^{sx}$ is a solution of (2.1.1), we have

$$\begin{aligned} x^{m-1} e^{sx} &= \int_{-\infty}^x K(x-y) y^{m-1} e^{sy} dy \\ &= \int_0^\infty K(z) (x-z)^{m-1} e^{s(x-z)} dz, \end{aligned}$$

i.e.,
$$x^{m-1} = \int_0^\infty K(z) (x-z)^{m-1} e^{-sz} dz.$$

Equating coefficients of powers of x , we obtain

$$\int_0^\infty K(z) e^{-sz} dz = 1,$$

$$\int_0^\infty K(z) z^\mu e^{-sz} dz = 0 \quad (\mu = 1, 2, \dots, m-1);$$

$$\text{i.e.,} \quad \kappa(s) = 1, \quad \kappa'(s) = \kappa''(s) = \dots = \kappa^{(m-1)}(s) = 0.$$

2.3. We write $\Re s = \sigma$, $\Im s = t$, so that $s = \sigma + it$.

LEMMA 2. $\kappa(\sigma + it) \rightarrow 0$ when $\sigma \rightarrow +\infty$, uniformly in t .

Put

$$\left. \begin{aligned} \kappa_1(s) &= \int_0^\infty K_1(x) e^{-sx} dx, \\ \kappa_2(s) &= \int_0^\infty K_2(x) e^{-sx} dx; \end{aligned} \right\} \quad (2.3.1)$$

we then have $\kappa(s) = \kappa_1(s) + \kappa_2(s)$. First,

$$|\kappa_1(\sigma + it)| \leq A \int_0^\infty e^{-\sigma x} dx = \frac{A}{\sigma} \rightarrow 0 \quad (\sigma \rightarrow \infty)$$

uniformly in t , i.e., the lemma holds for $\kappa_1(s)$.

Let $\delta > 0$. Then

$$\begin{aligned} \kappa_2(\sigma + it) &= \int_0^\delta K_2(x) e^{-(\sigma + it)x} dx + \int_\delta^\infty K_2(x) e^{-(\sigma + it)x} dx \\ &= I_1 + I_2, \end{aligned}$$

say. We have

$$|I_1| \leq \int_0^\delta |K_2(x)| dx < \frac{1}{2}\epsilon$$

for sufficiently small δ . Finally,

$$|I_2| \leq \int_\delta^\infty |K_2(x)| e^{-\sigma x} dx \leq e^{-\sigma\delta} \int_0^\infty |K_2(x)| dx < \frac{1}{2}\epsilon$$

for $\sigma \geq \sigma_0(\delta, \epsilon) = \sigma_0(\epsilon)$. Hence

$$|\kappa_2(\sigma + it)| < \epsilon$$

for $\sigma \geq \sigma_0(\epsilon)$, i.e., $\kappa_2(s)$ also satisfies the lemma.

LEMMA 3. The function $\kappa(s)$ can be defined on $\Re s = 0$, $s \neq 0$, in such a way that it is continuous in $\Re s \geq 0$, $s \neq 0$, and $s\kappa(s)$ (appropriately defined at $s = 0$) is continuous in $\Re s \geq 0$.

Since $K_2(x) \in L(0, \infty)$, the function $\kappa_2(s)$ defined by (2.3.1) is continuous in $\Re s \geq 0$.

Next, when $\Re s > 0$,

$$\begin{aligned}\kappa_1(s) &= \left[-\frac{1}{s} K_1(x) e^{-sx} \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-sx} dK_1(x) \\ &= \frac{1}{s} \int_0^\infty e^{-sx} dK_1(x),\end{aligned}$$

since $K_1(x)$ is of bounded variation and $K_1(0) = 0$. We define, when $t \neq 0$,

$$\kappa_1(it) = \frac{1}{it} \int_0^\infty e^{-itx} dK_1(x).$$

This completes the definition in the required manner, since the integral

$$\int_0^\infty e^{-sx} dK_1(x)$$

is uniformly convergent in $\Re s \geq 0$.

LEMMA 4. $\kappa(\sigma + it) \rightarrow 0$ when $|t| \rightarrow \infty$, uniformly in $\sigma \geq 0$.

(i) We first consider $\kappa_1(s)$. We have

$$\begin{aligned}|\kappa_1(s)| &\leq \frac{1}{|s|} \int_0^\infty e^{-\sigma x} |dK_1(x)| \\ &\leq \frac{1}{|t|} \int_0^\infty |dK_1(x)| = \frac{V}{|t|},\end{aligned}$$

where V is the total variation of $K_1(x)$ in $(0, \infty)$. The lemma therefore holds for $\kappa_1(s)$.

(ii) Given $\epsilon (> 0)$, there exists a function $K^*(x)$, belonging to $L(0, \infty)$ and to $V(0, \infty)$, such that $K^*(0) = 0$ and

$$\int_0^\infty |K_2(x) - K^*(x)| dx < \frac{1}{2} \epsilon.$$

By (i) above, there is a number $T > 0$ such that

$$\left| \int_0^\infty K^*(x) e^{-(\sigma + it)x} dx \right| = \left| \frac{1}{\sigma + it} \int_0^\infty e^{-(\sigma + it)x} dK^*(x) \right| < \frac{1}{2} \epsilon$$

for all $\sigma \geq 0$ and all t such that $|t| > T$. Hence

$$\begin{aligned}\left| \int_0^\infty K_2(x) e^{-(\sigma + it)x} dx \right| &\leq \int_0^\infty |K_2(x) - K^*(x)| e^{-\sigma x} dx + \left| \int_0^\infty K^*(x) e^{-(\sigma + it)x} dx \right| \\ &< \epsilon\end{aligned}$$

whenever $\sigma \geq 0$, $|t| > T$, i.e. the lemma holds also for $\kappa_2(s)$.

2.4. We now make the following further assumptions:

Hypothesis 2. (i) No root of the characteristic equation $\kappa(s) = 1$ lies on $\Re s = 0$; (ii) there exist positive numbers ϵ and δ such that $|\kappa(s) - 1| > \epsilon$ for all s such that $0 < |s| < \delta$.

LEMMA 5. If $\kappa(s)$ satisfies Hypothesis 2, the characteristic equation has only a finite number of roots in $\Re s \geq 0$.

By Lemma 2, there is a positive number σ_0 such that $|\kappa(s)| < \frac{1}{2}$ when $\Re s > \sigma_0$. By Lemma 4, there is a positive number T such that $|\kappa(s)| < \frac{1}{2}$ when $|\Im s| > T$ and $0 \leq \Re s \leq \sigma_0$. Hence every root of the characteristic equation in $\Re s \geq 0$ must lie in the rectangle $0 \leq \Re s \leq \sigma_0$, $-T \leq \Im s \leq T$.

Suppose that the equation has an infinity of roots in this rectangle. The set consisting of these roots must then have a limit-point, say s_0 . If $\Re s_0 > 0$, $\kappa(s) - 1$ has a singularity at $s = s_0$; but, since $\kappa(s)$ is regular in $\Re s > 0$, no such singularity can exist. If $\Re s_0 = 0$, $s_0 \neq 0$, then $\kappa(s_0) = 1$, since $\kappa(s)$ is continuous at s_0 in $\Re s \geq 0$; this is contrary to Hypothesis 2, (i). If $s_0 = 0$, $\kappa(s)$ takes the value 1 arbitrarily near $s = 0$, contrary to Hypothesis 2, (ii). These three contradictions together give the required result.

2.5. THEOREM 2. Suppose that $K(x)$ satisfies Hypothesis 1 and that the characteristic function $\kappa(s)$ satisfies Hypothesis 2. Let $s_1, s_2, \dots, s_n$ be the distinct roots of the characteristic equation $\kappa(s) = 1$ in $\Re s > 0$, and let $a_1, a_2, \dots, a_n$ be their respective multiplicities.

If (i) $f(x) \in L(-\infty, X)$ for every X , and (ii)

$$f(x) = \int_{-\infty}^x K(x-y)f(y)dy \quad (-\infty < x < \infty), \quad (2.5.1)$$

then
$$f(x) = \sum_{\nu=1}^n P_{\nu}(x)e^{s_{\nu}x} \quad (-\infty < x < \infty), \quad (2.5.2)$$

where $P_{\nu}(x)$ is a polynomial of degree at most $a_{\nu}-1$ ($\nu = 1, 2, \dots, n$); conversely, any expression of the form (2.5.2) satisfies (i) and is a solution of (2.5.1).

The converse statement follows at once from Theorem 1.

Let $f(x)$ be a solution of (2.5.1) belonging to $L(-\infty, 0)$. Put

$$\left. \begin{aligned} h(x) &= f(x) & (x \leq 0), \\ &= 0 & (x > 0), \end{aligned} \right\}$$

and

$$g(x) = h(x) - \int_{-\infty}^x K(x-y)h(y)dy \quad (-\infty < x < \infty).$$

Then

$$\left. \begin{aligned} g(x) &= 0 & (x \leq 0), \\ &= -\int_{-\infty}^0 K(x-y)f(y)dy & (x > 0). \end{aligned} \right\}$$

When $x > 0$,

$$\begin{aligned} g(x) &= -\int_{-x}^0 K_1(x-y)f(y)dy - \int_{-x}^0 K_2(x-y)f(y)dy \\ &= g_1(x) + g_2(x), \end{aligned}$$

say. We put $g_1(0) = g_2(0) = 0$. Since $f(x) \in L(-\infty, 0)$ and

$$K_1(x) \in V(0, \infty),$$

direct application of the definition of the variation of a function shows at once that $g_1(x) \in V(0, \infty)$. Also $g_2(x) \in L(0, \infty)$; for

$$\begin{aligned} \int_0^\infty |g_2(x)|dx &\leq \int_0^\infty dx \int_{-x}^0 |K_2(x-y)| \cdot |f(y)|dy \\ &\leq \int_{-\infty}^0 |f(y)|dy \int_0^\infty |K_2(x)|dx < \infty. \end{aligned}$$

Consequently $g(x)$ satisfies Hypothesis 1.

We now define the functions

$$\chi(s) = \int_{-\infty}^\infty h(x)e^{-sx}dx = \int_{-\infty}^0 f(x)e^{-sx}dx,$$

and

$$\gamma(s) = \int_0^\infty g(x)e^{-sx}dx.$$

Then $\chi(s)$ exists and is continuous in $\Re s \leq 0$ and is regular in $\Re s < 0$, and, since $g(x)$ satisfies Hypothesis 1, $\gamma(s)$ (appropriately defined on $\Re s = 0$) exists and is continuous in $\Re s \geq 0$, $s \neq 0$, and is regular in $\Re s > 0$, and $s\gamma(s)$ is continuous in $\Re s \geq 0$.

2.6. We now write $g(x) = g_3(x) + g_4(x)$, where

$$\left. \begin{aligned} g_3(x) &= -\int_{-\infty}^\infty K_1(x-y)h(y)dy, \\ g_4(x) &= h(x) - \int_{-x}^\infty K_2(x-y)h(y)dy. \end{aligned} \right\}$$

If necessary, we change the definitions of these functions at $x = 0$ so as to make $g_3(0) = 0$. It is easily seen that

$$g_3(x) \in V(-\infty, \infty) \quad \text{and} \quad g_4(x) \in L(-\infty, \infty).$$

Furthermore, if $\Re s = 0$, $s \neq 0$, we have

$$\gamma(s) = \frac{1}{s} \int_{-\infty}^{\infty} e^{-sx} dg_3(x) + \int_{-\infty}^{\infty} e^{-sx} g_4(x) dx;$$

for the expression on the right-hand side is equal to

$$\left. \begin{aligned} & \frac{1}{s} \int_{-\infty}^0 e^{-sx} dg_3(x) + \frac{1}{s} \int_0^{\infty} e^{-sx} dg_3(x) \\ & + \int_{-\infty}^0 e^{-sx} g_4(x) dx + \int_0^{\infty} e^{-sx} g_4(x) dx. \end{aligned} \right\} \quad (2.6.1)$$

Now, when $x < 0$, $g_3(x) = -g_4(x)$. Hence $g_3(x) \in L(-\infty, 0)$, and

$$\frac{1}{s} \int_{-\infty}^0 e^{-sx} dg_3(x) = \int_{-\infty}^0 e^{-sx} g_3(x) dx = - \int_{-\infty}^0 e^{-sx} g_4(x) dx.$$

The expression (2.6.1) therefore reduces to

$$\frac{1}{s} \int_0^{\infty} e^{-sx} dg_3(x) + \int_0^{\infty} e^{-sx} g_4(x) dx,$$

which is equal to $\gamma(s)$; for, when $x \geq 0$, $g_1(x) = g_3(x)$ and $g_2(x) = g_4(x)$.

We now proceed to evaluate $\gamma(s)$ when $\Re s = 0$, $s \neq 0$. We have

$$\begin{aligned} \gamma(s) &= \frac{1}{s} \int_{-\infty}^{\infty} e^{-sx} dg_3(x) + \int_{-\infty}^{\infty} e^{-sx} g_4(x) dx \\ &= \gamma_3(s) + \gamma_4(s), \end{aligned}$$

say. First,

$$\begin{aligned} \gamma_3(s) &= \frac{1}{s} \int_{-\infty}^{\infty} e^{-sx} dg_3(x) \\ &= -\frac{1}{s} \int_{-\infty}^{\infty} e^{-sx} d \left\{ \int_{-\infty}^{\infty} K_1(x-y) h(y) dy \right\} \\ &= -\frac{1}{s} \int_{-\infty}^{\infty} h(y) dy \int_{-\infty}^{\infty} e^{-sx} d_x K_1(x-y) \\ &= -\frac{1}{s} \int_{-\infty}^{\infty} h(y) e^{-sy} dy \int_{-\infty}^{\infty} e^{-sx} dK_1(x) \\ &= -\chi(s) \kappa_1(s). \end{aligned}$$

Next,

$$\begin{aligned}
 \gamma_4(s) &= \int_{-\infty}^{\infty} g_4(x) e^{-sx} dx \\
 &= \int_{-\infty}^{\infty} e^{-sx} dx \left\{ h(x) - \int_{-\infty}^{\infty} K_2(x-y) h(y) dy \right\} \\
 &= \chi(s) - \int_{-\infty}^{\infty} e^{-sx} dx \int_{-\infty}^{\infty} K_2(x-y) h(y) dy \\
 &= \chi(s) - \int_{-\infty}^{\infty} h(y) dy \int_{-\infty}^{\infty} e^{-sx} K_2(x-y) dx \\
 &= \chi(s) - \int_{-\infty}^{\infty} h(y) e^{-sy} dy \int_{-\infty}^{\infty} e^{-sx} K_2(x) dx \\
 &= \chi(s)[1 - \kappa_2(s)].
 \end{aligned}$$

To justify the above inversions of the order of integration we need only remark that all the repeated integrals which appear are absolutely convergent, so that Fubini's theorem and its appropriate analogue for Lebesgue-Stieltjes integrals can be applied.

Combining the above results, we have, when $\Re s = 0$, $s \neq 0$,

$$\gamma(s) = \chi(s)[1 - \kappa(s)]. \quad (2.6.2)$$

2.7. The distinct roots of the characteristic equation in $\Re s > 0$ are $s_1, s_2, \dots, s_n$, and their respective multiplicities are $a_1, a_2, \dots, a_n$. Put

$$P(s) = (s - s_1)^{a_1} (s - s_2)^{a_2} \dots (s - s_n)^{a_n}.$$

Then $[1 - \kappa(s)]/P(s)$ is regular in $\Re s > 0$ and has no zeros in $\Re s \geq 0$. Hence $P(s)/[1 - \kappa(s)]$ is regular in $\Re s > 0$ and continuous in $\Re s \geq 0$, except perhaps at $s = 0$; by Hypothesis 2, it is in any case bounded in the neighbourhood of $s = 0$.

Put

$$\left. \begin{aligned}
 \psi(s) &= \frac{\gamma(s) P(s)}{1 - \kappa(s)} \quad (\Re s \geq 0, \ s \neq 0), \\
 &= \chi(s) P(s) \quad (\Re s \leq 0).
 \end{aligned} \right\}$$

Now $s^2 \chi(s) P(s)$ is regular in $\Re s < 0$ and continuous in $\Re s \leq 0$, and $s^2 \gamma(s) P(s)/[1 - \kappa(s)]$ is regular in $\Re s > 0$ and continuous in $\Re s \geq 0$, if we give it the value 0 at $s = 0$. Hence, by (2.6.2) and the principle stated in §1.2, these two functions are analytic continuations of one another; $s^2 \psi(s)$ is consequently an integral function. It follows that $\psi(s)$ is regular in the whole finite plane, except that it may have a pole of order not more

than 2 at $s = 0$; since, however, $\psi(s) = \chi(s)P(s)$ is continuous at $s = 0$ in $\Re s \leq 0$, the origin cannot be a genuine pole of $\psi(s)$; $\psi(s)$ is therefore an integral function.

It follows from Lemmas 2 and 4 that, when $|s| \rightarrow \infty$ in $\Re s \geq 0$, $\gamma(s) \rightarrow 0$ and $\kappa(s) \rightarrow 0$. Hence

$$\psi(s) = \frac{\gamma(s)P(s)}{1 - \kappa(s)} = o(|s|^N) \quad (|s| \rightarrow \infty, \Re s \geq 0),$$

where N is the degree of $P(s)$. Secondly, when $|s| \rightarrow \infty$ in $\Re s \leq 0$, $\chi(s) \rightarrow 0$, so that

$$\psi(s) = \chi(s)P(s) = o(|s|^N) \quad (|s| \rightarrow \infty, \Re s \leq 0).$$

Consequently

$$\psi(s) = o(|s|^N) \quad (|s| \rightarrow \infty);$$

it follows at once† that $\psi(s)$ is a polynomial of degree at most $N - 1$.

2.8. We can now resolve the rational function

$$\chi(s) = \frac{\psi(s)}{P(s)}$$

into partial fractions, say

$$\chi(s) = \sum_{\nu, \rho} \frac{A_{\nu\rho}}{(s - s_\nu)^\rho},$$

where $\{s_\nu\}$ ($\nu = 1, 2, \dots, n$) is the set of distinct roots of the characteristic equation, and, for each value of ν , ρ runs from 1 to a_ν , the multiplicity of s_ν .

When $\sigma \leq 0$, we have

$$\chi(\sigma + it) = \int_{-\infty}^{\infty} h(x) e^{-\sigma x} e^{-itx} dx,$$

so that, apart from a constant factor, $\chi(\sigma + it)$ is the Fourier transform of $h(x)e^{-\sigma x}$, which belongs to $L(-\infty, \infty)$. Put

$$\left. \begin{aligned} p_{\nu\rho}(x) &= x^{\rho-1} e^{s_\nu x} & (x \leq 0), \\ &= 0 & (x > 0). \end{aligned} \right\}$$

Then $p_{\nu\rho}(x)e^{-\sigma x} \in L(-\infty, \infty)$, and its Fourier transform is of the form $A(s - s_\nu)^{-\rho}$. We can therefore find a finite linear combination

$$\sum_{\nu, \rho} C_{\nu\rho} p_{\nu\rho}(x) e^{-\sigma x}$$

whose Fourier transform is $\chi(\sigma + it)$. We now have two functions which both belong to $L(-\infty, \infty)$ and which possess the same Fourier transform;

† Titchmarsh, 40, 85.

the functions must then be equal to each other almost everywhere†. Hence

$$h(x)e^{-\sigma x} = \sum_{\nu, \rho} C_{\nu\rho} x^{\rho-1} e^{s_\nu x} e^{-\sigma x} \quad (x \leq 0),$$

$$i.e., \quad f(x) = \sum_{\nu, \rho} C_{\nu\rho} x^{\rho-1} e^{s_\nu x} \quad (x \leq 0).$$

Since there is no special virtue in choosing the origin as the point of division in the definition of $h(x)$, we can replace it by $x = X$ for arbitrary X . Hence $f(x)$ must be of the above form in $(-\infty, X)$ for all X , *i.e.*,

$$f(x) = \sum_{\nu, \rho} C_{\nu\rho} x^{\rho-1} e^{s_\nu x} \quad (-\infty < x < \infty).$$

2.9. Let us now consider the equation

$$f(x) = \int_{-\infty}^x K(x-y)f(y)dy, \quad (2.9.1)$$

where $K(x)e^{s_0 x}$ satisfies Hypothesis 1 for some complex value of s_0 . Put

$$K(x)e^{s_0 x} = H(x), \quad f(x)e^{s_0 x} = \theta(x).$$

Then (2.9.1) becomes

$$\theta(x)e^{-s_0 x} = \int_{-\infty}^x H(x-y)e^{-s_0 x}\theta(y)dy,$$

$$i.e., \quad \theta(x) = \int_{-\infty}^x H(x-y)\theta(y)dy. \quad (2.9.2)$$

To equation (2.9.2) we can apply Theorem 2, and so obtain those solutions $\theta(x)$ which belong to $L(-\infty, X)$ for all X . The corresponding solutions $f(x)$ of (2.9.1) are such that $f(x)e^{s_0 x} \in L(-\infty, X)$ for all X , where $\sigma_0 = \Re s_0$.

2.10. We now consider some examples which illustrate the results of the present chapter.

(i) We begin with the equation

$$\phi(\xi) = \lambda \int_0^\xi \frac{\phi(\eta)}{\xi + \eta} d\eta \quad (\xi > 0). \quad (2.10.1)$$

Put $\xi = e^x$, $\eta = e^y$, and $\phi(e^x) = e^{-x}f(x)$. Then (2.10.1) becomes

$$e^{-x}f(x) = \lambda \int_{-\infty}^x \frac{f(y)}{e^x + e^y} dy \quad (-\infty < x < \infty),$$

$$i.e., \quad f(x) = \lambda \int_{-\infty}^x \frac{f(y)}{1 + e^{-(x-y)}} dy. \quad (2.10.2)$$

† Bochner, 6, 47.

Here $K(x) = \lambda(1+e^{-x})^{-1} \in V(0, \infty)$, so that Hypothesis 1 is satisfied. We now compute the characteristic function. We have

$$\begin{aligned}\kappa(s) &= \lambda \int_0^\infty \frac{e^{-sx}}{1+e^{-x}} dx \\ &= \lambda \int_0^1 \frac{\xi^{s-1}}{1+\xi} d\xi \\ &= \frac{1}{2}\lambda[\psi(\tfrac{1}{2} + \tfrac{1}{2}s) - \psi(\tfrac{1}{2}s)]^\dagger,\end{aligned}$$

where

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

The characteristic equation $\kappa(s) = 1$ therefore reduces to

$$\psi(\tfrac{1}{2} + \tfrac{1}{2}s) - \psi(\tfrac{1}{2}s) = 2\lambda^{-1}. \quad (2.10.3)$$

Now
$$\int_0^\infty \frac{e^{-sx}}{1+e^{-x}} dx = \frac{1}{s} - \int_0^\infty \frac{e^{-sx}}{e^x+1} dx.$$

Thus, when $\Re s = 0$, $s \neq 0$, we have

$$\Re \left\{ \int_0^\infty \frac{e^{-sx}}{1+e^{-x}} dx \right\} = - \int_0^\infty \frac{\cos tx}{e^x+1} dx.$$

It can easily be shown that the right-hand side of the last equation is negative for all values of t . Hence, if $\Re \lambda > 0$, the characteristic equation can have no roots on $\Re s = 0$, so that Hypothesis 2 is satisfied. For every value of λ in the half-plane $\Re \lambda > 0$, (2.10.3) has at least one root s lying in $\Re s > 0$, and there is a corresponding solution $f(x) = Ae^{sx}$ of (2.10.2). The resulting solution of (2.10.1) is $\phi(\xi) = A\xi^{s-1}$. By Theorem 2, the solutions obtained in this way are the only ones for which $f(x) \in L(-\infty, X)$ for all X , i.e., $\phi(\xi) \in L(0, \Xi)$ for all $\Xi > 0$.

(ii) Now let us consider, for $\alpha > 0$, the equation[‡]

$$\phi(\xi) = \frac{\lambda}{\Gamma(\alpha)} \xi^{-\alpha} \int_0^\xi (\xi - \eta)^{\alpha-1} \phi(\eta) d\eta \quad (\xi > 0). \quad (2.10.4)$$

This may be regarded as a linear differential equation of (possibly fractional) order α , viz.,

$$\frac{d^\alpha}{d\xi^\alpha} [\xi^\alpha \phi(\xi)] = \lambda \phi(\xi).$$

[†] Whittaker and Watson, **45**, 262, Example 30.

[‡] This equation is mentioned by Titchmarsh, **41**, 310,

If we put $\xi = e^x$, $\eta = e^y$, $e^x \phi(e^x) = f(x)$, the equation becomes

$$f(x) = \frac{\lambda}{\Gamma(\alpha)} \int_{-\infty}^x [1 - e^{-(x-y)}]^{a-1} f(y) dy \quad (-\infty < x < \infty). \quad (2.10.5)$$

Here

$$K(x) = \frac{\lambda}{\Gamma(\alpha)} (1 - e^{-x})^{a-1},$$

which satisfies Hypothesis 1 for all $\alpha > 0$. Next,

$$\begin{aligned} \kappa(s) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^\infty e^{-sx} (1 - e^{-x})^{a-1} dx \\ &= \frac{\lambda}{\Gamma(\alpha)} \cdot \frac{\Gamma(s) \Gamma(\alpha)}{\Gamma(s+\alpha)} = \frac{\lambda \Gamma(s)}{\Gamma(s+\alpha)}. \end{aligned}$$

For all real $\lambda > 0$, the characteristic equation $\kappa(s) = 1$ has at least one real root in $\Re s > 0$. To each such root corresponds a solution $f(x) = A e^{sx}$ of (2.10.5) and a solution $\phi(\xi) = A \xi^{s-1}$ of (2.10.4). We do not go into further details here. It can easily be seen that, even for real λ , there are real values of α for which Hypothesis 2 (i) is not satisfied.

2.11. As an example of what may happen when Hypothesis 2 (ii) is false, let us take

$$K(x) = x^{-\frac{1}{2}} (1 - e^{-x}) \cos(\log x).$$

Then $K(x) \in V(0, \infty)$, so that Hypothesis 1 holds. We have

$$\begin{aligned} \kappa(s) &= \int_0^\infty x^{-\frac{1}{2}} (1 - e^{-x}) e^{-sx} \cos(\log x) dx \\ &= \int_0^\infty x^{-\frac{1}{2}} e^{-sx} \cos(\log x) dx - \int_0^\infty x^{-\frac{1}{2}} e^{-(s+1)x} \cos(\log x) dx \\ &= s^{-\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} e^{-x} \cos\left(\log \frac{x}{s}\right) dx - (s+1)^{-\frac{1}{2}} \int_0^\infty x^{-\frac{1}{2}} e^{-x} \cos\left(\log \frac{x}{s+1}\right) dx \\ &= s^{-\frac{1}{2}} [A \cos(\log s) + B \sin(\log s)] \\ &\quad + (s+1)^{-\frac{1}{2}} [D \cos \log(s+1) + E \sin \log(s+1)]. \end{aligned}$$

The constants A and B are respectively the real and imaginary parts of a value of the Γ -function, and so do not both vanish; consequently, when $s \rightarrow +0$, $\kappa(s) - 1$ oscillates between $+\infty$ and $-\infty$, and therefore has an infinity of zeros. Hence in this case the homogeneous integral equation with the kernel $K(x-y)$ has an infinity of linearly independent solutions

of the form Ae^{sx} ; further solutions may be obtained by forming infinite linear combinations.

3. Fredholm equations of Type 1.

3.1. We now consider equations of the form

$$f(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy \quad (-\infty < x < \infty), \quad (3.1.1)$$

where $K(x)$ is subjected to the following conditions:

Hypothesis 3. $K(x)$ can be expressed in the form

$$K(x) = K_1(x) + K_2(x) \quad (-\infty < x < \infty),$$

where $K_1(x)e^{|x|} \in V(-\infty, \infty)$ and $K_2(x)e^{|x|} \in L(-\infty, \infty)$.

For convenience, we also suppose that $K_1(0) = 0$.

The characteristic function $\kappa(s)$ is defined in $-1 < \Re s < 1$ by the equation

$$\kappa(s) = \int_{-\infty}^{\infty} K(x)e^{-sx}dx,$$

and the characteristic equation is, as before,

$$\kappa(s) = 1. \quad (3.1.2)$$

LEMMA 6. $\kappa(s)$ is a regular function of s in the strip $-1 < \Re s < 1$.

The proof is similar to that of Lemma 1 (§2.2).

THEOREM 3. Let $-1 < \Re s < 1$. A necessary and sufficient condition that e^{sx} , xe^{sx} , ..., $x^{m-1}e^{sx}$ should all be solutions of (3.1.1) is that s is an m -fold root of (3.1.2).

This is proved in exactly the same way as Theorem 1 (§2.2).

3.2. LEMMA 7. The function $\kappa(s)$ can be defined on the lines $\Re s = \pm 1$, $s \neq \pm 1$, in such a way that it is continuous in $-1 \leq \Re s \leq 1$, $s \neq \pm 1$, and $(s^2 - 1)\kappa(s)$, appropriately defined at $s = \pm 1$, is continuous in $-1 \leq \Re s \leq 1$.

We subdivide the expression defining $\kappa(s)$ as follows:

$$\begin{aligned} \kappa(s) &= \int_{-\infty}^0 K_1(x)e^{-sx}dx + \int_0^{\infty} K_1(x)e^{-sx}dx + \int_{-\infty}^{\infty} K_2(x)e^{-sx}dx \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. The integral I_3 is uniformly convergent in $-1 \leq \Re s \leq 1$. As in the proof of Lemma 3 (§2.3) we can show that

$$\left. \begin{aligned} (s-1) I_1 &= \int_{-\infty}^0 e^{(1-s)x} d[K_1(x) e^{-x}], \\ (s+1) I_2 &= \int_0^{\infty} e^{-(1+s)x} d[K_1(x) e^x]. \end{aligned} \right\}$$

Then the expression for $(s-1) I_1$ is uniformly convergent in $\Re s \leq 1$, and that for $(s+1) I_2$ is uniformly convergent in $\Re s \geq -1$. We can therefore complete the definition in the required manner.

LEMMA 8. $\kappa(\sigma+it) \rightarrow 0$ when $|t| \rightarrow \infty$, uniformly in $-1 \leq \sigma \leq 1$.

The proof is similar to that of Lemma 4 (§2.3).

3.3. We now make the following further assumptions:

Hypothesis 4. (i) No root of the characteristic equation $\kappa(s) = 1$ lies on either of the lines $\Re s = \pm 1$; (ii) there exist positive numbers ϵ and δ such that $|\kappa(s) - 1| > \epsilon$ for all s for which $0 < |s+1| < \delta$ or $0 < |s-1| < \delta$.

LEMMA 9. If $\kappa(s)$ satisfies Hypothesis 4, the characteristic equation has only a finite number of roots in $-1 \leq \Re s \leq 1$.

This is proved in the same way as Lemma 5 (§2.4).

3.4. THEOREM 4. Suppose that $K(x)$ satisfies Hypothesis 3, and that the characteristic function $\kappa(s)$ satisfies Hypothesis 4. Let $s_1, s_2, \dots, s_n$ be the distinct roots in $-1 < \Re s < 1$ of the characteristic equation $\kappa(s) = 1$, and let $a_1, a_2, \dots, a_n$ be their respective multiplicities.

If (i) $f(x) e^{-|x|} \in L(-\infty, \infty)$, and (ii)

$$f(x) = \int_{-\infty}^{\infty} K(x-y) f(y) dy, \quad (3.4.1)$$

$$\text{then} \quad f(x) = \sum_{\nu=1}^n P_{\nu}(x) e^{s_{\nu} x}, \quad (3.4.2)$$

where $P_{\nu}(x)$ is a polynomial of degree at most $a_{\nu}-1$; conversely, any expression of the form (3.4.2) satisfies (i) and is a solution of (3.4.1).

The converse statement follows at once from Theorem 3.

We only sketch the main lines of the proof, for in detail it is similar to that of Theorem 2 (§§2.5-2.8).

Let $f(x)$ be a solution of (3.4.1) such that $f(x)e^{-|x|} \in L(-\infty, \infty)$. Put

$$\begin{aligned} f_+(x) &= f(x) \quad (x \geq 0), \\ &= 0 \quad (x < 0), \\ f_-(x) &= f(x) - f_+(x), \\ g(x) &= f_+(x) - \int_{-\infty}^{\infty} K(x-y)f_+(y)dy. \end{aligned} \quad (3.4.3)$$

From (3.4.1) we obtain

$$\begin{aligned} g(x) &= - \int_0^{\infty} K(x-y)f(y)dy \quad (x < 0), \\ &= \int_{-\infty}^0 K(x-y)f(y)dy \quad (x \geq 0). \end{aligned}$$

Similarly

$$\begin{aligned} f_-(x) - \int_{-\infty}^{\infty} K(x-y)f_-(y)dy &= \int_0^{\infty} K(x-y)f(y)dy \quad (x < 0), \\ &= - \int_{-\infty}^0 K(x-y)f(y)dy \quad (x \geq 0). \end{aligned}$$

$$\text{Hence} \quad -g(x) = f_-(x) - \int_{-\infty}^0 K(x-y)f_-(y)dy. \quad (3.4.4)$$

It is easily seen that $g(x)$ satisfies Hypothesis 3.

We now define the functions

$$\begin{aligned} \phi_+(s) &= \int_{-\infty}^{\infty} f_+(x)e^{-sx}dx = \int_0^{\infty} f(x)e^{-sx}dx, \\ \phi_-(s) &= \int_{-\infty}^{\infty} f_-(x)e^{-sx}dx = \int_{-\infty}^0 f(x)e^{-sx}dx, \\ \gamma(s) &= \int_{-\infty}^{\infty} g(x)e^{-sx}dx. \end{aligned}$$

Then $\phi_+(s)$ exists and is continuous in $\Re s \geq 1$ and is regular in $\Re s > 1$; $\phi_-(s)$ exists and is continuous in $\Re s \leq -1$ and is regular in $\Re s < -1$; and, since $g(x)$ satisfies Hypothesis 3, $\gamma(s)$ (appropriately defined on $\Re s = \pm 1$) exists and is continuous in $-1 \leq \Re s \leq 1$, $s \neq \pm 1$, and is regular in $-1 < \Re s < 1$, and $(s^2-1)\gamma(s)$ is continuous in $-1 \leq \Re s \leq 1$.

Using equations (3.4.3) and (3.4.4), we can now show by the method of §2.6 that

$$\begin{aligned} \gamma(s) &= \phi_+(s)[1-\kappa(s)] \quad (\Re s = 1, s \neq 1), \\ -\gamma(s) &= \phi_-(s)[1-\kappa(s)] \quad (\Re s = -1, s \neq -1). \end{aligned}$$

We now put

$$P(s) = \prod_{\nu=1}^n (s-s_{\nu})^{a_{\nu}},$$

$$\left. \begin{aligned} \psi(s) &= \phi_+(s) & (\Re s \geq 1), \\ &= \frac{\gamma(s)}{1-\kappa(s)} & (-1 < \Re s < 1), \\ &= -\phi_-(s) & (\Re s \leq -1). \end{aligned} \right\}$$

We now show first that $P(s)\psi(s)$ is an integral function, and then that it is a polynomial. From this point on the proof goes precisely as in §§ 2.7-2.8.

3.5. We now consider the equation

$$f(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy, \quad (3.5.1)$$

where $K(x)$ satisfies the following condition:

For some complex s_0 and some $\alpha > 0$ we can write $K(x)$ as a sum,

$$K(x) = K_1(x) + K_2(x),$$

where $K_1(x)e^{s_0x+\alpha|x|} \in V(-\infty, \infty)$, $K_2(x)e^{s_0x+\alpha|x|} \in L(-\infty, \infty)$,

and $K_1(0) = 0$.

If we put $ax = \xi$, $ay = \eta$, $e^{s_0x}K(x) = \alpha H(ax)$, $e^{s_0x}f(x) = \theta(ax)$, equation (3.5.1) becomes

$$\theta(\xi) = \int_{-\infty}^{\infty} H(\xi-\eta)\theta(\eta)d\eta. \quad (3.5.2)$$

The function $H(x)$ satisfies Hypothesis 3, so we can now apply all the methods of this chapter to equation (3.5.2).

3.6. Our first example is the equation

$$\phi(\xi) = \lambda \int_0^{\infty} \frac{\phi(\eta)}{\xi+\eta} d\eta \quad (0 < \xi < \infty). \quad (3.6.1)$$

This expresses that the function $\phi(\xi)$ is an eigen-function of the Stieltjes transform. If we put $\xi = e^{2x}$, $\eta = e^{2y}$, $e^x\phi(e^{2x}) = f(x)$, the equation becomes

$$f(x) = \lambda \int_{-\infty}^{\infty} \frac{f(y)}{\cosh(x-y)} dy \quad (-\infty < x < \infty). \quad (3.6.2)$$

Hence $K(x) = \lambda \operatorname{sech} x$, and Hypothesis 3 clearly holds. Next,

$$\kappa(s) = \lambda \int_{-\infty}^{\infty} \frac{e^{-sx}}{\cosh x} dx = \frac{\pi\lambda}{\cos \frac{1}{2}\pi s},$$

as can easily be verified by contour integration. The characteristic equation is therefore

$$\cos \frac{1}{2}\pi s = \pi\lambda. \quad (3.6.3)$$

Provided that $\Re\lambda \neq 0$, Hypothesis 4 holds; hence, by Theorem 4, the roots of the characteristic equation give all solutions of (3.6.2) such that $f(x)e^{-|x|} \in L(-\infty, \infty)$, and therefore all solutions of (3.6.1) such that

$$\frac{\phi(\xi)}{1+\xi} \in L(0, \infty). \quad (3.6.4)$$

When $\Re\lambda > 0$, (3.6.3) has exactly 2 roots in $-1 < \Re s < 1$; these coincide if and only if $\lambda = 1/\pi$. When $\Re\lambda < 0$, (3.6.3) has no roots in the strip.

Hence (3.6.1) has two linearly independent solutions or no non-trivial solutions satisfying (3.6.4) according as $\Re\lambda > 0$ or $\Re\lambda < 0$; if $\lambda\pi = 1$, the general solution is

$$\phi(\xi) = A\xi^{-1} + B\xi^{-1} \log \xi.$$

Equation (3.6.1) is discussed by Hardy and Titchmarsh, 15; see also Titchmarsh, 41, 310.

3.7. We next discuss Picard's equation†, viz.,

$$f(x) = \lambda \int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy \quad (-\infty < x < \infty). \quad (3.7.1)$$

Here

$$K(x) = e^{-|x|},$$

$$\kappa(s) = \frac{2\lambda}{1-s^2} \quad (-1 < \Re s < 1).$$

The characteristic equation is

$$s^2 = 1 - 2\lambda.$$

Hypothesis 4 is satisfied if λ does not lie on the parabola $2\Re\lambda = [\Im\lambda]^2$; according as λ lies outside or inside the parabola, the number of roots of the characteristic equation in $-1 < \Re s < 1$ is 0 or 2. In the first case,

† Picard, 38, 39.

therefore, equation (3.7.1) has no non-zero solution such that

$$e^{-|x|}f(x) \in L(-\infty, \infty);$$

in the second case there are two linearly independent solutions. If $\lambda \neq \frac{1}{2}$, the general solution is

$$f(x) = Ae^{\sqrt{(1-2\lambda)}x} + Be^{-\sqrt{(1-2\lambda)}x};$$

if $\lambda = \frac{1}{2}$, it is

$$f(x) = A + Bx;$$

in each case A and B are arbitrary constants.

4. Wiener-Hopf equations of Type 1.

4.1. In the present chapter we consider equations of the form

$$f(x) = \int_0^\infty K(x-y)f(y)dy \quad (x \geq 0), \quad (4.1.1)$$

where $K(x)$ satisfies Hypothesis 3 (§3.1).

The characteristic function $\kappa(s)$ is defined by the equation

$$\kappa(s) = \int_{-\infty}^\infty K(x)e^{-sx}dx \quad (-1 < \Re s < 1),$$

and the characteristic equation is

$$\kappa(s) = 1. \quad (4.1.2)$$

The solutions of equation (4.1.1) are not, in general, linear combinations of exponential functions with polynomial coefficients; nevertheless, the roots of the characteristic equation continue to play an important part.

Lemmas 6, 7, 8, and 9, together with their proofs, can be carried over without alteration from §§3.1-3.3.

We need in this chapter a new restrictive assumption, namely

$$\text{HYPOTHESIS 5.} \quad \int_{-\infty}^\infty \frac{|\kappa(\sigma+it)|}{1+|t|} dt < \infty \quad (-1 < \sigma < 1).$$

This is quite a mild restriction, since we already know from Lemma 8 (§3.2) that $\kappa(\sigma+it) \rightarrow 0$ when $|t| \rightarrow \infty$, uniformly in $-1 \leq \sigma \leq 1$.

4.2. THEOREM 5. *If $K(x)$ satisfies Hypothesis 3 (§3.1), and the characteristic function $\kappa(s)$ satisfies Hypotheses 4 (§3.3) and 5, then equation (4.1.1) has only a finite number of linearly independent solutions $f(x)$ such that $f(x)e^{-x} \in L(0, \infty)$.*

Let $f(x)$ be a solution of (4.1.1) such that $f(x)e^{-x} \in L(0, \infty)$. Write

$$\begin{aligned} f(x) &= 0 \quad (x < 0), \\ g(x) &= f(x) - \int_{-\infty}^{\infty} K(x-y)f(y)dy. \end{aligned} \quad (4.2.1)$$

Then

$$\left. \begin{aligned} g(x) &= 0 & (x \geq 0), \\ &= -\int_0^{\infty} K(x-y)f(y)dy & (x < 0). \end{aligned} \right\}$$

It can easily be shown that

$$g(x) = g_1(x) + g_2(x),$$

where $g_1(x)e^{-x} \in V(-\infty, 0)$, and $g_2(x)e^{-x} \in L(-\infty, 0)$.

We now define the functions

$$\begin{aligned} \phi(s) &= \int_{-\infty}^{\infty} f(x)e^{-sx}dx, \\ \gamma(s) &= \int_{-\infty}^{\infty} g(x)e^{-sx}dx. \end{aligned}$$

$\phi(s)$ exists and is continuous in $\Re s \geq 1$, and is regular in $\Re s > 1$; $\gamma(s)$ (appropriately defined on $\Re s = 1$), exists and is continuous in $\Re s \leq 1$, $s \neq 1$, and is regular in $\Re s < 1$, and $(s-1)\gamma(s)$ is continuous in $\Re s \leq 1$.

Using equation (4.2.1), we can now show by the methods of §2.6 that

$$\gamma(s) = \phi(s)[1 - \kappa(s)] \quad (\Re s = 1, s \neq 1). \quad (4.2.2)$$

4.3. Let $s_1, s_2, \dots, s_N$ be the roots of the characteristic equation in $-1 < \Re s < 1$, each multiple root being repeated according to its multiplicity, and put

$$P(s) = \prod_{\nu=1}^N (s - s_{\nu}).$$

Define $\chi(s)$ by the equations

$$\left. \begin{aligned} \chi(s) &= \frac{\gamma(s)P(s)}{1 - \kappa(s)} \quad (-1 < \Re s \leq 1, s \neq 1), \\ &= \phi(s)P(s) \quad (\Re s \geq 1). \end{aligned} \right\} \quad (4.3.1)$$

Then, applying the arguments of §2.7 to equation (4.2.2), we see that $\chi(s)$ is regular in the half-plane $\Re s > -1$.

4.4. Now put

$$\tau(s) = \frac{1 - \kappa(s)}{P(s)} (s+1)^m (s-1)^{N-m} \quad (-1 < \Re s < 1), \quad (4.4.1)$$

where m is an integer still to be chosen. When $|t| \rightarrow \infty$, $\kappa(it) \rightarrow 0$, by Lemma 8 (§ 3.2); hence $\tau(it) \rightarrow 1$. The total increment of $\arg \tau(it)$ when t runs from $-\infty$ to $+\infty$ is therefore of the form $2k\pi$, for some integer k . By an appropriate choice of m , we can make $k=0$; for the total increment of $\arg \left(\frac{it+1}{it-1} \right)$ in $(-\infty, \infty)$ is exactly 2π .

When the above choice of m has been made, the branch of $\log \tau(s)$ which approaches zero when $t \rightarrow \infty$ also approaches zero when $t \rightarrow -\infty$; and $\log \tau(s)$ will be a regular single-valued function of s in $-1 < \Re s < 1$.

We can now choose β ($0 < \beta < 1$), such that all the roots of the characteristic equation lie in the strip $-\beta < \Re s < \beta$. We then define $\tau_1(s)$, $\tau_2(s)$ by the equations

$$\left. \begin{aligned} \tau_1(s) &= \exp \left\{ -\frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \frac{\log \tau(\zeta)}{\zeta-s} d\zeta \right\} & (\Re s > -\beta), \\ \tau_2(s) &= \exp \left\{ -\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\log \tau(\zeta)}{\zeta-s} d\zeta \right\} & (\Re s < \beta). \end{aligned} \right\} \quad (4.4.2)$$

By Hypothesis 5, the two integrals in (4.4.2) are absolutely convergent. Since $\log \tau(\sigma+it) \rightarrow 0$ when $|t| \rightarrow \infty$, uniformly in $-\beta \leq \sigma \leq \beta$, it follows from Cauchy's formula that

$$\log \tau(s) = \log \tau_1(s) - \log \tau_2(s),$$

$$i.e., \quad \tau(s) = \frac{\tau_1(s)}{\tau_2(s)} \quad (-\beta < \Re s < \beta).$$

An application of Cauchy's theorem shows that, so long as β satisfies the conditions stated in its definition, the functions $\tau_1(s)$ and $\tau_2(s)$ are independent of β . Hence $\tau_1(s)$ is regular in the half-plane $\Re s > -1$, $\tau_2(s)$ is regular in the half-plane $\Re s < 1$, and

$$\tau(s) = \frac{\tau_1(s)}{\tau_2(s)} \quad (-1 < \Re s < 1).$$

Furthermore,

$$\left. \begin{aligned} \tau_1(\sigma+it) &\rightarrow 1 & (\sigma \rightarrow +\infty), \\ \tau_2(\sigma+it) &\rightarrow 1 & (\sigma \rightarrow -\infty). \end{aligned} \right\}$$

4.5. We now put

$$\left. \begin{aligned} \rho_1(s) &= (s+1)^{-m} \tau_1(s) & (\Re s > -1), \\ \rho_2(s) &= (s-1)^{N-m} \tau_2(s) & (\Re s < 1). \end{aligned} \right\}$$

It then follows from (4.4.1) and (4.4.2) that

$$1 - \kappa(s) = P(s) \frac{\rho_1(s)}{\rho_2(s)} \quad (-1 < \Re s < 1). \quad (4.5.1)$$

Hence, by (4.3.1),

$$\chi(s) \rho_1(s) = \gamma(s) \rho_2(s) \quad (-1 < \Re s < 1). \quad (4.5.2)$$

Now $\chi(s) \rho_1(s)$ is regular in $\Re s > -1$, and $\gamma(s) \rho_2(s)$ is regular in $\Re s < 1$; the two sides of (4.5.2) therefore define an integral function $\psi(s)$. It is easily seen that

$$\psi(s) = o(|s|^{N-m}) \quad (|s| \rightarrow \infty);$$

$\psi(s)$ is consequently a polynomial of degree at most $N-m-1$.

When $\Re s \geq 1$, we have

$$\psi(s) = \phi(s) P(s) \rho_1(s),$$

$$\text{i.e.,} \quad \phi(s) = \frac{\psi(s)}{P(s) \rho_1(s)}. \quad (4.5.3)$$

We have now shown that, if $f(x)$ is a solution of (4.1.1) such that

$$f(x) e^{-x} \in L(0, \infty),$$

and

$$\phi(s) = \int_0^\infty e^{-sx} f(x) dx \quad (\Re s \geq 1),$$

then $\phi(s)$ is given by equation (4.5.3), where $\psi(s)$ is a polynomial of degree at most $N-m-1$. Since $f(x)$ is uniquely determined by $\phi(s)$, it follows that the number of linearly independent solutions of (4.1.1) such that $f(x) e^{-x} \in L(0, \infty)$ is at most $N-m$. We have therefore proved Theorem 5, and we have also obtained the formula (4.5.3) for the solutions of (4.1.1).

4.6. THEOREM 6. *If the functions $P(s)$ and $\rho_1(s)$ and the numbers N, m, β are defined as in the preceding sections, and $\psi(s)$ is an arbitrary polynomial of degree at most $N-m-1$, then to the function $\phi(s)$ defined by the equation*

$$\phi(s) = \frac{\psi(s)}{P(s) \rho_1(s)} \quad (\Re s \geq \beta), \quad (4.6.1)$$

there corresponds a solution $f(x)$ of equation (4.1.1) such that

$$f(x) e^{-x} \in L(0, \infty).$$

Conversely, every such solution of (4.1.1) is given by (4.6.1).

The converse has already been proved.

We first remark that $\phi(s)$ is regular in $\Re s \geq \beta$, and that $\phi(s) = O(|s|^{-1})$ as $|s| \rightarrow \infty$; it follows that, for all $\sigma \geq \beta$, $\phi(\sigma + it) \in L^2(-\infty, \infty)$. We can therefore define a function $f(x)$ such that, for all $\sigma \geq \beta$, $f(x)e^{-\sigma x} \in L^2(-\infty, \infty)$, by the equation

$$f(x)e^{-\sigma x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\sigma + it) e^{itx} dt, \quad (4.6.2)$$

the integral being interpreted in the mean-square sense if necessary. An application of Cauchy's theorem shows that $f(x)$ is independent of σ .

By the Parseval formula,

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{-2\sigma x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(\sigma + it)|^2 dt. \quad (4.6.3)$$

Since $|\phi(s)| \leq C|s|^{-1}$ in $\Re s \geq \beta$, the right-hand side of (4.6.3) remains bounded when $\sigma \rightarrow \infty$. It follows that $f(x) = 0$ almost everywhere in $(-\infty, 0)$; otherwise the left-hand side of (4.6.3) would increase indefinitely when $\sigma \rightarrow \infty$, a contradiction.

Choose α such that $\beta < \alpha < 1$, and define $\gamma(s)$ by the equation

$$\gamma(s) = \frac{\psi(s)}{\rho_2(s)} \quad (\Re s \leq \alpha). \quad (4.6.4)$$

Then $\gamma(s)$ is regular in $\Re s \leq \alpha$, and $\gamma(s) = O(|s|^{-1})$ as $|s| \rightarrow \infty$; hence, for all $\sigma \leq \alpha$, $\gamma(\sigma + it) \in L^2(-\infty, \infty)$. We can therefore define a function $g(x)$ such that, for all $\sigma \leq \alpha$, $g(x)e^{-\sigma x} \in L^2(-\infty, \infty)$, by the equation

$$g(x)e^{-\sigma x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\sigma + it) e^{itx} dx, \quad (4.6.5)$$

the integral being interpreted in the mean-square sense if necessary. By Cauchy's theorem, $g(x)$ does not depend on σ . We use the Parseval formula again to show that $g(x) = 0$ for almost all x in $(0, \infty)$.

Now choose σ_0 such that $\beta < \sigma_0 < \alpha$. Then $K(x)e^{-\sigma_0 x} \in L(-\infty, \infty)$. Furthermore $f(x)e^{-\sigma_0 x} \in L(-\infty, \infty)$; for

$$\begin{aligned} \left\{ \int_{-\infty}^{\infty} |f(x)| e^{-\sigma_0 x} dx \right\}^2 &= \left\{ \int_0^{\infty} |f(x)| e^{-\beta x} e^{-(\sigma_0 - \beta)x} dx \right\}^2 \\ &\leq \int_0^{\infty} |f(x)|^2 e^{-2\beta x} dx \int_0^{\infty} e^{-2(\sigma_0 - \beta)x} dx < \infty. \end{aligned}$$

Similarly

$$g(x)e^{-\sigma_0 x} \in L(-\infty, \infty).$$

It now follows from the standard Faltung theorems that

$$e^{-\sigma_0 x} \int_0^{\infty} K(x-y)f(y) dy \in L(-\infty, \infty),$$

and we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-itx} dx \int_0^{\infty} e^{-\sigma_0 x} K(x-y) f(y) dy &= \int_{-\infty}^{\infty} K(x) e^{-(\sigma_0+it)x} dx \int_0^{\infty} f(y) e^{-(\sigma_0+it)y} dy \\ &= \kappa(\sigma_0+it) \phi(\sigma_0+it). \end{aligned}$$

By (4.5.1), (4.6.1), and (4.6.4),

$$\begin{aligned} \kappa(s) \phi(s) &= \phi(s) - [1 - \kappa(s)] \phi(s) \\ &= \phi(s) - P(s) \frac{\rho_1(s)}{\rho_2(s)} \frac{\psi(s)}{P(s) \rho_1(s)} \\ &= \phi(s) - \gamma(s) \quad (\beta \leq \Re s \leq \alpha). \end{aligned}$$

Hence

$$\kappa(\sigma_0+it) \phi(\sigma_0+it) = \phi(\sigma_0+it) - \gamma(\sigma_0+it).$$

We now have two functions of x , both in $L(-\infty, \infty)$, which have the same Fourier transform, namely

$$e^{-\sigma_0 x} \int_0^{\infty} K(x-y) f(y) dy, \quad [f(x) - g(x)] e^{-\sigma_0 x}.$$

They must therefore be identical†, i.e.,

$$f(x) = g(x) + \int_0^{\infty} K(x-y) f(y) dy.$$

When $x \geq 0$, this reduces to

$$f(x) = \int_0^{\infty} K(x-y) f(y) dy,$$

i.e., $f(x)$ satisfies the given integral equation (4.1.1). The proof of Theorem 6 is now complete.

The remarks of § 3.5 apply with obvious modifications to the work of the present chapter.

4.7. We now apply our results to the equation

$$f(x) = \lambda \int_0^{\infty} e^{-|x-y|} f(y) dy. \quad (4.7.1)$$

This is known as the equation of Lalesco‡. We have

$$K(x) = \lambda e^{-|x|};$$

† Bochner, 6, 47.

‡ Lalesco, 32, 121-123.

hence

$$\kappa(s) = \frac{2\lambda}{1-s^2}.$$

The characteristic equation is therefore

$$s^2 + 2\lambda - 1 = 0,$$

and its roots are $\pm\sqrt{1-2\lambda}$. If $[\Im\lambda]^2 < 2\Re\lambda$, both these roots lie in the strip $-1 < \Re s < 1$; in this case $N = 2$ and $P(s) = s^2 + 2\lambda - 1$. If $[\Im\lambda]^2 > 2\Re\lambda$, the roots lie outside the strip, $N = 0$ and $P(s) = 1$; in this case there are no non-trivial solutions of (4.7.1) satisfying our conditions. If $[\Im\lambda]^2 = 2\Re\lambda$, Hypothesis 4 (§3.3) is not satisfied, and Theorem 6 is inapplicable.

From this point on we confine ourselves to the case when $[\Im\lambda]^2 < 2\Re\lambda$. Then

$$\begin{aligned}\tau(s) &= \frac{1-\kappa(s)}{P(s)} (s+1)^m (s-1)^{N-m} \\ &= \frac{(s+1)^m (s-1)^{2-m}}{s^2-1} \\ &= (s+1)^{m-1} (s-1)^{1-m}.\end{aligned}$$

To make $\log \tau(it) \rightarrow 0$ when $|t| \rightarrow \infty$, we must choose $m = 1$, so that $\tau(s) = 1$. Obviously $\tau_1(s) = \tau_2(s) = 1$; it follows that

$$\left. \begin{aligned}\rho_1(s) &= (s+1)^{-m} \tau_1(s) = (s+1)^{-1} \quad (\Re s > -1), \\ \rho_2(s) &= (s-1)^{N-m} \tau_2(s) = s-1 \quad (\Re s < 1).\end{aligned} \right\}$$

$\psi(s)$ is an arbitrary polynomial of degree $N-m-1$, *i.e.*, an arbitrary constant. Hence

$$\phi(s) = \frac{\psi(s)}{P(s)\rho_1(s)} = \frac{A(s+1)}{s^2+2\lambda-1}.$$

Consequently, if $\lambda \neq \frac{1}{2}$,

$$f(x) = A[\{1 + \sqrt{1-2\lambda}\} e^{x\sqrt{1-2\lambda}} - \{1 - \sqrt{1-2\lambda}\} e^{-x\sqrt{1-2\lambda}}];$$

if $\lambda = \frac{1}{2}$,

$$f(x) = A(1+x).$$

By Theorem 6, we have obtained all the solutions of (4.7.1) such that $f(x)e^{-x} \in L(0, \infty)$, *i.e.*, such that (4.7.1) has a meaning when the integration involved in it is taken in the sense of Lebesgue.

5. Volterra equations of Type 2.

5.1. The equations discussed in this chapter are formally of the same type as those of Chapter 2, but the restrictions on the behaviour of the kernel are less simple.

We consider the equation

$$f(x) = \int_{-\infty}^x K(x-y)f(y)dy, \quad (5.1.1)$$

where $K(x)$ satisfies the following conditions:

HYPOTHESIS 6. $K(x)$ can be expressed in the form

$$K(x) = K_1(x) + K_2(x) \quad (0 \leq x < \infty),$$

where

$$(i) \quad |K_1(x)| \leq A(1+x^k)^{-1} \quad (0 \leq x < \infty)$$

for some $k > 0$ (k is not necessarily an integer);

(ii) $K_1(x)$, $K_1'(x)$, ..., $K_1^{(r-1)}(x)$ are absolutely continuous in any finite interval (we take $r = k+1$ if k is an integer, $r = [k+2]$ otherwise);

(iii) $K_1(x)$, $K_1'(x)$, ..., $K_1^{(r)}(x)$ are of bounded variation in $(0, \infty)$;

$$(iv) \quad K_1(0) = K_1'(0) = \dots = K_1^{(r)}(0) = 0;$$

$$(v) \quad \int_0^\infty (1+x^r) |dK_1^{(r)}(x)| < \infty;$$

(vi) $K_2(x)$ vanishes outside some finite interval $(0, a)$;

(vii) $K_2(x) \in L(0, a)$.

These rather complicated conditions are by no means the only ones under which our results can be obtained. They are chosen as a sample, and they are those which, with appropriate modifications, are applied to the Milne equation in Chapter 8.

5.2. The characteristic function $\kappa(s)$ corresponding to the kernel $K(x-y)$ is defined as in § 2.2, viz., by the equation

$$\kappa(s) = \int_0^\infty K(x)e^{-sx}dx \quad (\Re s > 0). \quad (5.2.1)$$

Lemmas 1-5 and Theorem 1 are proved as in Chapter 2. If $k > 1$, Lemma 3 can be strengthened; $\kappa(s)$ is then continuous in $\Re s \geq 0$.

LEMMA 10. When $s \rightarrow 0$ in $\Re s \geq 0$, we have

$$\kappa^{(\rho)}(s) = O(|s|^{-r-1}) \quad (\rho = 0, 1, \dots, r).$$

We need consider only $\kappa_1(s)$; $\kappa_2^{(\rho)}(s)$ is continuous in $\Re s \geq 0$ for all ρ . By continued integration by parts, we obtain

$$\begin{aligned} \kappa_1(s) &= \frac{1}{s} \int_0^\infty K_1'(x) e^{-sx} dx \\ &= s^{-r-1} \int_0^\infty e^{-sx} dK_1^{(r)}(x). \end{aligned}$$

The result follows by successive differentiation with respect to s and application of Hypothesis 6, (v).

5.3. THEOREM 7. Suppose that $K(x)$ satisfies Hypothesis 6 and that the characteristic function $\kappa(s)$ satisfies Hypothesis 2 (§ 2.4). Let the distinct roots of the equation $\kappa(s) = 1$ in $\Re s > 0$ be $s_1, s_2, \dots, s_m$, of multiplicities $a_1, a_2, \dots, a_m$ respectively. If (i) $(1+|x|^k)^{-1}f(x) \in L(-\infty, X)$ for every X , and (ii)

$$f(x) = \int_{-\infty}^x K(x-y)f(y) dy, \quad (5.3.1)$$

then

$$f(x) = \sum_{\mu=1}^m Q_\mu(x) e^{s_\mu x}, \quad (5.3.2)$$

where $Q_\mu(x)$ is a polynomial of degree at most $a_\mu - 1$ ($\mu = 1, 2, \dots, m$); conversely, any expression of the form (5.3.2) satisfies (i) and is a solution of (5.3.1).

The converse statement follows at once from Theorem 1 (§ 2.2).

Let $f(x)$ be a solution of (5.3.1) such that

$$(1+|x|^k)^{-1}f(x) \in L(-\infty, 0).$$

Put

$$\begin{aligned} h(x) &= f(x) \quad (x \leq 0), \\ &= 0 \quad (x > 0), \end{aligned} \quad (5.3.3)$$

$$g(x) = h(x) - \int_{-\infty}^x K(x-y)h(y) dy. \quad (5.3.4)$$

It follows at once from (5.3.1) that

$$\begin{aligned} g(x) &= 0 \quad (x \leq 0), \\ &= - \int_{-\infty}^0 K(x-y)f(y) dy \quad (x > 0). \end{aligned} \quad (5.3.5)$$

Now put

$$\left. \begin{aligned} h_n(x) &= h(x) & (-n \leq x \leq 0), \\ &= 0 & (x < -n, x > 0). \end{aligned} \right\} \quad (n = 1, 2, \dots).$$

Then $h_n(x) \in L(-\infty, \infty)$ for all n . Next put

$$\left. \begin{aligned} g_n(x) &= 0 & (x \leq 0), \\ &= -\int_{-\infty}^0 K(x-y) h_n(y) dy & (x > 0). \end{aligned} \right\}$$

When $x > 0$, the expression

$$h_n(x) - \int_{-\infty}^{\infty} K(x-y) h_n(y) dy \quad (5.3.6)$$

is equal to $g_n(x)$. When $-n \leq x \leq 0$, it is equal to

$$\begin{aligned} & h(x) - \int_{-\infty}^{\infty} K(x-y) h(y) dy + \int_{-\infty}^{\infty} K(x-y) [h(y) - h_n(y)] dy \\ &= \int_{-\infty}^{-n} K(x-y) h(y) dy \\ &= p_n(x), \end{aligned}$$

say. We define $p_n(x)$ to be zero in $(-\infty, -n)$ and in $(0, \infty)$. When $x < -n$, the expression (5.3.6) vanishes identically, since $K(x-y) = 0$ when $x < y$.

Combining these definitions and results, we obtain, for $n = 1, 2, \dots$, and almost all x , the equation

$$g_n(x) + p_n(x) = h_n(x) - \int_{-\infty}^{\infty} K(x-y) h_n(y) dy. \quad (5.3.7)$$

To this fundamental equation we shall return later; we must first prove a number of lemmas.

5.4. LEMMA 11.

$$\int_{-\infty}^{\infty} \frac{|g(x)|}{1+|x|^{k+1}} dx = \int_0^{\infty} \frac{|g(x)|}{1+x^{k+1}} dx < \infty.$$

When $x > 0$, we have

$$\begin{aligned} &= -\int_{-\infty}^0 K_1(x-y) f(y) dy - \int_{-\infty}^0 K_2(x-y) f(y) dy \\ &= g^{(1)}(x) + g^{(2)}(x), \end{aligned}$$

say. First,

$$\begin{aligned}\int_0^\infty \frac{|g^{(1)}(x)|}{1+x^{k+1}} dx &\leq \int_0^\infty \frac{dx}{1+x^{k+1}} \int_{-\infty}^0 |K_1(x-y)| |f(y)| dy \\ &\leq A \int_{-\infty}^0 |f(y)| dy \int_0^\infty \frac{dx}{(1+x^{k+1})(1+|x-y|^k)}.\end{aligned}$$

Now
$$\int_{-\infty}^\infty \frac{dx}{(1+|x|^{k+1})(1+|x-y|^k)} \leq \frac{C}{1+|y|^k}, \quad (5.4.1)$$

where C depends only on k . For, taking $y \geq 0$, as we can do without loss of generality, we have

$$\begin{aligned}\int_{-\infty}^\infty \frac{dx}{(1+|x|^{k+1})(1+|x-y|^k)} &= \int_{-\infty}^{\frac{1}{2}y} + \int_{\frac{1}{2}y}^{\frac{3}{2}y} + \int_{\frac{3}{2}y}^\infty \\ &\leq \frac{1}{1+(\frac{1}{2}y)^k} \int_{-\infty}^\infty \frac{dx}{1+|x|^{k+1}} + \frac{y}{1+(\frac{1}{2}y)^{k+1}} \\ &\leq \frac{C}{1+y^k}.\end{aligned}$$

Hence
$$\int_0^\infty \frac{|g^{(1)}(x)|}{1+x^{k+1}} dx \leq C \int_{-\infty}^0 \frac{|f(y)|}{1+|y|^k} dy < \infty. \quad (5.4.2)$$

Secondly,

$$\begin{aligned}\int_0^\infty \frac{|g^{(2)}(x)|}{1+x^{k+1}} dx &\leq \int_0^\infty \frac{dx}{1+x^{k+1}} \int_{-\infty}^0 |K_2(x-y)| |f(y)| dy \\ &= \int_{-\infty}^0 |f(y)| dy \int_0^\infty \frac{|K_2(x-y)|}{1+x^{k+1}} dx \\ &= \int_{-\infty}^0 |f(y)| dy \int_{-y}^\infty \frac{|K_2(x)|}{1+(y+x)^{k+1}} dx \\ &\leq \int_{-a}^0 |f(y)| dy \int_0^a |K_2(x)| dx < \infty,\end{aligned} \quad (5.4.3)$$

since $K_2(x) = 0$ when $x > a$.

The result now follows from (5.4.2) and (5.4.3).

LEMMA 12.
$$\int_{-\infty}^0 \frac{|h_n(x) - h(x)|}{1+|x|^k} dx \rightarrow 0 \quad (n \rightarrow \infty).$$

This is obvious.

LEMMA 13.
$$\int_0^\infty \frac{|g_n(x) - g(x)|}{1+x^{k+1}} dx \rightarrow 0 \quad (n \rightarrow \infty).$$

For, if we apply (5.4.2) and (5.4.3) to $g_n(x) - g(x)$ instead of $g(x)$, we obtain

$$\int_0^\infty \frac{|g_n(x) - g(x)|}{1+x^{k+1}} dx \leq C \left\{ \int_{-\infty}^0 \frac{|h_n(y) - h(y)|}{1+|y|^k} dy + \int_{-a}^0 |h_n(y) - h(y)| dy \right\} \\ \rightarrow 0 \quad (n \rightarrow \infty).$$

LEMMA 14.

$$\int_{-\infty}^\infty \frac{|p_n(x)|}{1+|x|^{k+1}} dx = \int_{-n}^0 \frac{|p_n(x)|}{1+|x|^{k+1}} dx \rightarrow 0 \quad (n \rightarrow \infty).$$

We write

$$p_n(x) = \int_{-\infty}^{-n} K_1(x-y) h(y) dy + \int_{-\infty}^{-n} K_2(x-y) h(y) dy \\ = p_n^{(1)}(x) + p_n^{(2)}(x),$$

say. First,

$$\int_{-n}^0 \frac{|p_n^{(1)}(x)|}{1+|x|^{k+1}} dx \leq \int_{-n}^0 \frac{dx}{1+|x|^{k+1}} \int_{-\infty}^{-n} |K_1(x-y)| |h(y)| dy \\ \leq A \int_{-\infty}^{-n} |h(y)| dy \int_{-n}^0 \frac{dx}{(1+|x|^{k+1})(1+|x-y|^k)} \\ \leq C \int_{-\infty}^{-n} \frac{|h(y)|}{1+|y|^k} dy \\ \rightarrow 0 \quad (n \rightarrow \infty). \tag{5.4.4}$$

Secondly,

$$\int_{-n}^0 \frac{|p_n^{(2)}(x)|}{1+|x|^{k+1}} dx \leq \int_{-n}^0 \frac{dx}{1+|x|^{k+1}} \int_{-n-a}^{-n} |K_2(x-y)| |h(y)| dy \\ = \int_{-n-a}^{-n} |h(y)| dy \int_{-n}^0 \frac{|K_2(x-y)|}{1+|x|^{k+1}} dx \\ \leq \int_{-n-a}^{-n} |h(y)| dy \left\{ \int_y^{\frac{1}{2}y} + \int_{\frac{1}{2}y}^0 \frac{|K_2(x-y)|}{1+|x|^{k+1}} dx \right\} \\ = \int_{-n-a}^{-n} |h(y)| dy \int_y^{\frac{1}{2}y} \frac{|K_2(x-y)|}{1+|x|^{k+1}} dx,$$

provided that $n > 2a$. Consequently

$$\int_{-n}^0 \frac{|p_n^{(2)}(x)|}{1+|x|^{k+1}} dx \leq \int_{-n-a}^{-n} \frac{|h(y)|}{1+|\frac{1}{2}y|^{k+1}} dy \int_0^a |K_2(x)| dx$$

$$\rightarrow 0 \quad (n \rightarrow \infty), \quad (5.4.5)$$

The result now follows from (5.4.4) and (5.4.5).

5.5. In Hypothesis 6 we defined the positive integer r , equal to $k+1$ if k is an integer and to $[k+2]$ if k is not an integer. We write

$$b_\rho(u) = 1 - u + \frac{u^2}{2!} - \dots + (-1)^{\rho-1} \frac{u^{\rho-1}}{(\rho-1)!} \quad (\rho = 1, 2, \dots, r),$$

and we define the functions $\phi_\rho(s)$ ($\rho = 0, 1, \dots, r$) by the following equations:

$$\left. \begin{aligned} \phi_0(s) &= \int_{-\infty}^0 h(x) e^{-sx} dx, \\ \phi_1(s) &= \int_{-\infty}^{-1} h(x) \frac{e^{-sx}}{-x} dx + \int_{-1}^0 h(x) \frac{e^{-sx} - b_1(sx)}{-x} dx, \\ \dots & \dots \dots \dots \dots \dots \dots \dots \\ \phi_r(s) &= \int_{-\infty}^{-1} h(x) \frac{e^{-sx}}{(-x)^r} dx + \int_{-1}^0 h(x) \frac{e^{-sx} - b_r(sx)}{(-x)^r} dx \end{aligned} \right\} \quad (\Re s < 0),$$

$$\phi_r(s) = \int_{-\infty}^{-1} h(x) \frac{e^{-sx}}{(-x)^r} dx + \int_{-1}^0 h(x) \frac{e^{-sx} - b_r(sx)}{(-x)^r} dx \quad (\Re s \leq 0).$$

It is easily seen that all the functions $\phi_\rho(s)$ are regular in $\Re s < 0$, and that $\phi_{r-1}(s)$ and $\phi_r(s)$ are continuous in $\Re s \leq 0$. Furthermore,

$$\phi_\rho'(s) = \phi_{\rho-1}(s) \quad (\rho = 1, 2, \dots, r; \Re s < 0).$$

We define the functions $\gamma_\rho(s)$ ($\rho = 0, 1, \dots, r$) as follows:

$$\left. \begin{aligned} \gamma_0(s) &= \int_0^\infty g(x) e^{-sx} dx, \\ \gamma_1(s) &= \int_0^1 g(x) \frac{e^{-sx} - b_1(sx)}{-x} dx + \int_1^\infty g(x) \frac{e^{-sx}}{-x} dx, \\ \dots & \dots \dots \dots \dots \dots \dots \dots \\ \gamma_r(s) &= \int_0^1 g(x) \frac{e^{-sx} - b_r(sx)}{(-x)^r} dx + \int_1^\infty g(x) \frac{e^{-sx}}{(-x)^r} dx \end{aligned} \right\} \quad (\Re s > 0),$$

$$\gamma_r(s) = \int_0^1 g(x) \frac{e^{-sx} - b_r(sx)}{(-x)^r} dx + \int_1^\infty g(x) \frac{e^{-sx}}{(-x)^r} dx \quad (\Re s \geq 0).$$

All the functions $\gamma_\rho(s)$ are regular in $\Re s > 0$, and it follows from Lemma 11 that $\gamma_r(s)$ is continuous in $\Re s \geq 0$. Furthermore,

$$\gamma_\rho'(s) = \gamma_{\rho-1}(s) \quad (\rho = 1, 2, \dots, r; \Re s > 0).$$

We next define the functions $\phi_{n,\rho}(s)$ ($n = 1, 2, \dots; \rho = 0, 1, \dots, r$) which correspond to $h_n(x)$ ($n = 1, 2, \dots$) in the same way as the functions $\phi_\rho(s)$ do to $h(x)$. The functions $\phi_{n,\rho}(s)$ are all continuous in $\Re s \leq 0$.

LEMMA 15. $\phi_{n,r}(s) \rightarrow \phi_r(s) \quad (n \rightarrow \infty) \quad (5.5.1)$

uniformly in s in any bounded region contained in $\Re s \leq 0$.

We have

$$\begin{aligned} \phi_{n,r}(s) - \phi_r(s) &= \int_{-1}^0 [h_n(x) - h(x)] \frac{e^{-sx} - b_r(sx)}{(-x)^r} dx + \int_{-\infty}^{-1} [h_n(x) - h(x)] \frac{e^{-sx}}{(-x)^r} dx. \end{aligned}$$

Hence
$$|\phi_{n,r}(s) - \phi_r(s)| \leq C \int_{-\infty}^0 \frac{|h_n(x) - h(x)|}{1 + |x|^r} dx.$$

The result follows.

We now define functions $\gamma_{n,\rho}(s)$ ($n = 1, 2, \dots; \rho = 0, 1, \dots, r$) which correspond to $g_n(x)$ ($n = 1, 2, \dots$) in the same way as the functions $\gamma_\rho(s)$ do to $g(x)$.

LEMMA 16. $\gamma_{n,r}(s) \rightarrow \gamma_r(s) \quad (n \rightarrow \infty) \quad (5.5.2)$

uniformly in s in any bounded region contained in $\Re s \geq 0$.

This follows from Lemma 13 by an argument similar to the proof of Lemma 15.

Since $h_n(x) \in L(-\infty, 0)$, it follows as in § 2.5 that $g_n(x)$ is the sum of a function of V and a function of L . Hence $\gamma_{n,0}(s)$ can be defined when $\Re s = 0, s \neq 0$, in such a way that it is continuous in $\Re s \geq 0, s \neq 0$.

We now turn to $p_n(x)$, which vanishes outside $(-n, 0)$. We define

$$\begin{aligned} \varpi_{n,\rho}(s) &= \int_{-n}^{-1} p_n(x) \frac{e^{-sx}}{(-x)^\rho} dx + \int_{-1}^0 p_n(x) \frac{e^{-sx} - b_\rho(sx)}{(-x)^\rho} dx \\ &\quad (n = 1, 2, \dots; \rho = 0, 1, \dots, r), \end{aligned}$$

where $b_0(sx) \equiv 0$. Then

$$\varpi'_{n,\rho}(s) = \varpi_{n,\rho-1}(s) \quad (\rho = 1, \dots, r; n = 1, 2, \dots).$$

LEMMA 17. $\varpi_{n,r}(s) \rightarrow 0 \quad (n \rightarrow \infty) \quad (5.5.3)$

uniformly in s in any bounded region contained in $\Re s \leq 0$.

This follows at once from Lemma 14.

5.6. We now treat equation (5.3.7) by the methods of § 2.6. We obtain the result that, when $\Re s = 0$, $s \neq 0$,

$$\gamma_{n,0}(s) + \varpi_{n,0}(s) = \phi_{n,0}(s)[1 - \kappa(s)],$$

i.e.,

$$\frac{\gamma_{n,0}(s) + \varpi_{n,0}(s)}{1 - \kappa(s)} = \phi_{n,0}(s) \quad (n = 1, 2, \dots). \quad (5.6.1)$$

We write

$$P(s) = \prod_{\mu=1}^m (s - s_{\mu})^{\alpha_{\mu}},$$

and multiply (5.6.1) by $s^{\lambda} P(s)$, where λ is a certain positive integer. In the present circumstances we can take $\lambda = (r+1)^2$.

Writing

$$\tau(s) = \frac{s^{\lambda} P(s)}{1 - \kappa(s)}, \quad P_1(s) = s^{\lambda} P(s),$$

we obtain

$$P_1(s) \phi_{n,0}(s) = \tau(s) [\gamma_{n,0}(s) + \varpi_{n,0}(s)]. \quad (5.6.2)$$

We note that $\tau(s)$ is regular in $\Re s > 0$, and that, by Lemma 10 and our choice of λ , the functions $\tau^{(\rho)}(s)$ ($\rho = 0, 1, \dots, r$) are continuous in $\Re s \geq 0$ and vanish at $s = 0$.

Let $\Re s = 0$. Consider the expression

$$I_n(s) = \frac{1}{(r-1)!} \int_0^s (s-v)^{r-1} P_1(v) \phi_{n,0}(v) dv,$$

where the integration is taken along the imaginary axis from 0 to s . Integrating by parts r times, we obtain

$$I_n(s) = P_1(s) \phi_{n,r}(s) - \sum_{\rho=1}^r \int_0^s \binom{r}{\rho} \frac{(v-s)^{\rho-1}}{(\rho-1)!} P_1^{(\rho)}(v) \phi_{n,r}(v) dv.$$

Similarly, if we put

$$J_n(s) = \frac{1}{(r-1)!} \int_0^s (s-v)^{r-1} \tau(v) [\gamma_{n,0}(v) + \varpi_{n,0}(v)] dv,$$

we see that

$$J_n(s) = \tau(s) [\gamma_{n,r}(s) + \varpi_{n,r}(s)] - \sum_{\rho=1}^r \int_0^s \binom{r}{\rho} \frac{(v-s)^{\rho-1}}{(\rho-1)!} \tau^{(\rho)}(v) [\gamma_{n,r}(v) + \varpi_{n,r}(v)] dv.$$

By equation (5.6.2),

$$I_n(s) = J_n(s) \quad (\Re s = 0; \quad n = 1, 2, \dots). \quad (5.6.3)$$

Now let $n \rightarrow \infty$, and use Lemmas 15, 16, and 17. We obtain the result that, when $\Re s = 0$,

$$\begin{aligned} P_1(s) \phi_r(s) - \sum_{\rho=1}^r \int_0^s \binom{r}{\rho} \frac{(v-s)^{\rho-1}}{(\rho-1)!} P_1^{(\rho)}(v) \phi_r(v) dv \\ = \tau(s) \gamma_r(s) - \sum_{\rho=1}^r \int_0^s \binom{r}{\rho} \frac{(v-s)^{\rho-1}}{(\rho-1)!} \tau^{(\rho)}(v) \gamma_r(v) dv. \quad (5.6.4) \end{aligned}$$

The expression on the left-hand side of (5.6.4) is defined when $\Re s \leq 0$; that on the right-hand side when $\Re s \geq 0$. The two sides together therefore define a function $\psi_r(s)$ throughout the plane; and, by the principle enunciated in § 1.2, $\psi_r(s)$ is an integral function. Furthermore, it is easy to see that

$$\psi_r(s) = o(|s|^{\lambda+N+r}) \quad (|s| \rightarrow \infty),$$

where N is the degree of $P(s)$. It follows that $\psi_r(s)$ is a polynomial.

Now, when $\Re s < 0$,

$$\psi_r^{(\tau)}(s) = P_1(s) \phi_0(s) = s^\lambda P(s) \phi_0(s).$$

Hence

$$\phi_0(s) = \frac{\psi_r^{(\tau)}(s)}{s^\lambda P(s)} = \frac{\psi_0(s)}{s^\lambda P(s)},$$

say. Resolving this expression into partial fractions, and using the uniqueness theorem for Fourier transforms in the same way as in § 2.8, we see that

$$h(x) = \sum_{\mu=1}^m Q_\mu(x) e^{s_\mu x} + R(x) \quad (x \leq 0),$$

where $Q_\mu(x)$ is a polynomial of degree at most $a_\mu - 1$ ($\mu = 1, 2, \dots, m$), and $R(x)$ is a polynomial of degree at most $\lambda - 1$.

We now show that $R(x)$ must vanish identically. There are two cases to consider. If $k \leq 1$, our conclusion follows at once from the condition that

$$\int_{-\infty}^0 \frac{|h(x)|}{1+|x|^k} dx < \infty.$$

If $k > 1$, we deduce from this condition that the degree of $R(x)$ is less than $k - 1$. Since $h(x)$ and $\sum_{\mu=1}^m Q_\mu(x) e^{s_\mu x}$ are both solutions of (5.3.1), $R(x)$ must also be a solution. If $R(x)$ is not identically zero, it follows that $\kappa(0) = 1$, which contradicts Hypothesis 2 (§ 2.4).

Hence

$$f(x) = \sum_{\mu=1}^m Q_\mu(x) e^{s_\mu x} \quad (x \leq 0).$$

The proof can now be completed as in § 2.8.

Remarks similar to those of § 2.9 can be made here also.

5.7. As an example for the application of Theorem 7 let us consider the equation

$$f(x) = \frac{\lambda}{\Gamma(a)} \int_{-\infty}^x (x-y)^{a-1} f(y) dy, \quad (5.7.1)$$

where we suppose that $0 < a < 1$. This may be regarded as a differential equation of fractional order a , combined with the boundary condition that $f(x) \rightarrow 0$ when $x \rightarrow -\infty$.

We have
$$K(x) = \frac{\lambda}{\Gamma(a)} x^{a-1},$$

so that
$$\kappa(s) = \frac{\lambda}{\Gamma(a)} \int_0^\infty x^{a-1} e^{-sx} dx = \lambda s^{-a},$$

where s^{-a} denotes that branch of the function which is real and positive when s is real and positive.

The characteristic equation is

$$s^a = \lambda. \quad (5.7.2)$$

If $|\arg \lambda| < \frac{1}{2}\pi$ and $\lambda \neq 0$, Hypothesis 2 (§ 2.4) is satisfied and (5.7.2) has a unique root in the half-plane $\Re s > 0$. Since Hypothesis 6 is satisfied with $k = 1 - a$, we conclude from Theorem 7 that, under the above conditions, the only solutions $f(x)$ of (5.7.1) such that

$$(1 + |x|^{1-a})^{-1} f(x) \in L(-\infty, X)$$

for all X are $f(x) = A e^{sx}$, where s is the unique root of (5.7.2), and A is an arbitrary constant.

Hardy and Titchmarsh (16) have shown that the above solution is unique when λ is real and positive, and the integral is regarded as a Cauchy integral at both limits.

6. Fredholm equations of Type 2.

6.1. In this chapter we are concerned with equations of the form

$$f(x) = \int_{-\infty}^\infty K(x-y) f(y) dy, \quad (6.1.1)$$

where $K(x)$ satisfies the following conditions:

Hypothesis 7. $K(x)$ can be expressed in the form

$$K(x) = K_1(x) + K_2(x) = G_1(x) e^{-|x|} + K_2(x) \quad (-\infty < x < \infty),$$

where

$$(i) \quad |G_1(x)| \leq \frac{A}{1+|x|^k} \quad (-\infty < x < \infty)$$

for some $k > 0$ (k is not necessarily an integer),

$$(ii) \quad G_1(x), G_1'(x), \dots, G_1^{(r-1)}(x)$$

are absolutely continuous in every finite interval [we define r as in Hypothesis 6 (§ 5.1)],

$$(iii) \quad G_1(x), G_1'(x), \dots, G_1^{(r)}(x)$$

are of bounded variation in $(-\infty, \infty)$,

$$(iv) \quad G_1(0) = G_1'(0) = \dots = G_1^{(r)}(0) = 0,$$

$$(v) \quad \int_{-\infty}^{\infty} (1+|x|^r) |dG_1^{(r)}(x)| < \infty,$$

(vi) $K_2(x)$ vanishes outside some finite interval $(-a, a)$,

$$(vii) \quad K_2(x) \in L(-a, a).$$

6.2. We define the characteristic function $\kappa(s)$ as in § 3.1 by the equation

$$\kappa(s) = \int_{-\infty}^{\infty} K(x) e^{-sx} dx. \quad (6.2.1)$$

Then $\kappa(s)$ is regular in the strip $-1 < \Re s < 1$.

Lemmas 7, 8, and 9 are proved in the same way as in Chapter 3.

LEMMA 18. When $s \rightarrow \pm 1$ in $\Re s \geq 0$,

$$\kappa^{(\rho)}(s) = O(|s \mp 1|^{-r-\rho-1}) \quad (\rho = 0, 1, \dots, r).$$

This is proved by the same kind of argument as Lemma 10 (§ 5.2).

6.3. THEOREM 8. Suppose that $K(x)$ satisfies Hypothesis 7, and that $\kappa(s)$ satisfies Hypothesis 4 (§ 3.3). Let the distinct roots of the equation $\kappa(s) = 1$ in $-1 < \Re s < 1$ be $s_1, s_2, \dots, s_m$, of multiplicities $a_1, a_2, \dots, a_m$ respectively. If (i)

$$\frac{f(x) e^{-|x|}}{1+|x|^k} \in L(-\infty, \infty),$$

$$\text{and (ii)} \quad f(x) = \int_{-\infty}^{\infty} K(x-y) f(y) dy, \quad (6.3.1)$$

then
$$f(x) = \sum_{\mu=1}^m Q_{\mu}(x) e^{s_{\mu}x}, \quad (6.3.2)$$

where $Q_{\mu}(x)$ is a polynomial of degree at most $a_{\mu}-1$ ($\mu=1, 2, \dots, m$); conversely, any expression of the form (6.3.2) satisfies (i) and is a solution of (6.3.1).

The converse statement is, as usual, immediate.

Let $f(x)$ be a solution of (6.3.1) satisfying (i) above. Write

$$\begin{aligned} f_+(x) &= f(x) \quad (x \geq 0), \\ &= 0 \quad (x < 0), \\ f_-(x) &= f(x) - f_+(x), \\ g(x) &= f_+(x) - \int_0^{\infty} K(x-y) f_+(y) dy. \end{aligned} \quad (6.3.3)$$

Then

$$\begin{aligned} g(x) &= \int_{-\infty}^0 K(x-y) f(y) dy \quad (x \geq 0), \\ &= -\int_0^{\infty} K(x-y) f(y) dy \quad (x < 0). \end{aligned} \quad (6.3.4)$$

We prove as in §3.4 that

$$-g(x) = f_-(x) - \int_{-\infty}^0 K(x-y) f_-(y) dy. \quad (6.3.5)$$

6.4. Now put

$$\begin{aligned} h_n(x) &= f_-(x) = f(x) \quad (-n \leq x \leq 0), \\ &= 0 \quad (x < -n, \ x > 0). \end{aligned} \quad \left. \vphantom{\begin{aligned} h_n(x) &= f_-(x) = f(x) \quad (-n \leq x \leq 0), \\ &= 0 \quad (x < -n, \ x > 0). \end{aligned}} \right\}$$

Then $h_n(x) e^{|x|\epsilon} \in L(-\infty, \infty)$ for all n . Next put

$$\begin{aligned} g_n(x) &= -\int_0^{\infty} K(x-y) f(y) dy \quad (x \leq 0), \\ &= \int_{-\infty}^0 K(x-y) h_n(y) dy \quad (x > 0). \end{aligned} \quad \left. \vphantom{\begin{aligned} g_n(x) &= -\int_0^{\infty} K(x-y) f(y) dy \quad (x \leq 0), \\ &= \int_{-\infty}^0 K(x-y) h_n(y) dy \quad (x > 0). \end{aligned}} \right\}$$

When $x > 0$, the expression

$$h_n(x) - \int_{-\infty}^{\infty} K(x-y) h_n(y) dy \quad (6.4.1)$$

is equal to $-g_n(x)$. When $-n \leq x \leq 0$, it is equal to

$$\begin{aligned} f_-(x) - \int_{-\infty}^x K(x-y)f_-(y)dy + \int_{-x}^{\infty} K(x-y)[f_-(y) - h_n(y)]dy \\ = -g_n(x) + \int_{-\infty}^{-n} K(x-y)f_-(y)dy \\ = -g_n(x) + p_n(x), \end{aligned}$$

say. When $x < -n$, it is equal to

$$\begin{aligned} - \int_{-n}^0 K(x-y)f_-(y)dy = f_-(x) - \int_{-\infty}^0 K(x-y)f_-(y)dy \\ - f_-(x) + \int_{-x}^{-n} K(x-y)f_-(y)dy \\ = -g_n(x) - f_-(x) + \int_{-\infty}^{-n} K(x-y)f_-(y)dy \\ = -g_n(x) + q_n(x), \end{aligned}$$

say. We define $p_n(x)$ and $q_n(x)$ to be zero outside $(-n, 0)$ and $(-\infty, -n)$ respectively.

Combining these definitions and results, we obtain, for all n and almost all x , the equation

$$-g_n(x) + p_n(x) + q_n(x) = h_n(x) - \int_{-\infty}^{\infty} K(x-y)h_n(y)dy. \quad (6.4.2)$$

This fundamental equation is to be converted into a relation expressing that two functions of s are analytic continuations of each other across the line $\Re s = -1$.

6.5. LEMMA 19.

$$\int_{-\infty}^{\infty} |g(x)| \frac{e^{|x|}}{1+|x|^{k+1}} dx < \infty. \quad (6.5.1)$$

We express $g(x)$ in the form (6.3.4), and split it up into the parts depending on K_1 and K_2 respectively:

$$g(x) = g^{(1)}(x) + g^{(2)}(x).$$

Then

$$\left. \begin{aligned} e^{|x|} g^{(1)}(x) &= - \int_0^{\infty} e^{x-y} K_1(x-y) e^{-y} f(y) dy \quad (x \leq 0), \\ &= \int_{-\infty}^0 e^{x-y} K_1(x-y) e^y f(y) dy \quad (x > 0). \end{aligned} \right\}$$

Hence

$$\begin{aligned}
 \int_{-\infty}^{\infty} |g^{(1)}(x)| \frac{e^{|x|}}{1+|x|^{k+1}} dx &\leq \int_{-\infty}^{\infty} \frac{dx}{1+|x|^{k+1}} \int_{-\infty}^{\infty} e^{|x-y|} |K_1(x-y)| e^{-|y|} |f(y)| dy \\
 &= \int_{-\infty}^{\infty} e^{-|y|} |f(y)| dy \int_{-\infty}^{\infty} \frac{e^{|x-y|} |K_1(x-y)|}{1+|x|^{k+1}} dx \\
 &\leq A \int_{-\infty}^{\infty} e^{-|y|} |f(y)| dy \int_{-\infty}^{\infty} \frac{dx}{(1+|x|^{k+1})(1+|x-y|^k)} \\
 &\leq C \int_{-\infty}^{\infty} |f(y)| \frac{e^{-|y|}}{1+|y|^k} dy < \infty;
 \end{aligned}$$

we have here used (5.4.1).

Secondly,

$$\begin{aligned}
 \int_{-\infty}^{\infty} |g^{(2)}(x)| \frac{e^{|x|}}{1+|x|^{k+1}} dx &\leq \int_{-\infty}^{\infty} e^{-|y|} |f(y)| dy \int_{-\infty}^{\infty} \frac{e^{|x-y|} |K_2(x-y)|}{1+|x|^{k+1}} dx \\
 &= \int_{-\infty}^{\infty} e^{-|y|} |f(y)| dy \int_{-a}^a \frac{e^{|x|} |K_2(x)|}{1+|x+y|^{k+1}} dx \\
 &= \left\{ \int_{-\infty}^{-2a} + \int_{-2a}^{2a} + \int_{2a}^{\infty} \right\} e^{-|y|} |f(y)| dy \int_{-a}^a \frac{e^{|x|} |K_2(x)|}{1+|x+y|^{k+1}} dx \\
 &= I_1 + I_2 + I_3,
 \end{aligned}$$

say. Then

$$I_1 + I_3 \leq C \int_{-\infty}^{\infty} \frac{e^{-|y|} |f(y)|}{1+|\frac{1}{2}y|^{k+1}} dy \int_{-a}^a |K_2(x)| dx < \infty.$$

I_2 is taken over a finite region, and is trivially finite.

LEMMA 20.
$$\int_{-\infty}^0 |h_n(x) - f(x)| \frac{e^{-|x|}}{1+|x|^k} dx \rightarrow 0 \quad (n \rightarrow \infty).$$

This is obvious.

LEMMA 21.

$$\int_{-\infty}^{\infty} \frac{|g_n(x) - g(x)|}{1+|x|^{k+1}} e^{|x|} dx = \int_0^{\infty} \frac{|g_n(x) - g(x)|}{1+|x|^{k+1}} e^{|x|} dx \rightarrow 0 \quad (n \rightarrow \infty).$$

This follows at once from the proof of Lemma 19.

LEMMA 22.
$$\int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|^{k+1}} |p_n(x)| dx \rightarrow 0 \quad (n \rightarrow \infty).$$

We write

$$\begin{aligned} p_n(x) &= \int_{-\infty}^{-n} K_1(x-y) f_-(y) dy + \int_{-\infty}^{-n} K_2(x-y) f_-(y) dy \\ &= p_n^{(1)}(x) + p_n^{(2)}(x), \end{aligned}$$

say. First,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|^{k+1}} |p_n^{(1)}(x)| dx &= \int_{-n}^0 \frac{e^{-|x|}}{1+|x|^{k+1}} |p_n^{(1)}(x)| dx \\ &\leq \int_{-n}^0 \frac{e^{-|x|}}{1+|x|^{k+1}} dx \int_{-\infty}^{-n} |K_1(x-y)| |f_-(y)| dy \\ &= \int_{-\infty}^{-n} |f_-(y)| e^{-|y|} dy \int_{-n}^0 \frac{e^{|x-y|} |K_1(x-y)|}{1+|x|^{k+1}} dx \\ &\leq C \int_{-\infty}^{-n} \frac{e^{-|y|} |f_-(y)|}{1+|y|^k} dy \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Secondly,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|^{k+1}} |p_n^{(2)}(x)| dx &\leq \int_{-n}^0 \frac{e^x}{1+|x|^{k+1}} dx \int_{-n-a}^{-n} |K_2(x-y)| |f(y)| dy \\ &= \int_{-n-a}^{-n} |f(y)| dy \int_{-n}^0 \frac{e^x}{1+|x|^{k+1}} |K_2(x-y)| dx \\ &\leq \int_{-n-a}^{-n} |f(y)| dy \int_y^{iy} \frac{e^x}{1+|x|^{k+1}} |K_2(x-y)| dx, \end{aligned}$$

provided that $n > 2a$. Consequently

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|^{k+1}} |p_n^{(2)}(x)| dx &\leq \int_{-n-a}^{-n} \frac{e^{\frac{1}{2}y}}{1+|\frac{1}{2}y|^{k+1}} dy \int_0^a |K_2(x)| dx \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

LEMMA 23.

$$\int_{-\infty}^{-n} \frac{|e^{-|x|}}{1+|x|^{k+1}} |q_n(x)| dx \rightarrow 0 \quad (n \rightarrow \infty).$$

We have, when $x < -n$,

$$\begin{aligned} q_n(x) &= -f_-(x) + \int_{-\infty}^{-n} K_1(x-y) f_-(y) dy + \int_{-\infty}^{-n} K_2(x-y) f_-(y) dy \\ &= -f_-(x) + q_n^{(1)}(x) + q_n^{(2)}(x), \end{aligned}$$

say. First,

$$\int_{-\infty}^{-n} \frac{e^{-|x|}}{1+|x|^{k+1}} |f_-(x)| dx \rightarrow 0 \quad (n \rightarrow \infty).$$

Secondly,

$$\begin{aligned} \int_{-\infty}^{-n} \frac{e^{-|x|}}{1+|x|^{k+1}} |q_n^{(1)}(x)| dx &\leq A \int_{-\infty}^{-n} |f_-(y)| dy \int_{-\infty}^y \frac{e^{2x-y} dx}{(1+|x|^{k+1})(1+|x-y|^k)} \\ &\quad + A \int_{-\infty}^{-n} |f_-(y)| dy \int_y^{-n} \frac{e^y dx}{(1+|x|^{k+1})(1+|x-y|^k)} \\ &\leq A \int_{-\infty}^{-n} e^{-|y|} |f_-(y)| dy \int_{-\infty}^{-n} \frac{dx}{(1+|x|^{k+1})(1+|x-y|^k)} \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Finally,

$$\begin{aligned} \int_{-\infty}^{-n} \frac{e^{-|x|}}{1+|x|^{k+1}} |q_n^{(2)}(x)| dx &\leq \int_{-\infty}^{-n} \frac{e^{-|x|} dx}{1+|x|^{k+1}} \int_{-\infty}^{-n} |K_2(x-y)| |f(y)| dy \\ &\leq \int_{-\infty}^{-n} e^{-|y|} |f(y)| dy \int_{-\infty}^{-n} \frac{e^{|x-y|}}{1+|x|^{k+1}} |K_2(x-y)| dx \\ &\leq \int_{-\infty}^{-n} e^{-|y|} |f(y)| dy \int_{-a}^a \frac{e^{|x|}}{1+|x+y|^{k+1}} |K_2(x)| dx. \end{aligned}$$

When $n > 2a$ we have $|x+y| > \frac{1}{2}|y|$ in the domain of integration of this double integral. Consequently

$$\begin{aligned} \int_{-\infty}^{-n} \frac{e^{-|x|}}{1+|x|^{k+1}} |q_n^{(2)}(x)| dx &\leq \int_{-\infty}^{-n} \frac{e^{-|y|}}{1+|\frac{1}{2}y|^{k+1}} |f(y)| dy \int_{-a}^a e^{|x|} |K_2(x)| dx \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

6.6. Now that the necessary lemmas have been proved, it is clear that we can adapt the work of §§ 5.5–5.6 to our present purposes, just as we adapted the methods of Chapter 2 in Chapter 3. The reasoning would be based on equation (6.4.2) and on a similar equation which enables us to carry out an analytic continuation across the line $\Re s = 1$. In this way the proof of Theorem 8 can be completed.

Remarks similar to those of § 3.5 can be made here also. The results of the present chapter are applied in Chapter 8.

7. Wiener-Hopf equations of Type 2.

7.1. In this chapter we return to the form of equation considered in Chapter 4, viz.,

$$f(x) = \int_0^\infty K(x-y)f(y) dy \quad (x \geq 0). \quad (7.1.1)$$

This time we assume that $K(x)$ satisfies Hypothesis 7 (§6.1), and that $\kappa(s)$ satisfies Hypotheses 4 (§3.3) and 5 (§4.1).

The necessary preliminary lemmas are proved in the usual way.

7.2. THEOREM 9. *Under the above assumptions, the equation (7.1.1) has only a finite number of linearly independent solutions $f(x)$ such that*

$$\frac{e^{-x}f(x)}{1+x^k} \in L(0, \infty). \quad (7.2.1)$$

Suppose that $f(x)$ is such a solution, and define $f(x)$ to be zero when $x < 0$. Put

$$g(x) = f(x) - \int_{-\infty}^{\infty} K(x-y)f(y)dy. \quad (7.2.2)$$

Then

$$\left. \begin{aligned} g(x) &= - \int_0^{\infty} K(x-y)f(y)dy & (x < 0), \\ &= 0 & (x \geq 0). \end{aligned} \right\}$$

Next, for $n = 1, 2, \dots$, put

$$\left. \begin{aligned} h_n(x) &= f(x) & (0 \leq x \leq n), \\ &= 0 & (x < 0, x > n); \end{aligned} \right\}$$

and

$$\left. \begin{aligned} g_n(x) &= - \int_0^{\infty} K(x-y)h_n(y)dy & (x < 0), \\ &= 0 & (x \geq 0). \end{aligned} \right\}$$

When $x < 0$, the expression

$$h_n(x) - \int_{-\infty}^{\infty} K(x-y)h_n(y)dy \quad (7.2.3)$$

is equal to $g_n(x)$. When $0 \leq x \leq n$, it is equal to

$$\begin{aligned} f(x) - \int_0^{\infty} K(x-y)f(y)dy + \int_0^{\infty} K(x-y)[f(y) - h_n(y)]dy &= \int_n^{\infty} K(x-y)f(y)dy \\ &= p_n(x), \end{aligned}$$

say. When $x > n$, it is equal to

$$\begin{aligned} - \int_0^n K(x-y)f(y)dy &= f(x) - \int_0^{\infty} K(x-y)f(y)dy - f(x) + \int_n^{\infty} K(x-y)f(y)dy \\ &= -f(x) + \int_n^{\infty} K(x-y)f(y)dy \\ &= q_n(x), \end{aligned}$$

say. We define $p_n(x)$ and $q_n(x)$ to be zero outside $(0, n)$ and (n, ∞) respectively.

Combining these definitions and results, we obtain

$$g_n(x) + p_n(x) + q_n(x) = h_n(x) - \int_{-\infty}^{\infty} K(x-y) h_n(y) dy. \quad (7.2.4)$$

We can now treat this equation by the methods of Chapters 5 and 6, obtaining finally a relation which expresses that two functions of $s = \sigma + it$ are analytic continuations of one another across the line $\Re s = 1$. In this way we see that the function $\chi(s)$ defined by (4.3.1) is regular in the half-plane $\Re s > -1$. The argument of §§ 4.4–4.5 can now be carried through without alteration.

7.3. Theorem 6 (§ 4.6) holds in the present case, and it is proved by the same argument as in § 4.6. It follows that, if a solution $f(x)$ of (7.1.1) satisfies (7.2.1), it even satisfies the condition $e^{-x}f(x) \in L(0, \infty)$; in fact, as can be seen from the proof of Theorem 6, still more is true.

The results of the present chapter are applied in Chapter 8.

8. *The integral equations of E. A. Milne.*

8.1. The first equation which we consider in this chapter appears in the literature in two different forms, viz.,

$$f(x) = \int_{-\infty}^{\infty} K(x-y) f(y) dy, \quad (8.1.1)$$

where

$$K(x) = \frac{1}{2} \int_{|x|}^{\infty} \frac{e^{-y}}{y} dy;$$

and

$$F'(x) = \int_0^{\infty} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt, \quad (8.1.2)$$

where the integral on the right-hand side of (8.1.2) is interpreted as

$$\lim_{\epsilon \rightarrow +0} \int_{\epsilon}^{\infty} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt.$$

If $F(x)$ satisfies (8.1.2) and is subjected to certain restrictions as to order of magnitude and smoothness, then $f(x) = F'(x)$ is a solution of (8.1.1), and conversely. The two equations can be connected in another way, which is discussed in detail later.

It is easy to see that $f(x) = Ax + B$, where A and B are arbitrary constants, is a solution of (8.1.1). For

$$\begin{aligned} \int_{-\infty}^{\infty} K(x-y)(Ay+B)dy &= \frac{1}{2} \int_{-\infty}^x (Ay+B)dy \int_{x-y}^{\infty} \frac{e^{-z}}{z} dz + \frac{1}{2} \int_x^{\infty} (Ay+B)dy \int_{y-x}^{\infty} \frac{e^{-z}}{z} dz \\ &= \frac{1}{2} \int_0^{\infty} \frac{e^{-z}}{z} dz \int_{x-z}^{x+z} (Ay+B)dy \\ &= (Ax+B) \int_0^{\infty} e^{-z} dz = Ax+B. \end{aligned}$$

It can be shown without difficulty that this is the general polynomial solution of (8.1.1). Our object is to show, under the lightest possible conditions, that there are no other solutions.

The general polynomial solution of (8.1.2) is

$$F(x) = Ax^2 + Bx + C,$$

where A , B , and C are arbitrary constants.

8.2. We shall now show how the solving of (8.1.2) can be reduced to that of (8.1.1).

THEOREM 10. *If $F(x)$ satisfies (8.1.2) for almost all x , $F(x)$ is absolutely continuous in every finite interval, and*

$$\int_{-\infty}^{\infty} \frac{e^{-|t|} |F(t)|}{1+|t|} dt < \infty,$$

then there is a constant A such that $f(x) = F(x) - Ax^2$ satisfies (8.1.1).

We have

$$\begin{aligned} F'(x) &= \int_0^{\epsilon} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt + \int_{\epsilon}^{\infty} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt \\ &= f_1(x) + f_2(x), \end{aligned} \tag{8.2.1}$$

say. Take $y > 0$; this involves no loss of generality. Then

$$\begin{aligned} \int_0^y f_1(x) dx &= \int_0^y dx \int_0^{\epsilon} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt \\ &= \int_0^y dx \int_0^{\epsilon} \frac{e^{-t}}{2t} dt \int_{x-t}^{x+t} F'(u) du \\ &= \int_0^{\epsilon} \frac{e^{-t}}{2t} dt \int_0^y dx \int_{x-t}^{x+t} F'(u) du. \end{aligned}$$

Hence

$$\begin{aligned}
 \left| \int_0^y f_1(x) dx \right| &\leq \int_0^\epsilon \frac{e^{-t}}{2t} dt \int_{-t}^{y+t} |F'(u)| du \int_{u-t}^{u+t} dx \\
 &\leq \int_0^\epsilon e^{-t} dt \int_{-\epsilon}^{y+\epsilon} |F'(u)| du \\
 &= (1-e^{-\epsilon}) \int_{-\epsilon}^{y+\epsilon} |F'(u)| du \\
 &\rightarrow 0 \quad (\epsilon \rightarrow +0).
 \end{aligned} \tag{8.2.2}$$

This argument incidentally shows that the integral for $f_1(x)$ exists as a Lebesgue integral for almost all x .

Next,

$$\begin{aligned}
 \int_0^y f_2(x) dx &= \int_0^y dx \int_\epsilon^\infty \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt \\
 &= \int_0^y dx \int_{-\infty}^{-\epsilon} \frac{e^t}{2t} F(x+t) dt + \int_0^y dx \int_\epsilon^\infty \frac{e^{-t}}{2t} F(x+t) dt \\
 &= \int_0^y dx \int_{-\infty}^{x-\epsilon} \frac{e^{u-x}}{2(u-x)} F(u) du + \int_0^y dx \int_{x+\epsilon}^\infty \frac{e^{x-u}}{2(u-x)} F(u) du \\
 &= I_1 + I_2,
 \end{aligned}$$

say. We begin by transforming I_1 :

$$\begin{aligned}
 I_1 &= \int_{-\infty}^{-\epsilon} F(u) du \int_0^y \frac{e^{u-x}}{2(u-x)} dx + \int_{-\epsilon}^{y-\epsilon} F(u) du \int_{u+\epsilon}^y \frac{e^{u-x}}{2(u-x)} dx \\
 &= - \int_{-\infty}^{-\epsilon} F(u) du \int_{-u}^{y-u} \frac{e^{-t}}{2t} dt - \int_{-\epsilon}^{y-\epsilon} F(u) du \int_\epsilon^{y-u} \frac{e^{-t}}{2t} dt \\
 &= \int_{-\infty}^{-\epsilon} [K(y-u) - K(u)] F(u) du + \int_{-\epsilon}^{y-\epsilon} [K(y-u) - K(\epsilon)] F(u) du.
 \end{aligned}$$

Similarly

$$I_2 = \int_\epsilon^{y+\epsilon} [K(\epsilon) - K(u)] F(u) du + \int_{y+\epsilon}^\infty [K(y-u) - K(u)] F(u) du.$$

Hence

$$\begin{aligned}
 \int_0^y f_2(x) dx &= \int_{-\infty}^{y-\epsilon} K(y-u) F(u) du + \int_{y+\epsilon}^\infty K(y-u) F(u) du \\
 &\quad - \int_{-\infty}^{-\epsilon} K(u) F(u) du - \int_\epsilon^\infty K(u) F(u) du \\
 &\quad + K(\epsilon) \left\{ \int_{y-\epsilon}^{y+\epsilon} F(u) du - \int_{-\epsilon}^\epsilon F(u) du \right\}.
 \end{aligned} \tag{8.2.3}$$

Now let $\epsilon \rightarrow +0$ in (8.2.1). Using (8.2.2) and (8.2.3), we obtain

$$\int_0^y F'(u) du = \int_{-\infty}^{\infty} K(y-u) F(u) du + \text{const.},$$

i.e.,
$$F(y) = \int_{-\infty}^{\infty} K(y-u) F(u) du + B,$$

say. By elementary computations we can easily show that

$$\int_{-\infty}^{\infty} K(y-u) u^2 du = y^2 + C,$$

where $C \neq 0$. Choosing $A = -BC^{-1}$, we have

$$\int_{-\infty}^{\infty} K(y-u) [F(u) - Au^2] du = F(y) - B - Ay^2 - AC = F(y) - Ay^2,$$

i.e., $F(y) - Ay^2$ satisfies (8.1.1).

8.3. THEOREM 11. If (i) $F(x)$ is absolutely continuous in every finite interval, (ii)

$$\int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|} |F(x)| dx < \infty,$$

and (iii)
$$F'(x) = \int_0^{\infty} \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt,$$

then, for some constants A, B, C ,

$$F(x) = Ax^2 + Bx + C.$$

THEOREM 12. If (i)

$$\int_{-\infty}^{\infty} \frac{e^{-|x|}}{1+|x|} |f(x)| dx < \infty,$$

and (ii)
$$f(x) = \int_{-\infty}^{\infty} K(x-y) f(y) dy,$$

where

$$K(x) = \frac{1}{2} \int_{|x|}^{\infty} \frac{e^{-y}}{y} dy,$$

then, for some constants A, B ,

$$f(x) = Ax + B.$$

By Theorem 10, Theorem 11 follows at once from Theorem 12, which it therefore suffices to prove. We first evaluate the characteristic function

$\kappa(s)$. We have

$$\begin{aligned}\kappa(s) &= \int_{-\infty}^{\infty} e^{-sx} K(x) dx \quad (-1 < \Re s < 1) \\ &= \int_{-\infty}^0 e^{-sx} dx \int_{-x}^{\infty} \frac{e^{-y}}{2y} dy + \int_0^{\infty} e^{-sx} dx \int_x^{\infty} \frac{e^{-y}}{2y} dy \\ &= \int_0^{\infty} \frac{e^{-y}}{2y} dy \int_{-y}^0 e^{-sx} dx + \int_0^{\infty} \frac{e^{-y}}{2y} dy \int_0^y e^{-sx} dx \\ &= \frac{1}{2s} \int_0^{\infty} [e^{-(1-s)y} - e^{-(1+s)y}] \frac{dy}{y} \\ &= \frac{1}{2s} \log \frac{1+s}{1-s},\end{aligned}$$

where we choose that branch of $\log(1 \pm s)$ which vanishes at $s = 0$.

$K(x)$ is an even function with a logarithmic discontinuity at the origin, and it behaves like $|x|^{-1}e^{-|x|}$ when $x \rightarrow \pm \infty$. Hence we can write

$$K(x) = K_1(x) + K_2(x),$$

where $K_2(x) \in L(-a, a)$ for some finite a , $K_2(x) = 0$ when $|x| > a$, and $|K_1(x)| \leq A(1+|x|)^{-1}e^{-|x|}$. To see that the remaining conditions of Hypothesis 7 (§ 6.1) can be made to hold, with $k = 1$, $r = 2$, we note that, when $x > a$,

$$\begin{aligned}2G_1(x) &= 2e^x K_1(x) = e^x \int_x^{\infty} \frac{e^{-y}}{y} dy \\ &= \frac{1}{x} - e^x \int_x^{\infty} \frac{e^{-y}}{y^2} dy.\end{aligned}$$

The characteristic equation is

$$\kappa(s) = \frac{1}{2s} \log \frac{1+s}{1-s} = 1.$$

In the neighbourhood of $s = 0$, $\kappa(s) = 1 + O(s^2)$; hence $s = 0$ is a double root. It is shown by Hopf (28, p. 74) that the equation has no other root in the strip $-1 < \Re s < 1$, and it is easy to see that the conditions of Hypothesis 4 (§ 3.3) are satisfied. We can now apply Theorem 8 (§ 6.3); Theorem 12 follows at once.

We note that Theorem 12 is best possible, in the sense that the exhaustiveness of the known solution is proved when it is supposed only that the equation has a meaning when the integration is taken in the sense of Lebesgue.

8.4. We now proceed to consider the second pair of equations, namely

$$f(x) = \int_0^\infty K(x-y)f(y)dy, \quad (8.4.1)$$

where

$$K(x) = \frac{1}{2} \int_{|x|}^\infty \frac{e^{-y}}{y} dy;$$

and

$$F'(x) = \int_0^x \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt + \int_x^\infty \frac{F(x+t)}{2t} e^{-t} dt, \quad (8.4.2)$$

where the first integral on the right-hand side of (8.4.2) is interpreted as

$$\lim_{\epsilon \rightarrow +0} \int_\epsilon^x \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt.$$

We begin with a discussion of equation (8.4.1).

THEOREM 13. *Equation (8.4.1) has precisely one linearly independent solution $f(x)$ such that*

$$(1+x)^{-1} e^{-x} f(x) \in L(0, \infty).$$

An explicit expression for this solution will be found in the course of the proof.

The kernel $K(x)$ has already been considered in this chapter; the characteristic function is

$$\kappa(s) = \frac{1}{2s} \log \frac{1+s}{1-s}.$$

Hypotheses 4, 5, and 7 (§§ 3.3, 4.1, 6.1 respectively) are clearly satisfied, with $k=1$; hence, as we proved in Chapter 7, we can apply Theorems 9 (§ 7.2) and 6 (§ 4.6). We must now evaluate $P(s)$, $\rho_1(s)$, N and m (cf. §§ 4.4–4.5). Clearly

$$P(s) = s^2, \quad N = 2.$$

Hence

$$\tau(s) = \frac{1}{s^2} \left(1 - \frac{1}{2s} \log \frac{1+s}{1-s} \right) (s+1)^m (s-1)^{2-m}.$$

The integer m must be chosen in such a way that the total increment of $\arg \tau(it)$ as t runs from $-\infty$ to $+\infty$ is equal to zero. A little consideration shows that we have to take $m=1$. Consequently

$$\tau(s) = \frac{s^2-1}{s^2} \left(1 - \frac{1}{2s} \log \frac{1+s}{1-s} \right).$$

For β we can take any value in the open interval $(0, 1)$. The function $\tau_1(s)$ is defined as in equation (4.4.2), *i.e.*,

$$\tau_1(s) = \exp \left\{ -\frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \frac{\log \tau(\zeta)}{\zeta-s} d\zeta \right\}.$$

Since $\tau_1(s)$ is independent of β , it is regular in the half-plane $\Re s > -1$. Next,

$$\rho_1(s) = (s+1)^{-m} \tau_1(s) = \frac{\tau_1(s)}{s+1}.$$

$\psi(s)$ is an arbitrary polynomial of degree $N-m-1=0$, *i.e.*, an arbitrary constant A . Hence

$$\phi(s) = \frac{A(s+1)}{s^2 \tau_1(s)} \quad (\Re s \geq \beta).$$

It follows that equation (8.4.1) has exactly one linearly independent solution $f(x)$, given by

$$e^{-\sigma x} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \phi(\sigma+it) dt \quad (\sigma \geq \beta),$$

where $\phi(s)$ is defined as above, and the integral is interpreted, if necessary, as a mean-square limit.

8.5. We now show how the solution of (8.4.2) can be reduced to that of (8.4.1).

THEOREM 14. *Let $f(x)$ be a non-trivial solution of (8.4.1) of the form described in Theorem 13. If (i) $F(x)$ is absolutely continuous in every finite interval, (ii)*

$$\int_0^\infty \frac{e^{-x}}{1+x} |F(x)| dx < \infty,$$

and (iii)

$$F'(x) = \int_0^x \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt + \int_x^\infty \frac{F(x+t)}{2t} e^{-t} dt,$$

then

$$F(x) = Cf(x) + D \int_0^x f(y) dy,$$

for some constants C, D . Conversely, any expression of this form satisfies (i), (ii), and (iii).

We write

$$F'(x) = f_1(x) + f_2(x), \tag{8.5.1}$$

where

$$\left. \begin{aligned} f_1(x) &= \int_0^\epsilon \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt, \\ f_2(x) &= \int_\epsilon^x \frac{F(x+t) - F(x-t)}{2t} e^{-t} dt + \int_x^\infty \frac{F(x+t)}{2t} e^{-t} dt. \end{aligned} \right\}$$

By the argument used in § 8.2, we can show that

$$\lim_{\epsilon \rightarrow +0} \int_0^y f_1(x) dx = 0. \quad (8.5.2)$$

Now

$$\begin{aligned} \int_0^y f_2(x) dx &= \int_0^y dx \int_\epsilon^\infty \frac{e^{-t}}{2t} F(x+t) dt + \int_0^\epsilon dx \int_x^\infty \frac{e^{-t}}{2t} F(x-t) dt \\ &\quad - \int_\epsilon^y dx \int_\epsilon^x \frac{e^{-t}}{2t} F(x-t) dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. In § 8.2 it was shown that I_1 (there called I_2) is given by

$$I_1 = \int_\epsilon^{y+\epsilon} [K(\epsilon) - K(u)] F(u) du + \int_{y+\epsilon}^\infty [K(y-u) - K(u)] F(u) du.$$

Next,

$$\begin{aligned} I_2 &= \int_0^\epsilon dx \int_{x-\epsilon}^0 \frac{e^{u-x}}{2(x-u)} F(u) du \\ &= \int_{-\epsilon}^0 F(u) du \int_0^{u+\epsilon} \frac{e^{u-x}}{2(x-u)} dx \\ &= \int_{-\epsilon}^0 F(u) du \int_{-u}^\epsilon \frac{e^{-t}}{2t} dt \\ &= \int_{-\epsilon}^0 [K(-u) - K(\epsilon)] F(u) du. \end{aligned}$$

Finally,

$$\begin{aligned} I_3 &= \int_\epsilon^y dx \int_0^{x-\epsilon} \frac{e^{u-x}}{2(u-x)} F(u) du \\ &= \int_0^{y-\epsilon} F(u) du \int_{u+\epsilon}^y \frac{e^{u-x}}{2(u-x)} dx \\ &= - \int_0^{y-\epsilon} F(u) du \int_\epsilon^{y-u} \frac{e^{-t}}{2t} dt \\ &= \int_0^{y-\epsilon} [K(y-u) - K(\epsilon)] F(u) du. \end{aligned}$$

We therefore have

$$\begin{aligned} \int_0^y f_2(x) dx &= \int_0^{y-\epsilon} K(y-u) F(u) du + \int_{y+\epsilon}^{\infty} K(y-u) F(u) du \\ &\quad + \int_{-\epsilon}^0 K(u) F(u) du - \int_{\epsilon}^{\infty} K(u) F(u) du \\ &\quad + K(\epsilon) \left\{ \int_{y-\epsilon}^{y+\epsilon} F(u) du - \int_{-\epsilon}^{\epsilon} F(u) du \right\}. \quad (8.5.3) \end{aligned}$$

Now let $\epsilon \rightarrow +0$ in (8.5.1). Using (8.5.2) and (8.5.3), we obtain

$$\int_0^y F'(x) dx = \int_0^{\infty} K(y-u) F(u) du + \text{const.},$$

$$\text{i.e.,} \quad F(y) = \int_0^{\infty} K(y-u) F(u) du + B. \quad (8.5.4)$$

$$\text{Write} \quad F_0(x) = \int_0^x f(y) dy,$$

where $f(x)$ is a (not identically zero) solution of (8.4.1). Then

$$f(x) = \int_0^{\infty} K(x-y) f(y) dy = \int_{-\infty}^x f(x-u) K(u) du.$$

Hence

$$\begin{aligned} F_0(x) &= \int_0^x dy \int_{-\infty}^y f(y-u) K(u) du \\ &= \int_{-\infty}^0 K(u) du \int_0^x f(y-u) dy + \int_0^x K(u) du \int_u^x f(y-u) dy \\ &= \int_{-\infty}^x K(u) F_0(x-u) du - \int_{-\infty}^0 K(u) F_0(-u) du \\ &= \int_0^{\infty} K(x-y) F_0(y) dy + A, \end{aligned}$$

say. We must now show that $A \neq 0$. Suppose that $A = 0$. Then

$$F_0(x) = \int_0^{\infty} K(x-y) F_0(y) dy,$$

i.e., $F_0(x)$ is a solution, not identically zero, of (8.4.1). Hence, by Theorem 13, $F_0(x) = \lambda f(x)$, for some constant $\lambda \neq 0$. Differentiating, we

obtain $f(x) = \lambda f'(x)$, whence it follows that $f(x) = ae^{x/\lambda}$ for some $a \neq 0$, and that

$$F_0(x) = \lambda a(e^{x/\lambda} - 1),$$

contradicting the result that $F_0(x) = \lambda f(x)$. Hence we must have $A \neq 0$.

Now write

$$F_1(x) = F(x) - A^{-1} B F_0(x).$$

Then
$$\int_0^\infty K(x-y) F_1(y) dy = F(x) - B - BA^{-1} F_0(x) + B$$
$$= F_1(x),$$

i.e., $F_1(x)$ is a solution of (8.4.1), and is therefore of the form $Cf(x)$. Hence

$$F(x) = Cf(x) + A^{-1} B \int_0^x f(y) dy,$$

the required result.

We must now prove the converse results. Let

$$F(x) = Cf(x) + D \int_0^x f(y) dy,$$

where C and D are arbitrary constants. We first show that $F(x)$ is absolutely continuous in every finite interval. This is clearly true of $\int_0^x f(y) dy$, so we need only consider $f(x)$. We have

$$\begin{aligned} f(x) &= \int_0^\infty K(x-y) f(y) dy \\ &= \int_0^\infty K(x-y) dy \int_0^\infty K(y-z) f(z) dz \\ &= \int_0^\infty K_2(x, z) f(z) dz, \end{aligned}$$

where
$$K_2(x, z) = \int_0^\infty K(x-y) K(y-z) dy.$$

If we continue this process, we obtain

$$f(x) = \int_0^\infty K_n(x, z) f(z) dz;$$

for sufficiently large n , $K_n(x, z)$ is smooth enough for differentiation under the integral sign in the last equation to be permissible. We do not

go into the details of the proof of this fact; the proof is tedious, and offers no difficulties of principle.

The proof of the estimation (ii) runs on obvious lines.

We now prove (iii). It has already been shown that

$$F(x) = A + \int_0^\infty K(x-y) F(y) dy$$

for some constant A . Hence

$$\begin{aligned} F(x) &= A + \frac{1}{2} \int_0^x F(y) dy \int_{x-y}^\infty \frac{e^{-u}}{u} du + \frac{1}{2} \int_x^\infty F(y) dy \int_{y-x}^\infty \frac{e^{-u}}{u} du \\ &= A + \frac{1}{2} \int_0^x \frac{e^{-u}}{u} du \int_{x-u}^{x+u} F(y) dy + \frac{1}{2} \int_x^\infty \frac{e^{-u}}{u} du \int_0^{x+u} F(y) dy. \end{aligned}$$

Since $F(x)$ is absolutely continuous in every finite interval, we can justify the formal differentiation of the last equation by reversing the argument used in the first half of this section. The differentiation gives

$$\begin{aligned} F'(x) &= \frac{e^{-x}}{2x} \int_0^{2x} F(y) dy + \int_0^x \frac{F(x+u) - F(x-u)}{2u} e^{-u} du \\ &\quad - \frac{e^{-x}}{2x} \int_0^{2x} F(y) dy + \int_x^\infty \frac{F(x+u)}{2u} e^{-u} du \\ &= \int_0^x \frac{F(x+u) - F(x-u)}{2u} e^{-u} du + \int_x^\infty \frac{F(x+u)}{2u} e^{-u} du, \end{aligned}$$

i.e., $F(x)$ satisfies (8.4.2).

This concludes the proof of Theorem 14.

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THE MEAN VALUES OF CERTAIN DIRICHLET SERIES, I

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[Received 16 September, 1939.—Read 15 December, 1939.]

1. *Introduction.*

Suppose that the series

$$(1) \quad f(s) = \sum_{n=1}^{\infty} a_n l_n^{-s} \quad (s = \sigma + it)$$

has a half-plane of convergence and that

$$0 < l_1 < l_2 < \dots < l_n < l_{n+1} \dots \quad (l_n \rightarrow \infty).$$

Let σ_a denote the smallest number such that $f(s)$ is of finite order for* $\sigma \geq \sigma_a + \epsilon > \sigma_a$ and all its singularities in $\sigma \geq \sigma_a + \epsilon$ are contained in a rectangle R of finite area. We suppose that, in the integral

$$\int_{-T}^T |f(\sigma + it)|^2 dt,$$

that part of the range of integration which belongs to R is omitted. Under these conditions Carlson† has proved

THEOREM A. *If*

$$(2) \quad \int_{-T}^T |f(\sigma + it)|^2 dt = O(T)$$

for $\sigma > \sigma_b$, then

$$(3) \quad \int_{-T}^T |f(\sigma + it)|^2 dt \sim 2T \sum_{n=1}^{\infty} |a_n|^2 l_n^{-2\sigma}$$

*for $\sigma > \max(\sigma_a, \sigma_b)$.** Here and throughout this paper ϵ denotes an arbitrary positive number.

† F. Carlson, 1.

The mean-order function $\lambda(\sigma)$ is defined as the lower bound of numbers ξ for which

$$\int_{-T}^T |f(\sigma + it)|^2 dt = O(T^\xi).$$

It is well known* that $\lambda(\sigma)$ is a continuous convex function of σ . Let σ_m denote the abscissa of mean convergence, *i.e.* the least number such that (3) holds for $\sigma > \sigma_m$. Concerning σ_m Carlson has proved† the following analogue of the Schnee-Landau theorem.

THEOREM B. If (i) $a_n = O(l_n^{h+\epsilon})$,

$$(ii) \quad \log(l_n/l_{n-1}) > l_n^{-(l+\epsilon)}$$

for $n \geq n_0(\epsilon)$, where $l \geq 0$,

$$(iii) \quad L = \frac{1}{2}(\bar{\sigma} + h + l), \quad q = L - \frac{L - \eta}{\lambda(\eta)},$$

where $\bar{\sigma}$ is the abscissa of absolute convergence of (1) and $\eta > \sigma_a$, then

$$\sigma_m \leq q.$$

It is our aim to prove the corresponding theorem when the condition (i) above is replaced by the condition

$$(i)' \quad \sum_{l_n \leq x} |a_n|^2 = O(x^{a+\epsilon}),$$

or the equivalent condition (i)'' $\sum_1^\infty |a_n|^2 l_n^{-s}$ is convergent for $\sigma > a$. We shall then apply the theorem to Hecke's Dirichlet series with signature‡ $\{\lambda, k, \gamma\}$, and deduce, for instance, that, if

$$f(s) = \sum_{n=1}^\infty a_n n^{-s}$$

is regular at $s = k$ and has the signature $\{\lambda, k, \gamma\}$, where $0 < \lambda < 2$, then

$$\int_{-T}^T |f(\sigma + it)|^2 dt \sim 2T \sum_{n=1}^\infty |a_n|^2 n^{-2\sigma}$$

for $\sigma > \frac{1}{2}k$.

* Cf. F. Carlson, 2, or G. H. Hardy, 1.

† F. Carlson, 3.

‡ This is defined in § 4.

2. *The main theorem.*THEOREM 1. *If*

$$(i) \quad \sum_{l_n \leq x} |a_n|^2 = O(x^{a+\epsilon}),$$

$$(ii) \quad \log(l_n/l_{n-1}) > l_n^{-(l+\epsilon)}$$

for $n \geq n_0$, where $l \geq 0$,

$$(iii) \quad L = \frac{1}{2}(a+l), \quad q = L - \frac{L-\eta}{\lambda(\eta)},$$

where $\eta > \sigma_a$, then $\sigma_m \leq q$, i.e.

$$(3) \quad \int_{-T}^T |f(\sigma+it)|^2 dt \sim 2T \sum_{n=1}^{\infty} |a_n|^2 l_n^{-2\sigma}$$

for $\sigma > q$.

First of all we shall state a few lemmas.

LEMMA 1. *If the sequence $l_1, l_2, l_3, \dots$ satisfies the condition (ii) of Theorem 1 and*

$$N(x) = \sum_{l_n \leq x} 1,$$

then

$$N(x) = O(x^{l+\epsilon})$$

and

$$\log(l_m/l_n) > (m-n)l_m^{-(l+\epsilon)}$$

if $m > n \geq n_0$.

For the proofs we refer to Landau, 1, 2, 841.

LEMMA 2. *If $0 < l_1 < l_2 \dots l_n < l_{n+1} \dots, l_n \rightarrow \infty$, and*

$$C(x) = \sum_{l_n \leq x} c_n = O(x^c),$$

where $c_1, c_2, \dots$ are positive, then as $\delta \rightarrow +0$

$$\sum_{n=1}^{\infty} c_n l_n^{-\sigma} \exp(-\delta l_n^\beta) = O(\delta^{(\sigma-c)/\beta}),$$

provided that $\sigma < c$ and $\beta > 0$.

Proof. We have, if $\delta > 0$,

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n l_n^{-\sigma} \exp(-\delta l_n^\beta) \\ &= \int_0^\infty x^{-\sigma} \exp(-\delta x^\beta) dC(x) \\ &= \beta \delta \int_0^\infty C(x) x^{\beta-\sigma-1} \exp(-\delta x^\beta) dx + \sigma \int_0^\infty C(x) x^{-\sigma-1} \exp(-\delta x^\beta) dx \\ &= O\left(\delta \int_0^\infty t^{(c-\sigma)/\beta} e^{-\delta t} dt\right) + O\left(\int_0^\infty t^{(c-\sigma)/\beta-1} e^{-\delta t} dt\right) \\ &= O(\delta^{\sigma-c/\beta}) \end{aligned}$$

if $\sigma < c$ and $\beta > 0$.

LEMMA 3. *If*

$$A(x) = \sum_{l_n \leq x} |a_n|^2 = O(x^{a+\epsilon}),$$

then, as $\delta \rightarrow +0$,

$$\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ m \neq n}}^{\infty} \frac{|a_m a_n| \exp\{-\delta(l_m^\beta + l_n^\beta)\}}{(l_m l_n)^\sigma |\log(l_m/l_n)|} = O(\delta^{2(\sigma-L-2\epsilon)/\beta}),$$

provided that $\sigma \leq L$ and $\beta > 0$.

Proof. The sum in question is equal to

$$2 \sum_{1 \leq n < m < \infty} \frac{|a_m a_n| \exp\{-\delta(l_m^\beta + l_n^\beta)\}}{(l_m l_n)^\sigma |\log(l_m/l_n)|}.$$

Let S_1 and S_2 denote the sum of the terms for which $l_n \leq \frac{1}{2}l_m$ and the sum of the remaining terms for which $\frac{1}{2}l_m < l_n < l_m$. Since $\log(l_m/l_n) \geq \log 2$ in S_1 we have

$$\begin{aligned} S_1 &= O\left\{\left(\sum_{n=1}^{\infty} |a_n| l_n^{-\sigma} \exp(-\delta l_n^\beta)\right)^2\right\} \\ &= O\left\{\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{l_n^{2\sigma-L}} \exp(-\delta l_n^\beta)\right) \left(\sum_{n=1}^{\infty} \frac{\exp(-\delta l_n^\beta)}{l_n^L}\right)\right\} \\ &= O(\delta^{2(\sigma-L-\epsilon)/\beta}) \end{aligned}$$

by Lemmas 1 and 2 if $\sigma \leq L$. To evaluate S_2 we must first consider

$$B(x) = \sum_{l_m \leq x} b_m, \quad \text{where} \quad b_m = \sum_{l_n < l_m} \frac{|a_m a_n|}{m-n}.$$

Putting $m = n + g$, we have

$$\begin{aligned} B(x) &= \sum_{n < n+g \leq N(x)} \sum_{g < N(x)} \frac{|a_{n+g} a_n|}{g} = \sum_{g < N(x)} \frac{1}{g} \sum_{n \leq N(x)-g} |a_{n+g} a_n| \\ &< \sum_{g < N(x)} \frac{1}{g} \left\{ \sum_{n \leq N(x)-g} |a_{n+g}|^2 \sum_{n \leq N(x)} |a_n|^2 \right\}^{\frac{1}{2}} \\ &= A(x) \sum_{g < N(x)} \frac{1}{g} = O(x^{\epsilon+2\epsilon} \log N(x)) \\ &= O(x^{a+2\epsilon}) \end{aligned}$$

since $N(x) = O(x^{l+\epsilon})$. Hence by Lemmas 1 and 2

$$\begin{aligned} S_2 &= O \left(\sum_{\frac{1}{2}l_m < l_n < l_m} \sum_{n \leq N} \frac{|a_m a_n| \exp(-\delta l_m^\beta)}{l_m^{2\sigma-l-\epsilon}(m-n)} \right) \\ &= O \left(\sum_{m=2}^{\infty} b_m l_m^{-2\sigma+l+\epsilon} \exp(-\delta l_m^\beta) \right) \\ &= O(\delta^{2(\sigma-L-2\epsilon)/\beta}). \end{aligned}$$

3. Proof of the main theorem.

First of all the series for $f(s)$ is absolutely convergent for $\sigma > L$. For if the abscissa of absolute convergence is positive its value is

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log \left(\sum_{n \leq N} |a_n| \right)}{\log l_N} &\leq \frac{1}{2} \lim_{N \rightarrow \infty} \frac{\log \left(N \sum_{n \leq N} |a_n|^2 \right)}{\log l_N} \\ &\leq \frac{1}{2} \lim_{N \rightarrow \infty} \frac{\log (l_N^{1+\epsilon} l_N^{a+\epsilon})}{\log l_N} \\ &= L + \epsilon. \end{aligned}$$

It follows that (3) holds for $\sigma > L$. We have*, if $c > L$, $c > \sigma$ and $\beta > 0$,

$$\begin{aligned} \frac{1}{2\pi i \beta} \int_{c-i\infty}^{c+i\infty} \Gamma \left(\frac{\omega-s}{\beta} \right) f(w) \delta^{(s-w)/\beta} dw &= \sum_{n=1}^{\infty} a_n l_n^{-s} \exp(-\delta l_n^\beta) \\ &= f(\delta, s), \end{aligned}$$

* See, e.g., E. C. Titchmarsh, **1**, 301.

say. Suppose that $s (= \sigma + it)$ does not lie in the rectangle R and that $\frac{1}{2}T \leq |t| \leq T$. We replace the above path of integration by one made up of the following parts:

- (a) C_1 : the segment $w = c - iv$, $2T < v < \infty$,
- (b) C_2 : the segment $w = u - 2iT$, where $\eta < u \leq c$,
- (c) C_3 : the segments $w = \eta + iv$, where $H \leq |v| \leq 2T$ and H is chosen large enough to ensure that C_3 does not overlap R ,
- (d) C_R : a simple contour of finite length round the rectangle R to the "right", joining the points $\eta - iH$, $\eta + iH$,
- (e) C_5 : the segment $w = u + 2iT$, where $\eta < u \leq c$,
- (f) C_6 : the segment $w = c + iv$, where $2T < v < \infty$.

Let C denote the contour composed of these parts. Then, if $\sigma - \beta < \eta < \sigma$, we have

$$f(\delta, s) = f(s) + \frac{1}{2\pi i \beta} \int_C \Gamma\left(\frac{w-s}{\beta}\right) f(w) \delta^{(s-w)/\beta} dw,$$

for the integrand has then a single pole at $w = s$ with residue $f(s)$. Since $\Gamma(x+iy) = O(e^{-\frac{1}{2}\pi|y|} |y|^{x-\frac{1}{2}})$, we have*

$$\begin{aligned} \int_{C_1} \Gamma\left(\frac{w-s}{\beta}\right) f(w) \delta^{(s-w)/\beta} dw &= O\left\{\delta^{(\sigma-c)/\beta} \int_{2T}^{\infty} e^{-\frac{1}{2}\pi|v+t|/\beta} v^K dv\right\} \\ &= O(\delta^{(\sigma-c)/\beta} T^K e^{-\pi T/2\beta}) = O(e^{-KT}) \end{aligned}$$

if $\delta > T^{-K}$. Similarly the integral along C_6 is of the same order. Next we have

$$\begin{aligned} \int_{C_2} \Gamma\left(\frac{w-s}{\beta}\right) f(w) \delta^{(s-w)/\beta} dw &= O(e^{-\pi|2T+t|/2\beta} T^K \delta^{-K}) \\ &= O(e^{-KT}). \end{aligned}$$

The same result holds for the integral along C_5 . Finally

$$\begin{aligned} \int_{C_R} \Gamma\left(\frac{w-s}{\beta}\right) f(w) \delta^{(s-w)/\beta} dw &= O\left(\int_{C_R} e^{-\pi|v-t|/2\beta} |v-t|^K v^K \delta^{(\sigma-u)/\beta} |dw|\right) \\ &= O(\delta^{-K} |t|^K e^{-\pi|t|/2\beta}) \\ &= O(e^{-KT}) \end{aligned}$$

* K denotes a positive number, not necessarily the same at each occurrence, depending possibly on σ , η , c , β but independent of t , v , δ , T .

if $\frac{1}{2}T < t < T$ and $\delta > T^{-K}$. Hence

$$\begin{aligned} f(s) &= f(\delta, s) - \frac{1}{2\pi i \beta} \int_{\eta-2iT}^{\eta+2iT} \Gamma\left(\frac{w-s}{\beta}\right) f(w) \delta^{(s-w)/\beta} dw + O(e^{-KT}) \\ &= f(\delta, s) - \phi(\delta, s) + O(e^{-KT}), \end{aligned}$$

say. But, by Minkowski's inequality,

$$\begin{aligned} \left\{ \int_{\frac{1}{2}T}^T |f(s)|^2 dt \right\}^{\frac{1}{2}} &< \left\{ \int_{\frac{1}{2}T}^T |f(\delta, s)|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_{\frac{1}{2}T}^T |\phi(\delta, s)|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_{\frac{1}{2}T}^T |O(e^{-KT})|^2 dt \right\}^{\frac{1}{2}} \\ &= I_1^{\frac{1}{2}} + I_2^{\frac{1}{2}} + O(e^{-KT}), \end{aligned}$$

say. Now

$$\begin{aligned} I_1 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \bar{a}_n (l_m l_n)^{-\sigma} \exp\{-\delta(l_m^{\beta} + l_n^{\beta})\} \int_{\frac{1}{2}T}^T (l_m/l_n)^{it} dt \\ &= \frac{1}{2}T \sum_{n=1}^{\infty} |a_n|^2 l_n^{-2\sigma} \exp(-2\delta l_n^{\beta}) + O\left\{ \sum_{m \neq n} \frac{|a_m a_n| \exp\{-\delta(l_m^{\beta} + l_n^{\beta})\}}{(l_m l_n)^{\sigma} |\log(l_m/l_n)|} \right\} \\ &= O(T) + O(\delta^{2(\sigma-L-2\epsilon)/\beta}) \end{aligned}$$

by Lemma 3, provided that $\frac{1}{2}\alpha < \sigma < L$. Also, by Schwarz's inequality,

$$\begin{aligned} |\phi(\delta, s)|^2 &\leq \frac{\delta^{2(\sigma-\eta)/\beta}}{4\pi^2 \beta^2} \int_{-2T}^{2T} \left| \Gamma\left(\frac{w-s}{\beta}\right) \right| dv \int_{-2T}^{2T} \left| \Gamma\left(\frac{w-s}{\beta}\right) \right| |f(w)|^2 dv \\ &= O\left(\delta^{2(\sigma-\eta)/\beta} \int_{-2T}^{2T} e^{-K|v-t|} dv \int_{-2T}^{2T} e^{-K|v-t|} |f(w)|^2 dv \right) \\ &= O\left(\delta^{2(\sigma-\eta)/\beta} \int_{-2T}^{2T} e^{-K|v-t|} |f(\eta+iv)|^2 dv \right). \end{aligned}$$

Hence

$$\begin{aligned} I_2 &= O\left(\delta^{2(\sigma-\eta)/\beta} \int_{-2T}^{2T} |f(\eta+iv)|^2 dv \int_{\frac{1}{2}T}^T e^{-K|v-t|} dt \right) \\ &= O(\delta^{2(\sigma-\eta)/\beta} T^{\lambda(\eta)+\epsilon}). \end{aligned}$$

Let us choose δ so that

$$\delta^{2(\eta-L)} = T^{\beta\lambda(\eta)}.$$

If we put $q = L - (L - \eta)/\lambda(\eta)$, this gives

$$I_1 = O(T) + O(T^{(L-\sigma+2\epsilon)/(L-q)}) = O(T)$$

provided that $\eta < L$, $\sigma > q$, and that ϵ is small enough. Similarly

$$I_2 = O(T^{(L-\sigma)/(L-q)+\epsilon}) = O(T)$$

under the same conditions. Hence, if $\eta < L$ and $\sigma > q$,

$$\int_{\frac{1}{2}T}^T |f(\sigma + it)|^2 dt = O(T).$$

Thus

$$\begin{aligned} \int_H^T |f(\sigma + it)|^2 dt &< \sum_{n=0}^{K \log T} \int_{2^{-n-1}T}^{2^{-n}T} |f(\sigma + it)|^2 dt \\ &= \sum_n O(2^{-n} T) = O(T). \end{aligned}$$

And similarly

$$\int_{-T}^{-H} |f(\sigma + it)|^2 dt = O(T)$$

if $\sigma > q$. The truth of our theorem now follows by Theorem A.

4. Applications to Dirichlet series with functional equations.

We first consider the class of Dirichlet series discussed by Hecke*. Suppose that the ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is a function with the signature $\{\lambda, k, \gamma\}$, where $\lambda > 0$, $k > 0$ and $\gamma = \pm 1$. That is, we suppose that

(a) $(s-k)f(s)$ is an integral function of finite order,

(b) $R(s) = \gamma R(k-s)$, where

$$(4) \quad R(s) = (2\pi/\lambda)^{-s} \Gamma(s) f(s).$$

For these functions we have the following theorem:

THEOREM 2. If $f(s)$ has the signature $\{\lambda, k, \gamma\}$ and

$$(5) \quad \sum_{n \leq x} |a_n|^2 = O(x^{k+\epsilon}),$$

then

$$(6) \quad \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^2 dt \sim \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}$$

for $\sigma > \frac{1}{2}k$, and

$$(7) \quad \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^2 dt \sim \left(\frac{2\pi}{\lambda}\right)^{2\sigma-k} \frac{T^{2k-4\sigma}}{2k-4\sigma+1} \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2k-2\sigma}}$$

for $\sigma < \frac{1}{2}k$.

* E. Hecke, 1, 2, 3.

Proof. Let $F(\sigma) = \sum |a_n|^2 n^{-2\sigma}$

and

$$j(t) = \int_{-t}^t |f(k - \sigma - iu)|^2 du,$$

so that

$$j(t) \sim 2t F(k - \sigma)$$

if $\sigma < k - \sigma_m$. Then, by the functional equation (b),

$$\begin{aligned} (8) \quad & \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^2 dt \\ &= \left(\frac{2\pi}{\lambda}\right)^{2\sigma-k} \frac{1}{2T} \int_{-T}^T \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 |f(k-s)|^2 ds \\ &= \left(\frac{2\pi}{\lambda}\right)^{2\sigma-k} \frac{1}{2T} \int_0^T t^{2k-4\sigma} \{1 + O(t^{-2})\} j'(t) dt \\ &= \left(\frac{2\pi}{\lambda}\right)^{2\sigma-k} \left[T^{2k-4\sigma} \frac{j(T)}{2T} - \frac{k-2\sigma}{T} \int_0^T t^{2k-4\sigma-1} j(t) dt \right] + O(T^{2k-4\sigma-2}) \\ &\sim \left(\frac{2\pi}{\lambda}\right)^{2\sigma-k} \frac{F(k-\sigma)}{2k-4\sigma+1} T^{2k-4\sigma} \end{aligned}$$

provided that $\sigma < k - \sigma_m$. It remains to show that $\sigma_m = \frac{1}{2}k$. First of all, by (8), since $\lambda(\sigma)$ is continuous,

$$\lambda(\sigma) = \begin{cases} 2k-4\sigma+1 & \text{if } \sigma \leq k-\sigma_m, \\ 1 & \text{if } \sigma \geq \sigma_m. \end{cases}$$

And, since $\lambda(\sigma)$ is convex*, we have

$$\lambda(\sigma) \geq 1 \quad \text{for all } \sigma,$$

so that

$$4\sigma_m - 2k + 1 = \lambda(k - \sigma_m) \geq 1,$$

i.e. $\sigma_m \geq \frac{1}{2}k$. On the other hand, taking $\eta = k - \sigma_m$ in Theorem 1, we have

$$\sigma_m \leq \frac{1}{2}(k+1) - \frac{\frac{1}{2}(k+1) - (k - \sigma_m)}{4\sigma_m - 2k + 1},$$

i.e.

$$(2\sigma_m - k)^2 \leq 0$$

so that

$$\sigma_m = \frac{1}{2}k.$$

We may extend the application of Theorem 1 to a more general class of Dirichlet series.

* F. Carlson, 2, or G. H. Hardy, 1.

THEOREM 3. Suppose that the functions

$$(9) \quad f_1(s) = \sum_{n=1}^{\infty} a_n l_n^{-s}, \quad f_2(s) = \sum_{n=1}^{\infty} b_n k_n^{-s},$$

both satisfy the conditions of Theorem 1, so that

$$(10) \quad \log(l_n/l_{n-1}) > l_n^{-(l+\epsilon)}, \quad \log(k_n/k_{n-1}) > k_n^{-(l+\epsilon)}$$

for $n \geq n_0(\epsilon)$, and

$$(11) \quad \sum_{l_n \leq x} |a_n|^2 = O(x^{\beta+\epsilon}), \quad \sum_{k_n \leq x} |b_n|^2 = O(x^{\beta+\epsilon}).$$

Suppose also that $f_1(s)$ and $f_2(s)$ satisfy the functional equation

$$(12) \quad f_1(s) = H(s)f_2(\alpha-s),$$

where

$$(13) \quad H(s) = O(|t|^{c(\frac{1}{2}\alpha-\sigma)}),$$

$c > 0$, and

$$(14) \quad 1/H(s) = O(|t|^{c(\sigma-\frac{1}{2}\alpha)}).$$

Then

$$(15) \quad \int_{-T}^T |f_1(\sigma+it)|^2 dt \sim 2T \sum_{n=1}^{\infty} |a_n|^2 l_n^{-2\sigma}$$

for $\sigma > \max(\frac{1}{2}\alpha, L-1/c)$, where $L = \frac{1}{2}(\beta+l)$, and a similar result holds for $f_2(s)$.

Proof. Let $\lambda_1(\sigma)$, $\lambda_2(\sigma)$ be the mean order functions of $f_1(s)$, $f_2(s)$, and let σ_1 , σ_2 be the abscissae of mean convergence. By (12) and (13),

$$(16) \quad \begin{cases} \lambda_1(\sigma) \leq c(\alpha-2\sigma)+1 & \text{for } \sigma \leq \alpha-\sigma_2, \\ \lambda_1(\sigma) = 1 & \text{for } \sigma \geq \sigma_1, \end{cases}$$

and, by (12) and (14),

$$(17) \quad \begin{cases} \lambda_2(\sigma) \leq c(\alpha-2\sigma)+1 & \text{for } \sigma \leq \alpha-\sigma_1, \\ \lambda_2(\sigma) = 1 & \text{for } \sigma \geq \sigma_2. \end{cases}$$

By (16), $1 \leq \lambda_1(\alpha-\sigma_2) \leq c(2\sigma_2-\alpha)+1$,

so that $\sigma_2 \geq \frac{1}{2}\alpha$. Similarly $\sigma_1 \geq \frac{1}{2}\alpha$. Again, by Theorem 1,

$$\sigma_1 \leq L - \frac{L-\alpha+\sigma_2}{c(2\sigma_2-\alpha)+1},$$

so that

$$(18) \quad \sigma_1 - (L - 1/c) \leq \frac{\sigma_2 - (L - 1/c)}{c(2\sigma_2 - a) + 1}.$$

Similarly

$$(19) \quad \sigma_2 - (L - 1/c) \leq \frac{\sigma_1 - (L - 1/c)}{c(2\sigma_1 - a) + 1}.$$

Hence, by (18) and (19), since $c(2\sigma_2 - a) \geq 0$,

$$\sigma_1 - (L - 1/c) \leq \frac{\sigma_1 - (L - 1/c)}{\{c(2\sigma_1 - a) + 1\} \{c(2\sigma_2 - a) + 1\}}.$$

It follows that either $\sigma_1 = \sigma_2 = \frac{1}{2}a$ or $\sigma_1 \leq L - 1/c$ and $\sigma_2 \leq L - 1/c$.

THEOREM 4. *If $f(s)$ has the signature $\{\lambda, k, \gamma\}$, where $0 < \lambda < 2$, and $f(s)$ is regular at $s = k$, then*

$$\int_{-T}^T |f(\sigma + it)|^2 dt \sim 2T \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}$$

for $\sigma > \frac{1}{2}k$.

Proof. Under these conditions Hecke* has proved that

$$\sum_{n \leq x} |a_n|^2 = O(x^k).$$

Hence the result follows from Theorem 2.

We describe $F(\tau)$ as an integral modular form of the kind† $(-k, S)$, if

(a) $F(\tau)$ is regular inside the upper half-plane,

(b) S is a subgroup‡ of the modular group and for every substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of S we have

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = \omega(c\tau + d)^k F(\tau) \quad (|\omega| = 1),$$

(c) for every substitution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the modular group there is a power series expansion

$$(c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\sum_{n=0}^{\infty} b_n e^{2\pi i n \tau / N}\right) e^{2\pi i \rho \tau},$$

where N is a positive integer and $\rho \geq 0$.

* E. Hecke, 3, 80 and 2, 8.

† See E. Hecke, 3, 81.

‡ Of finite index.

If $\rho > 0$, then we call $F(\tau)$ a cusp form (*Spitzenform*).

THEOREM 5. Suppose that

$$F(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$$

is a cusp form of the kind $(-k, S)$ and that the corresponding Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has the signature $\{\lambda, k, \gamma\}$. Then

$$\int_{-T}^T |f(\sigma + it)|^2 dt \sim 2T \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}$$

for $\sigma > \frac{1}{2}k$.

Proof. In this case also Hecke has proved* that

$$\sum_{n \leq x} |a_n|^2 = O(x^k),$$

so the result follows from Theorem 2.

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